ON REFLEXIVE GEOMETRY*

BY

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This memoir is in continuation of two in these Transactions (vol. I, p. 97 and vol. IV, p. 1). It follows especially § 3 of the first memoir. The object of the three is to illustrate an algebraic method of handling planar displacements, which applies more immediately to elementary geometry than does the usual coördinate geometry. While the latter heads directly for projective geometry, the present method heads for inversive geometry. It may perhaps be called *reflexive geometry*.

§ 1. *Outline of the method.*

This is not the place to expound the beginnings of the method, but an outline may be given. On analysing displacements of a plane on a plane, on the hypothesis that translations exist, we find easily that they amount to

(1) reflexion in a line chosen once for all,

(2) rotations about a point, chosen once for all, and

(3) translations.

We take the point on the line and call them base-point and base-line.

Let now the displaced object have symmetry; and passing at once to the extreme case let it be a point. Displacements previously different become the the same, and equations arise.

The subsequent details are merely an enlargement of the planimetric interpretation of the fundamental operations of algebra, as given in works on the theory of functions and in some works on algebra.

The notation used is a superposed bar for reflexion in the base-line (thus $\bar{x}$ is the image in the base-line, or "conjugate" of x) and the letter t or τ for rotation about the base-point. The symbol 0 denotes that the moving particle is at the base-point, the symbol 1 that it has undergone a unit translation along the base-line, the symbol t that it is somewhere on the circle with center 0 and radius the unit of length. This circle is called the *base-circle*, and the special complex number t is called a *turn*.

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§ 2. *The cyclogen.*

Consider the series of self-conjugate equations:

$$a + \bar{a}t = 0,$$

$$a + 2\mu t + \bar{a}t^2 = 0,$$

$$a + bt + \bar{b}t^2 + \bar{a}t^3 = 0,$$

$$a + bt + 2\mu t^2 + \bar{b}t^3 + \bar{a}t^4 = 0,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

where μ is real. Such an equation is of the same generality as the equation with real coefficients usually studied in the theory of equations. In fact by mapping the base-line on the base-circle they are interchanged. To the real roots of the latter correspond the turn roots of the former, to conjugate roots of the latter correspond inverse points as to the base-circle.

Thus every equation of our series, whose degree is odd, has one turn as a root. We take as the standard case an equation all of whose roots are turns.

Now regard in the above equations a pair of conjugate coefficients as variable. We have then for given t a line, for varying t a curve of lines.

If we denote by n the class of the curve and by m the number of lines of the curve in a given direction, then the curve may be denoted by C^n_m. The simpler curves are:

C^1_1, a point;	C^3_3, a cardioid;
C^2_0, a segment;	C^4_0, a segment*;
C^2_2, a circle;	C^4_2, a parastroid;
C^3_1, a deltoid;	C^4_4, a paranephroid.

These curves may be called *cyclogens*.

We are concerned here solely with the curves C^n_n for which the end coefficients are variables. If we mean by the *aspect* of a curve the number of parallel tangents, then these curves are cyclogens of full aspect. They might be called cardioids, or better ennacardioids. Thus C^4_4 is the tetracardioid. Denoting the curve C^n_n now simply by C^n, the series of ennacardioids begins with the point C^1, the circle C^2, the cardioid C^3. We call the equation by which a cyclogen was defined its line-equation, having no other line-equation in the context. It comes under what Laguerre (*Works*, vol. 2, p. 190), called the "équation mixte."

* Described with superposed harmonic motions.

§ 3. *The ennacardioid.*

Taking now the case of C^n, we apply to it STUDY's theory of osculants. We write the line equation in abridged form

(1) $$(\alpha t)^n = 0,$$

where $(\alpha t) = \alpha_1 + \alpha_2 t$, $\alpha_1^r \alpha_2^{n-r}$ is a coefficient conjugate to $\alpha_1^{n-r}\alpha_2^r$ and $x = \alpha_1^n$.

The equation

$$(\alpha t_1)(\alpha t)^{n-1} = 0$$

defines a (first) osculant; it is a curve C^{n-1}. An osculant of this is a second osculant of C^n. And so on till we come to the completely polarized form

(2) $$(\alpha t_1)(\alpha t_2)\cdots(\alpha t_n) = 0,$$

which shows that the n osculant points, formed from $n-1$ of the n t's, lie on a *line*.

The relation of x to t, deduced from

$$(\alpha t)^n = 0$$

is

(3) $$\alpha_1(\alpha t)^{n-1} = 0.$$

This is the map-equation of C^n. Its polars are map-equations of the osculants.

If in a map-equation we calculate $D_t x$ and make it zero, we obtain values of t which, when turns, give cusps. The cusps of $(\alpha t)^n$ are then given by

(4) $$\alpha_1\alpha_2(\alpha t)^{n-2} = 0,$$

a self-conjugate equation. Thus C^n can have $n-2$ cusps. To avoid periphrasis I will use the projective view and say that there are always $n-2$ cusps, real or imaginary, though, strictly, imaginary points have no foothold in the reflexive geometry, which is a chapter of mathematics preceding projective geometry.

The cusps themselves are

$$\alpha_1(\alpha t)^{n-1} = 0$$

where

$$\alpha_1\alpha_2(\alpha t)^{n-2} = 0,$$

or are

(5) $$\alpha_1^2(\alpha t)^{n-2} = 0.$$

The map equation of an osculant is

$$\alpha_1(\alpha t_1)(\alpha t)^{n-2} = 0$$

and this equation holds when (4) and (5) do. That is: *The osculant is on the cusps of C^n.*

The osculants are thus defined by the facts of touching the curve and being on the $n-2$ cusps.

§ 4. *The envelopes of osculants.*

Denoting the osculant

$$(\alpha t_1)^r(\alpha t)^{n-r}=0$$

by O_r, it has an envelope and a cusp-locus. The envelope contains the original curve C^n and the curve χ_r. It is to be shown that *the cusp-locus of* O_r *is* χ_{r+1}. The map-equation of O_r is

$$\alpha_1(\alpha t_1)^r(\alpha t)^{n-r-1}=0.$$

Now to obtain the envelope of

$$x=f(t,\,t_1)$$

the rule is that

(6) $$D_t x/t : D_{t_1} x/t_1, \text{ is real.}$$

Here then

$$\alpha_1\alpha_2(\alpha t_1)^r(\alpha t)^{n-r-2}/t : \alpha_1\alpha_2(\alpha t_1)^{r-1}(\alpha t)^{n-r-1}/t_1, \text{ is rea}$$

But the equation

$$\alpha_1\alpha_2(\alpha t_1)^r(\alpha t)^{n-r-2}=0$$

is self-conjugate. Hence either $t=t_1$, which gives C^n, or

$$\alpha_1\alpha_2(\alpha t_1)^r(\alpha t)^{n-r-2}=0,$$

which manifestly gives the cusp-locus, or

$$\alpha_1\alpha_2(\alpha t_1)^{r-1}(\alpha t)^{n-r-1}=0,$$

which gives χ_r. Thus χ_{r+1} is given by

(7) $$\begin{cases}\alpha_1(\alpha t_1)^{r+1}(\alpha t)^{n-r-2}=0,\\ \alpha_1\alpha_2(\alpha t_1)^r(\alpha t)^{n-r-2}=0.\end{cases}$$

The former equation may be replaced by

$$\alpha_1^2(\alpha t_1)^r(\alpha t)^{n-r-2}=0,$$

and this again by

$$\alpha_1(\alpha t_1)^r(\alpha t)^{n-r-1}=0.$$

This proves the theorem.

There are then associated with C^n a series of curves

χ_1 the cusps,

χ_2 the cusp locus of O_1 and envelope of O_2,

χ_3 the cusp locus of O_2 and envelope of O_3;

and writing again the equations for χ_r:

$$\alpha_1\alpha_2(\alpha t_1)^{r-1}(\alpha t)^{n-r-1}=0,$$
$$\alpha_1^2(\alpha t_1)^{r-1}(\alpha t)^{n-r-1}=0,$$

we see that since we may interchange t and t_1 the curves χ_r and $\chi_{r'}$ are the same if

$$r + r' = n.$$

Thus for the curve C^4 there are

χ_1 the cusps,

χ_2 a circle or arc of a circle (according as the cusps are not real or real) which is the cusp locus of osculant cardioids and the envelope of osculant circles;

and for the curve C^5, which has as osculants C^4, C^3 and C^2,

χ_1 the cusps,

χ_2 which is at once the cusp locus of C^4, the envelope of C^3, the cusp locus of C^3, and the envelope of C^2.

§ 5. *Note on the rational plane curve in general.*

For rational plane curves in general, given for convenience of statement in lines, it is equally true that the cusp locus of O_r is the envelope of O_{r+1}. We know from STUDY's theory that the cusps of the given curve are on O_1, and the cusps of O_r on O_{r+1}. Let O_r be given by

$$\xi = (\alpha s)^r(\alpha t)^{n-r},$$

$$\eta = (\beta s)^r(\beta t)^{n-r},$$

$$\zeta = (\gamma s)^r(\gamma t)^{n-r},$$

where

$$\alpha t = \alpha_1 + \alpha_2 t.$$

The cusp parameters are given by

$$|\xi \quad \eta \quad \zeta| = 0,$$

that is by

$$\begin{vmatrix} (\alpha s)^r(\alpha t)^{n-r} & \cdots \\ \alpha_2(\alpha s)^r(\alpha t)^{n-r-1} & \cdots \\ \alpha_2^2(\alpha s)^r(\alpha t)^{n-r-2} & \cdots \end{vmatrix} = 0$$

or by

$$\begin{vmatrix} \alpha_1^2 & \beta_1^2 & \gamma_1^2 \\ \alpha_2^2 & \beta_2^2 & \gamma_2^2 \\ \alpha_1\alpha_2 & \beta_1\beta_2 & \gamma_1\gamma_2 \end{vmatrix} (\alpha s)^r(\alpha t)^{n-r-2} \cdots = 0.$$

The envelope of O_{r+1}, or

$$\xi = (\alpha s)^{r+1}(\alpha t)^{n-r-1}, \text{ etc.},$$

is given by

$$|\xi \quad D_s\eta \quad D_t\zeta| = 0,$$

or by

$$\begin{vmatrix} (\alpha s)^{r+1}(\alpha t)^{n-r-1}\cdots \\ \alpha_2(\alpha s)^r \ (\alpha t)^{n-r-1}\cdots \\ \alpha_2(\alpha s)^{r+1}(\alpha t)^{n-r-2}\cdots \end{vmatrix} = 0;$$

and this on throwing out the factor $s - t$ gives the same result as before.

Thus for example with the rational curve of class 4 is associated a curve χ_2 on which lie the cusps of osculant cubics, and which has as tritangent conics all osculant conics.

§ 6. *Construction of the ennacardioid.*

The number of coördinates specifying a C^n is $n + 1$. For the number of arbitrary real constants in $(\alpha t)^n$ is $n - 1$, and the base-point is here a definite point as to the curve — say its center.

Hence when $2(n - 2) = n + 1$, or $n = 5$, the cusps determine a C^n, though not perhaps uniquely. Where $n > 5$ the cusps are not independent.

Two osculants, say

$$(\alpha t_1)(\alpha t)^{n-1} \qquad \text{and} \qquad (\alpha t_2)(\alpha t)^{n-1},$$

have a common osculant

$$(\alpha t_1)(\alpha t_2)(\alpha t)^{n-2}.$$

But conversely if two C^{n-1}'s have a common osculant, they are osculants of a C^n. For an osculant of one C^{n-1} can touch the other; if in addition it is on the cusps, this is $n - 3$ conditions and the coördinates of the two and

$$2n - (n - 3) \qquad \text{or} \qquad n + 3.$$

But this is also the specification of C^n and two osculants.

The tangent at t of the osculant at t_1, C^{n-1}, is

$$(\alpha t_1)(\alpha t)^{n-1} = 0.$$

This is satisfied when $\alpha_1(\alpha t)^{n-1} = 0$, and this gives the point t of C^n. Varying t, we obtain a rigid pencil of lines, each touching some osculant C^{n-1}. Conversely, consider any two C^{n-1}'s. Let an angle slide around them. The locus of its vertex is easily seen to be a polynomic or curve of the form

$$x = Pt + Q(1/t)$$

where P and Q are polynomials. And in fact any polynomic can be so constructed.

But when the two C^{n-1}'s have a common osculant C^{n-2}, and are therefore osculants of a C^n, the locus of the vertex can be this C^n.

Thus granting that we can construct one and therefore two C^n's of which an osculant C^{n-1} is given, then we mark on each line of C^{n-1} the corresponding points of the two C^n's, and the lines of these C^n's at these points meet on a C^{n+1}. Thus the curves can be drawn in succession; the first step is to take two intersecting circles, and draw lines on a common point; the tangents of the circles at the other intersections meet on a cardioid.

§ 7. *The C^{n-1} of an n-line.*

It was shown in these Transactions (vol. 1, p. 102) that (with the notation there used) the circles of 3 from 4 lines are osculants of a C^3,

$$x = a_1 - 2a_2 t + a_3 t^2,$$

where $|a_2| = |a_3|$.

Calling this the cardioid of the 4-line we have now the theorem:

The cardioids of 4 from 5 lines are osculants of a C^4; and in particular meet at two points (the cusps).

For the curve for 5 lines

$$x = a_1 - 3a_2 t + 3a_3 t^2 - a_4 t^3$$

is a C^4.

And so in general:

The C^{n-1}'s of $n-1$ from n lines are osculants of the C^n of the n lines, and in particular are on the $n-2$ cusps.

Conversely, a C^n and $n+1$ osculants determine at once $n+1$ lines, for a C^n and n osculants determine the line

$$(\alpha t_1)(\alpha t_2)\cdots(\alpha t_n) = 0.$$

This line may be found thus. We are given a C^n and n osculant C^{n-1}'s. Every two C^{n-1}'s have a common osculant C^{n-2}. Hence a selected C^{n-1} has $n-1$ osculant C^{n-2}'s; and so finally we come to a C^3 with three osculant C^2's whose common osculants are points on the line in question.

There is then for an n-line a curve C^{n-1}. There is also a definite circle called (vol. 1, p. 99) the centre-circle. I will call it now the centric circle and its center the centric of the n-line. I proceed to give a general meaning to these two curves.

§ 8. *The images of a point in 4 lines.*

To connect by a curve the images of a point x_0 in four given lines we employ the process of interpolation. This is not a definite process, but employed as follows what results is *the conic on x_0 and its images.*

Write the lines

$$x - a_2 s_1 + a_3 s_2 - \bar{x} s_3 = 0,$$

where the 3 turns whose symmetric functions are s_1, s_2, s_3 are selected from 4 whose product is -1.

Then the image of x_0 is

$$x = a_2 s_1 - a_3 s_2 + \bar{x}_0 s_3.$$

We have to symmetrize this.

Assume two turns τ, τ_1 such that

$$x = A\tau^3 + A_1\tau_1^3 - A\prod^3(\tau - t_i) - A_1\prod^3(\tau_1 - t_i), \tag{8}$$

where

$$A\tau^2 + A_1\tau_1^2 = a_2,$$

$$A\tau + A_1\tau_1 = a_3,$$

$$A + A_1 = \bar{x}_0.$$

Then

$$\bar{x}_0\tau\tau_1 - a_3(\tau + \tau_1) + a_2 = 0,$$

or

$$x_0 - a_2(\tau + \tau_1) + a_3\tau\tau_1 = 0.$$

Thus x_0 is any point inside the C^3

$$x_1 - 2a_2\tau + a_3\tau^2 = 0.$$

We have then, symmetrizing (8),

$$x = A\tau^3 + A_1\tau_1^3 - A\,\frac{\prod^4(\tau - t_i)}{\tau - t_i} - A_1\,\frac{\prod^4(\tau_1 - t_i)}{\tau_1 - t_i}, \tag{9}$$

where A and A_1 are known.

This is the map-equation of a hyperbola. It evidently is on the 4 images of x_0, but further it is on x_0 itself.

For eliminating A and A_1 the equation is

$$\begin{vmatrix} x & \tau^3 - \dfrac{\Pi}{\tau - t} & \tau_1^3 - \dfrac{\Pi_1}{\tau_1 - t} \\ a_2 & \tau^2 & \tau_1^2 \\ a_3 & \tau & \tau_1 \end{vmatrix} = 0. \tag{10}$$

The terms independent of t give x_0. But the terms in t are

$$\Delta = \begin{vmatrix} 0 & -\dfrac{\Pi}{\tau - t} & -\dfrac{\Pi_1}{\tau_1 - t} \\ a_2 & \tau^2 & \tau_1^2 \\ a_3 & \tau & \tau_1 \end{vmatrix},$$

and the conjugate of this is

$$\Delta t / \tau^3\tau_1^3.$$

Hence $\Delta = 0$ is a self-conjugate equation, giving a value of t which is a turn. That is, we have the hyperbola on x_1 and its images.

The clinants of the asymptotes are found at once to be $1/\tau$ and $1/\tau_1$.

There are now two special conics:

1) the rectangular hyperbola, when

$$\tau + \tau_1 = 0,$$

the point x_1 is then given by

$$x + a_3 \tau\tau_1$$

and is on the centric circle;

2) the parabola, when

$$\tau = \tau_1,$$

the point is then on the cardioid. Thus the conic on a point x_0 and its images in 4 given lines is a rectangular hyperbola when x_0 is on a circle and a parabola when x_0 is on a cardioid.

§ 9. *The images of a point in n lines.*

The generalization is now immediate. Thus if 5 lines be written

$$x - a_2 s_1 + a_3 s_2 - a_4 s_3 + \bar{x} s_4 = 0,$$

where the s_i are symmetric functions of 4 from 5 turns whose product is 1, the images of of x_0 in the 5 are on

$$\begin{vmatrix} x & \tau^4 - \dfrac{\Pi}{\tau - t} & \tau_1^4 - \dfrac{\Pi_1}{\tau_1 - t} & \tau_2^4 - \dfrac{\Pi_2}{\tau_2 - t} \\ a_2 & \tau^3 & \tau_1^3 & \tau_2^3 \\ a_3 & \tau^2 & \tau_1^2 & \tau_2^2 \\ a_4 & \tau & \tau_1 & \tau_2 \end{vmatrix} = 0$$

where

$$x_0 - a_2(\tau + \tau_1 + \tau_2) + a_3(\tau_1\tau_2 + \tau_2\tau + \tau\tau_1) - a_4 \tau\tau_1\tau_2 = 0.$$

We have then a pencil of cubic curves J^3 passing through the images of x_0, and twice through x_0.

If we make $\tau = \omega\tau_1 = \omega^2\tau_2$ where $\omega^2 + \omega + 1 = 0$, then

$$x_0 - a_4 \tau^3 = 0,$$

that is, for points x_0 on the centric circle the cubic has equispaced asymptotes.

If we make $\tau = \tau_1 = \tau_2$ the point x_0 is on the C^4 of the 5 lines and the J^3 meets infinity (regarding infinity as a line, i. e., speaking projectively) at one point only.

And so in general, calling a curve of order m with a multiple point of order $m - 1$ a JONQUIÈRES *curve* J^m:

Let a J^{n-2} be drawn with multiple point at x_0 and on the images of x_0 in n given lines. If the J^{n-2} have equispaced asymptotes then x_0 is on the centric circle of the n-line; and if the J^{n-2} have coincident points at infinity then x_0 is on the C^{n-1} of the n-line.

There are naturally other special cases. Important among these as giving the CLIFFORD theorem is the one where the system of J^{n-2}'s (for n odd) reduces to a $J^{(n-1)/2}$. This involves the vanishing of minors of the elements in t in the determinant and brings us back to the analysis given in these Transactions, loc. cit., p. 104.

The statement of CLIFFORD's theorem in terms of C^{n-1} is: The curve C^{2m-1} has one osculant C^m which reduces to a point; the parameters are the canonizant of the cusp-parameters. For the curve C^{2m} the locus of points which are degenerate osculant C^m's is a circle.

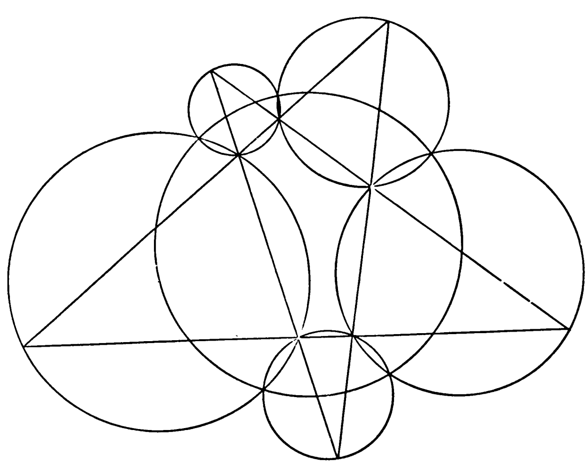
§ 10. *The case when the C^{n-1} of an n-line is a circle.*

We have seen that the image-curve for a point on the centric circle is an equispaced J^{n-2}. Let the point be the centric of $n-1$ of the lines. The remaining line and therefore the image of x_0 in it becomes arbitrary. Hence the centric of $n-1$ lines is the multiple point of not one but a pencil of equispaced J^{n-2}'s on the images of x_0. But the pencil meets again in $(n-2)^2-(n-3)^2-(n-1)$ or $n-4$ points. Hence the $n-1$ lines belong to a symmetric system of $2n-5$ lines. That is, *m lines belong to a set of $2m-3$ lines.*

Any m of these have the same centric, and therefore the centric circle of $m+1$ is a point. The analytic condition is then, since the centric circle is

$$x = a_1 - a_2 t,$$

that a_2 formed for any $m+1$ lines in 0. This requires for the $2m-3$ lines



that $a_2 = a_3 = \cdots = a_{m-2} = 0$. The radius of the centric circle of any m lines is $|a_{m-1}|$, and therefore any m have the same centric circle. Since all a's except a_{m-1} vanish, the conditions are summed up by saying that the C^{2m-4} of the lines is a circle.

Thus in particular 4 lines determine a fifth; the 5 forming say a *pentacle*, such that the centres of the 10 3-lines are on a circle. The figure is drawn simply by placing a ring of 5 circles with centres on a given circle and each intersecting the next on this circle. The 5 other intersections of the adjacent circles being joined in order form the pentacle, and the salient thing is that *the intersections of non-adjacent sides are also on the respective 5 circles.*

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