DENUMERANTS OF DOUBLE DIFFERENTIANTS

BY

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I. Introduction.

1.† In 1856 ‡ CAYLEY gave (without rigorous proof) a formula for the number of linearly independent \(yx\)-differentiants (as I call them) of a given type belonging to a single binary quantic,—a formula that he derived from the assumption that the necessary and sufficient conditions, as they naturally appear, are linearly independent; this formula was first proved in 1877 § by SLYLESTER, who extended it to a system of any number of binary quantics. The formula as originally given by CAYLEY expresses the number in question as the excess of the number of terms in the general form of the type of the differentiants over the number of terms in the general form of a certain other type, precisely as in formula (37) of this paper (when \(x\) and \(y\) are the only variables); but the number of terms in a general binary form is readily expressible as the sum of the numbers of partitions of a certain kind of certain numbers, and it was the formula in terms of such partition-numbers that SLYLESTER proved and extended. About 1885 it occurred to me, while giving a course of lectures on invariants at the Johns Hopkins University, that CAYLEY's original formula and SLYLESTER's proof of it,—indeed, the latter's extension of it, when expressed in the original terms,—are valid for \(yx\)-differentiants (simple differentials, as I call them below), even if \(x\) and \(y\) are only two out of any number of variables. In 1892 || I discovered operators that produce all the differentiants of a given type and certain combinations of characters, namely, all forms that are at once \(yx\)-, \(zx\)-, \(sx\)-, \(xs\)-, \(ux\)-differentiants and all forms that are at once \(xy\)-, \(xz\)-, \(xs\)-, \(ux\)-differentiants. Suspecting that the method by which these operators were obtained could be utilized for the discovery and proof of a formula for the number of linearly independent forms of any given type that possess any given combination of differentiant characters, I have devoted much time during the

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† I shall refer to these numbered sections as § 1, etc. For Table of Contents see p. 70.
last fourteen years to attempts to obtain such formulæ. I have succeeded in
the cases of double differentiants of all kinds, as the present paper will show;
the extension of the formulæ to triple and higher differentiants is almost self-
evident, but the proof of the extended formulæ presents a difficulty that I have
not yet overcome.

When I began these attempts I was not aware of what Deruyts had done in
a similar line;* but his work differs from mine in essential respects, both as to
methods and results. He has, to be sure, given a formula for the denumerant
of the semi-invariants (chain-differentiants of order 0 — that is, not involving
the variables explicitly) which is formally identical with the formula (51) of this
paper for three variables, but my formula is proved for xyz-differentiants of any
type, even if this type implies any number of variables other than \(x, y\) and \(z\),
and the explicit presence of all these variables in the differentiant, while his
formula supposes that the variables do not enter explicitly and that \(x, y,\) and \(z\)
are the only variables in the system. Moreover, I have obtained formulæ for
other combinations of differentiant characters not considered by Deruyts. But
the most important differences in our work will be found in the methods used.
Deruyts has employed the methods of "sources" and of "symbolic representa-
tion," while my main object has been to obtain formulæ by direct consideration
of the "actual" forms, without recourse to those (from my point of view) indi-
rect methods. At all events, it must be for the advantage of mathematical
science that various methods should be exploited.

2. We assume a system of quantics (homogeneous polynomials), which we
shall call simply the quantics, finite in number and of finite orders \(m, m', m'', \ldots\),
respectively, in a finite number \(n\) of variables \(x, y, z, \ldots\). Any such quantic
of order \(m\) is assumed in the form

\[
\sum_{g! \, h! \, i! \ldots} {m! \over g! \, h! \, i! \ldots} \, a_{g, h, i, \ldots} \, x^g y^h z^i \ldots,
\]

where the summation extends to all integral values (positive and 0) of \(g, h, i, \ldots\),
whose sum is \(m\). For convenience, we speak of \(a_{g, h, i, \ldots}\) freed from the cor-
responding polynomial coefficient as a coefficient of the quantic. If it were
necessary to represent the coefficients of the different quantics, we should denote
those of the first quantic by \(a\), those of the second by \(b\), those of the third by \(c,\n\text{etc.}, each with its proper suffixes; the coefficients of the quantics will be used
as so many sets of additional variables (though not so called) having no deter-
mined numerical values. To each coefficient is assigned a weight in each var-

* Essai d'une théorie générale des formes algébriques, Mémoires de la société royale des

† Of course the sum of the suffixes of any coefficient is equal to the order of the quantic to

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Of Double Differentials

The weight of any coefficient in either variable is the sum of all its suffixes excepting that which denotes the power of the variable in question in the term of the quantic to which the coefficient belongs; thus, in (1) the weight of \( a_{p,n,i} \) in \( z \) is \( m - i \). The weight of any variable in itself is 1 and its weight in any other variable is 0. The weight of any product of variables and coefficients in any variable is the sum of the weights of its factors in that variable; thus the weight of any power of a variable or coefficient is the weight of that variable or coefficient multiplied by the exponent of the power.

The order of any product of powers of variables and coefficients is the sum of the exponents of the powers of the variables and its degree in the coefficients of either quantic is the sum of the exponents of the powers of such coefficients in it. If the terms of a polynomial in the variables and coefficients of the quantics are all of one order and of one degree in the coefficients of each quantic of the system (even if of different degrees in the coefficients of the different quantics), the polynomial is homogeneous of that order and those degrees. If the terms of a polynomial are all of one weight in each variable (even if of different weights in the different variables), the polynomial is isobaric of those weights in the several variables. Any polynomial is a sum of homogeneous isobaric polynomials. The order, degrees in the coefficients of the several quantics, and weights in the several variables, of a homogeneous isobaric polynomial, or form, together constitute the type of that form. One consequence (and advantage) of the definition of weight here given (which has not been adopted by all writers) is that each of the quantics is isobaric in each of the variables of a weight equal to its order. The numbers that characterize the type of any form are connected by a simple relation; namely, if the quantics are of orders \( m, m', m'', \ldots \), respectively, in \( n \) variables \( x, y, z, \ldots \), and any form of degrees \( j, j', j'', \ldots \), in the coefficients of these quantics, respectively, and of order \( \partial \) in the variables, is of weights \( w_x, w_y, w_z, \ldots \) in these variables, respectively, then

\[
(2) \quad w_x + w_y + w_z + \cdots = (n - 1) \sum mj + \partial,
\]

where \( \sum mj = mj + mj' + mj'' + \cdots \); so that the weight in either variable is completely determined when the other characteristics of its type are given.

If a linear transformation is imposed upon the variables, that is, if the \( n \) variables \( x, y, z, \ldots \) are expressed as homogeneous linear functions of \( n \) new variables \( x', y', z', \ldots \), and these expressions are substituted for \( x, y, z, \ldots \) in any one of the quantics, this becomes a quantic of the same order in \( x', y', z', \ldots \), whose coefficients are homogeneous linear functions of the original coefficients of the quantic and homogeneous functions of the parameters of transformation of a degree equal to the order of the quantic: the coefficients of the new quantic in \( x', y', z', \ldots \) are the transformed coefficients of the quantic in question. It is assumed that the determinant of the parameters of transformation is not 0, so...
that the new variables \(x', y', z', \ldots\) can be expressed as homogeneous linear functions of the old variables \(x, y, z, \ldots\). Any polynomial in the variables and the coefficients of the quantics is transformed by replacing the variables by the expressions of the corresponding new variables in terms of the old ones and the coefficients of the quantics by the expressions of the corresponding transformed coefficients in terms of the original coefficients. A homogeneous polynomial is thus transformed into a homogeneous polynomial of the same order and the same degrees; but the weights are not generally preserved by an arbitrary linear transformation,—indeed an isobaric form does not remain isobaric in general.

The linear transformation by which the variable \(x\) is increased by a multiple of the variable \(y\), that is, the transformation \(x = x' + \lambda y', y = y', z = z', \ldots\), shall be called a \(yx\)-shear; and similarly for other pairs of variables. A shear involves a single parameter (\(\lambda\), in the example just given), which we shall generally suppose to have an arbitrary value. A polynomial in the variables and the coefficients of the quantics that is not altered by an arbitrary \(yx\)-shear (that is, by a \(yx\)-shear whose parameter is arbitrary) shall be called a \(yx\)-differentiant, and similarly for other pairs of variables. A polynomial subject to only one such condition is a simple differentiant; if subject to more than one such condition, it is a multiple differentiant, namely, double, triple, etc., according to the number of independent conditions of this kind to which it is subject (we shall see that certain combinations of simple differentiant conditions imply others).

The aggregate of differentiant conditions to which a polynomial is subject constitute its character. The object of this paper is the determination of the number of linearly independent double differentiants of a given type, more specifically, the number of \(xy\) and \(xz\)-differentiants, of \(xz\) and \(yz\)-differentiants, of \(xy\)- and \(yz\)-differentiants (which are, as we shall see, also \(xz\)-differentiants), and of \(xy\)- and \(zs\)-differentiants. There are as many kinds of simple shears and, therefore, of simple differentiants and simple characters as there are pairs of variables, if we have regard to the sequence of variables in the pairs (so that \(xy\) and \(yx\) are regarded as different pairs), that is \(n(n-1)\) kinds.

A very important class of multiple differentiants is that defined by a set of simple characters such that the second letter of each defining pair of variables is the same as the first letter of the next pair, when the pairs are taken in the proper sequence; such differentiants may be called chain-differentiants (SYLVESTER called them “seminvariants”). Any chain-differentiant character is completely determined by the letters in it and their sequence in the chain, so that it may be unambiguously designated by this sequence; thus an \(xy\)-, \(yz\)-, \(zs\)-, \(tu\)-differentiant may be called simply an \(xyzs\ldots\) \(tu\)-differentiant. Any simple differentiant character may be regarded as a chain-differentiant character of two letters. Of course, the number of letters in any given differentiant character
has no particular relation to the whole number of variables in the quantics and differentiants in question.

The name "differentiant" is derived from the fact that a $yx$-differentiant not explicitly involving the variables is a function of the differences of the roots of the equations in $x:y$ formed by equating the quantics to 0, if $x$ and $y$ are the only variables ($n = 2$).

3. The result of imposing the $yx$-shear $x = x' + \lambda y', y = y', z = z', \ldots$ on any polynomial $\phi$ is

$$\phi + \lambda \cdot \widehat{yx} \cdot \phi + \frac{\lambda^2}{2!} \cdot \widehat{yx^2} \cdot \phi + \frac{\lambda^3}{3!} \cdot \widehat{yx^3} \cdot \phi + \ldots,$$

the operator $\widehat{yx}$ being defined by

$$\widehat{yx} \cdot \phi \equiv \sum \sum h \cdot a_{g+1, h-1, i} \cdots \frac{\partial \phi}{\partial a_{g, h, i, \ldots}} - y \cdot \frac{\partial \phi}{\partial x},$$

where the double summation extends to the coefficients $a_{g, h, i, \ldots}$, $b_{g, h, i, \ldots}$, $c_{g, h, i, \ldots}$, $\ldots$ of all the quantics and to all values of the suffixes of the coefficients of each quantic whose sum is the order of that quantic. From (3) it is evident that the necessary and sufficient condition that $\phi$ shall be a $yx$-differentiant is the identity

$$\widehat{yx} \cdot \phi \equiv 0;$$

that is, the condition is that the multipliers of the different products of powers of the variables and of the coefficients of the quantics that occur in $\widehat{yx} \cdot \phi$ (these multipliers are homogeneous linear functions of the multipliers of such products in $\phi$) shall all be 0, a condition that gives as many homogeneous linear equations between the multipliers of the terms of $\phi$ as there are terms of $\widehat{yx} \cdot \phi$, but it does not yet appear that these equations are linearly independent. From (4) it appears that $\widehat{yx}$ leaves unchanged the order and degrees of any form, as well as its weights in all the variables excepting $x$ and $y$, diminishes the weight in $x$ by 1, and increases the weight in $y$ by 1, so that (5) must be satisfied by each of the homogeneous isobaric parts of $\phi$ of different types if it is satisfied by $\phi$ as a whole; that is, every $yx$-differentiant is a sum of homogeneous isobaric $yx$-differentiants, — and the same is true of differentiants of any simple or multiple character. It appears from (5) that a $yx$-differentiant is a polynomial that is annihilated by the shear-operator $\widehat{yx}$.

Being concerned with linearly independent differentiants alone, we shall henceforth assume that every polynomial with which we have to do is homogeneous and isobaric, that is, is a form. Moreover, the forms to be considered at any one time will all be obtained by applying operators analogous to $\widehat{yx}$ to forms of one and the same given type; the orders and degrees of all forms thus obtained

*The sign $\equiv$ is used throughout this paper to denote identity.*
are the same, as are also the weights in the several variables excepting those variables that occur in the shear-operators used in obtaining the forms, and the sum of the weights in these excepted variables is the same for all the forms. In giving the type of a form it will, then, suffice to give its weights in those variables in which the forms to be considered are not all of the same weight; thus, if the forms considered are all of a given order, of given degrees, and of given weights in all the variables excepting \( x, y, \) and \( z, \) — so that the weights in these three variables are the only characteristics of the type that vary from form to form, — the type of any particular form may be represented by \((w_x, w_y, w_z)\), where \(w_x, w_y, w_z\) are its weights in \( x, y, z, \) respectively; and similarly for any number of variables. It is to be observed that the expression of the type of a form by its weights in certain variables does not imply anything with regard to the whole number of variables, but only that the weights in all the variables that do not enter into this expression of the type are the same for all the forms considered. To avoid confusion, we suppose the variables to be arranged in a certain sequence \( x, y, z, \ldots \) and we write the weights in any expression of a type in the corresponding sequence.

4. By a complete system of forms of given type or types and given differentiant characters, subject to other conditions or not, we mean a system of such linearly independent forms satisfying the given conditions that any form of the given type or types satisfying the conditions can be expressed as a linear function of the forms of the system. The forms of any complete system of one type can, evidently, be replaced by the same number of linearly independent linear functions of them, chosen arbitrarily, without affecting the completeness of the system or the character of its forms (because the shear-operators are distributive over their operands). By the rank of any form relatively to any given shear-operator we mean the exponent of the highest power of that operator that does not annihilate the form; thus, \( \phi \) is of rank \( r \) qua \( x y \) if \( x y^r \cdot \phi \neq 0 \) but \( x y^{r+1} \cdot \phi = 0 \). A complete system of forms of a given type and given characters shall be called reduced qua \( x y \) if its forms are so taken as to include the greatest possible number of forms of rank not greater than \( r \) qua \( x y \) for each value of \( r \) from 0 to the highest rank of any form \( \dagger \) of the system, inclusive. A complete system of forms of a given type and given characters reduced qua \( x y \) can be constructed thus: out of the \((N)\) forms \( \psi \) of any complete system of the given type and characters construct the greatest possible number \((N_0)\) of linearly independent linear functions \( \phi_0 \) of rank 0 qua \( x y \) and select as many \((N'_0 = N - N_0)\) other forms \( \psi_0 \) of the system that are linearly independent of

* Qua signifies relatively to, with respect to, according to.

† We employ this abbreviated expression instead of the more cumbersome "rank of the forms that are of highest rank."
each other and the \( \phi_0 \)'s as may be necessary to constitute a complete system with the \( \phi_0 \)'s; out of these \( (N'_0) \) other forms \( \psi_0 \) construct the greatest possible number \( (N'_1) \) of linearly independent linear functions \( \phi_1 \) of rank 1 qua \( xy \) and select as many \( (N'_1' = N'_0 - N'_1) \) other forms \( \psi_1 \) from among the \( \psi_0 \)'s that are linearly independent of each other and the \( \phi_1 \)'s as may be necessary to constitute a complete system with the \( \phi_0 \)'s and \( \phi_1 \)'s; out of the \( (N'_1) \) forms \( \psi_1 \) construct the greatest possible number \( (N'_2) \) of linearly independent linear functions \( \phi_2 \) of rank 2 qua \( xy \) and select as many \( (N'_2' = N'_1 - N'_2) \) other forms \( \psi_2 \) from among the \( \psi_1 \)'s that are linearly independent of each other and the \( \phi_2 \)'s as may be necessary to constitute a complete system with the \( \phi_0 \)'s, \( \phi_1 \)'s and \( \phi_2 \)'s; and so proceed until the forms \( \psi_{k-1} \) last selected are all expressible as linear functions of the forms \( \phi_k \) last constructed. The forms \( \phi_0, \phi_1, \phi_2, \ldots, \phi_k \) thus constructed constituted a complete system of forms of the given type and given characters reduced qua \( xy \). Of course, the forms \( \psi_{k-1} \) are forms \( \phi_k \).

The peculiarity of a system reduced qua \( xy \) is that any linear function of the forms of such a system is of exactly the rank qua \( xy \) of the forms of highest rank in it. For no linear function of forms of a system reduced qua \( xy \) that are all of the rank \( r \) can be annihilated by \( xy^r \) (otherwise, the system would not contain the greatest possible number of forms of rank less than \( r \), that is, would not be reduced), but every linear function whose terms are all of the rank \( r \) is annihilated by \( xy^{r+1} \); so that the result of applying \( xy^r \) to a linear function of forms of the reduced system of which some are of rank \( r \) and others of rank less than \( r \) is the same as the result of applying \( xy^r \) to the aggregate of those terms of the linear function whose rank is \( r \) alone, and this result is not identically 0.

If \( \phi \) is a form of rank \( r \) qua \( xy \), then \( xy^r \cdot \phi \equiv 0 \) but \( xy^{r+1} \cdot \phi \equiv 0 \), that is, \( xy^r \cdot \phi \) is an \( xy \)-differentiant (of rank 0 qua \( xy \)). The form derived from any form \( \phi \) by operating upon it with the power of \( xy \) whose exponent is the rank qua \( xy \) of the form \( \phi \) in question shall be called the derivative qua \( xy \) of that form \( \phi \); the derivative qua \( xy \) of any form is, then, an \( xy \)-differentiant, and the derivatives qua \( xy \) of the forms of a system of one type reduced qua \( xy \) are linearly independent \( xy \)-differentiants (namely, the derivatives of forms of the same type that are of different ranks are themselves of different types, so that a linear relation between the forms of a reduced system implies a linear relation between the derivatives of forms that are of one rank, and this again implies a linear function of these forms that is of lower rank than they, which is impossible, as above shown).

5. In the present investigation we are concerned chiefly with what we may call denumerants, that is, expressions for the numbers of linearly independent forms of given types that satisfy certain conditions. A denumerant shall be
represented by $N$ followed by the type with the conditions attached to the $N$ as suffixes and exponents. Conditions shall be expressed thus:

that the forms in question have certain differentiant characters shall be indicated by the pairs and sequences of variables that define these characters, attached to $N$ as suffixes (thus, the suffix $xy$ shall indicate that the forms are $xy$-differentiants);

that the forms are of given ranks relatively to certain shear-operators in a system reduced relatively to each of these operators shall be indicated by the highest powers of the respective operators that do not annihilate the forms, attached to $N$ as exponents (thus, the exponent $yz$ indicates that the forms are of rank $r$ qua $yz$ in a system reduced qua $yz$);

other conditions will generally be attached to $N$ as suffixes;

several suffixes or exponents attached to the same $N$ shall be separated by commas; and

the absence of suffix or exponent shall indicate that no condition such as would generally be written in that place is imposed.

Thus,

$N(w_x, w_y, \ldots)$ is the number of linearly independent forms of the type $(w_x, w_y, \ldots)$, which is also the number of terms in the general form of this type;

$N_{xy}(w_x, w_y, \ldots)$ is the number of linearly independent $xy$-differentiants of the type $(w_x, w_y, \ldots)$;

$N_{xy,zz}(w_x, w_y, w_z, \ldots)$ is the number of linearly independent $xy$- and $zz$-differentiants of the type $(w_x, w_y, w_z, \ldots)$;

$N_{xq}(w_x, w_y, \ldots)$ is the number of forms of a complete system of type $(w_x, w_y, \ldots)$ reduced qua $xy$ that are of rank $r$ qua $xy$;

$N_{xy,zz}(w_x, w_y, w_z, \ldots)$ is the number of $xy$- and $zz$-differentiants of rank $r$ qua $yz$ in a complete system of type $(w_x, w_y, w_z, \ldots)$ reduced qua $yz$;

$N_{xyz}(w_x, w_y, w_z, \ldots)$ is the number of linearly independent $xyz$-differentiants of the type $(w_x, w_y, w_z, \ldots)$;

$N_{K}(w_x, w_y, \ldots)$ is the number of linearly independent forms of the type $(w_x, w_y, \ldots)$ that satisfy the conditions $K$ (whatever they may be);

$N_{K,xy}(w_x, w_y, \ldots)$ is the number of linearly independent $xy$-differentiants of the type $(w_x, w_y, \ldots)$ that satisfy the further conditions $K$; etc. Evidently, the exponent $xy^0$ is equivalent to the suffix $xy$, so that

$N_{xy,zz}(w_x, w_y, w_z) = N_{xyz}(w_x, w_y, w_z)$.

The conditions to which the forms under consideration are subject are generally that the forms shall have certain differentiant characters and that certain numerical characteristics (rank, etc.) of the forms shall have given values or values lying between given limits; it will be convenient to speak of the former as differentiant conditions and of the latter as numerical conditions; there can even be no objection to speaking of the differentiant characters assumed and the
values or limitations of value of the numerical characteristics as themselves conditions, because they determine the conditions.

If $\phi$ is a form of type $(w_x, w_y, \ldots)$, $\overline{xy} \cdot \phi$ is a form of type $(w_x + 1, w_y - 1, \ldots)$, — provided it is not identically 0; accordingly, using the symbol $\overline{xy}$ to represent the operator that changes the type of any form $\phi$ into the type of $\overline{xy} \cdot \phi$, we have (as definition of the symbol $\overline{xy}$).

$$\overline{xy} \cdot (w_x, w_y, \ldots) = (w_x + 1, w_y - 1, \ldots),$$

so that

$$\overline{xy}^k \cdot (w_x, w_y, \ldots) = (w_x + k, w_y - k, \ldots).$$

The types obtained from any given type by applying powers of $\overline{xy}$ whose exponents are positive integers or 0 shall be called types subordinate to the given type qua $\overline{xy}$; thus, the types $(w_x + k, w_y - k, \ldots)$ for all positive integral values of $k$ and 0 are types subordinate to $(w_x, w_y, \ldots)$. Of course the same notation and nomenclature applies to the type-operators $xy$, etc., that correspond to all shear-operators $\overline{xy}$, etc.

Any denumerant may be regarded as a function of the type, and we shall apply the type-operators $\overline{xy}$, etc., to any denumerant as to a function of the type, observing that they affect only the type, but neither the suffixes nor the exponents of $N$; thus,

$$\overline{xy}^k \cdot N_{xz}(w_x, w_y, w_z) = N_{xz}(w_x + k, w_y - k, w_z),$$

$$(1 - \overline{xy})(1 - xz) \cdot N(w_x, w_y, w_z) = N(w_x, w_y, w_z) - N(w_x + 1, w_y - 1, w_z) - N(w_x + 1, w_y - 1, w_z - 1) + N(w_x + 2, w_y - 1, w_z - 1),$$

etc.

It is to be observed that the operators $\overline{xy}$, etc., like the operators $\overline{xy}$, etc., are distributive over their operands, but, unlike the latter, the former operators are commutative, so that, in their effects on any function of a type,

$$\overline{xy} \cdot yz = 1, \quad \overline{xy} \cdot yz = \overline{yz} \cdot \overline{xy} = xx, \quad etc.$$  

As $\overline{xy}^k \cdot \phi \equiv 0$ if $k$ is greater than the weight of $\phi$ in $y$, we must take $\overline{xy}^k \cdot N(w_x, w_y, \ldots) = 0$ if $w_y < k$, whatever suffixes and exponents $N$ may have; that is, the result of applying any product of powers of type-operators to a denumerant must be taken to be 0 if it makes the weight in any variable negative.* But it may be that some less value of $k$ than $w_y + 1$ will make $\overline{xy}^k \cdot N(w_x, w_y, \ldots) = 0$, because, with the notation of § 2, the weight in $y$ of any form can be 0 only if the weights in each of the other variables is as least as great as $\sum mj$ for the form. The result of applying a product of powers of type-operators to a denumerant may be 0 also because there are no forms of the

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*Any form of weight 0 in either variable has, evidently, every differentiant character that is defined by a pair in which the variable in question occupies the second place.
resultant type that satisfy the conditions imposed by the suffixes and exponents of $N$ in the case in question; thus,

$$y^2 \cdot N_{xy, zz}(w_x, w_y, w_z) = N_{xy, zz}(w_x, w_y + k, w_z - k) = 0$$

if $w_x - w_y < k$, because there are no $xy$-differentiants whose weight in $y$ is greater than that in $x$, as we shall see in § 8.

6. We shall make use of the notations

$$(6) \quad \alpha^{(i)} = \alpha(\alpha - 1)(\alpha - 2) \cdots (\alpha - i + 1), \quad i^{(i)} = i!$$

for any value of $\alpha$ and any positive integer $i$; from this follows

$$(7) \quad \alpha^{(i)} = \alpha(\alpha - 1)(\alpha - 2) \cdots (\alpha - i + 1) = \frac{\alpha^{(i+1)}}{\alpha - i},$$

by means of which the symbol $\alpha^{(i)}$ can be extended to negative integral values of $i$ and 0; thus, still for a positive integer $i$,

$$(8) \quad \alpha^{(-i)} = \frac{1}{(\alpha + 1)(\alpha + 2)(\alpha + 3) \cdots (\alpha + i)} = \frac{1}{(\alpha + i)^{i}}, \quad \alpha^{(0)} = 1.$$  

We shall write also, for any integer $i$ (positive, negative or 0),

$$(9) \quad \frac{\alpha^{(i)}}{i!} = \binom{\alpha}{i},$$

which is the coefficient of $\alpha^i$ in the development of $(1 + x)^\alpha$ according to positive or negative powers of $x$, according as $i$ is positive or negative. In particular, if $\alpha$ is a positive integer or 0,

$$(10) \quad \binom{\alpha}{i} = 0 \text{ unless } 0 \leq i \leq \alpha;$$

but if $\alpha$ is a negative integer,

$$(11) \quad \binom{\alpha}{i} = 0 \text{ for } \alpha < i \leq -1;$$

while

$$(12) \quad i! = \infty \text{ if } i \leq -1.$$

It is evident from (7) and (8) that

$$(13) \quad (-\alpha)^{(i)} = (-1)^i(\alpha + i - 1)^{(i)}, \quad \alpha^{(-i)} = (-1)^i \frac{1}{(\alpha + 1)^{i}}.$$  

If $\alpha$ and $i$ are both integers,

$$(14) \quad \alpha^{(i)} = 0 \text{ if } 0 \leq \alpha < i, \text{ and only then.}$$
We shall have occasion to use the formula of summation

\[ S_{r, q}^{m, n} = \sum_{i=0}^{n} (-1)^i \cdot \binom{n}{i} \cdot \frac{(p + q - m + 1)^n}{(q + m + 1)} \cdot \frac{(p - n - 1)}{(q + m + 1)}, \]

where \( m, n, p, q \) are given integers, of which \( n \) is not negative; from which follows

\[ S_{r, q}^{m, n} \neq 0 \text{ if } 1 \leq m \leq q \text{ and } m + n - 1 \leq p. \]

7. The results of applying the shear-operators defined by (4) for different pairs of letters successively are connected by these simple relations, where \( \phi \) is any form of type \((w_x, w_y, w_z, w_t, \cdots)\):

\[ xy \cdot yx \cdot \phi = yx \cdot xy \cdot \phi + (w_x - w_y) \cdot \phi, \]

\[ xy \cdot zx \cdot \phi = zx \cdot xy \cdot \phi + xy \cdot \phi, \]

\[ xy \cdot yz \cdot \phi = yz \cdot xy \cdot \phi - zx \cdot \phi, \]

\[ xy \cdot zs \cdot \phi = zs \cdot xy \cdot \phi, \]

where \( x, y, z, s \) are any four different variables. The last three identities of (17) show that any two shear-symbols and their powers are commutative unless they have a common letter that occupies different places in them. Repetitions of the first three formulae of (17) give

\[ \sum_{i=0}^{\alpha} \binom{\alpha}{i} (w_x - w_y + \alpha - \beta)^i \cdot xy^{a-i} \cdot \phi, \]

\[ \sum_{i=0}^{\alpha} \binom{\alpha}{i} \cdot zx^{b-i} \cdot yx^{a-i} \cdot \phi, \]

\[ \sum_{i=0}^{\alpha} (-1)^i \binom{\alpha}{i} \cdot yz^{b-i} \cdot xz^{a-i} \cdot \phi, \]

where each sum extends to all integral values of \( i \) from 0 to the smaller of the two numbers \( \alpha \) and \( \beta \).

8. Any form \( \phi \) of type \((w_x, w_y)\) is annihilated by a certain lowest power of \( \hat{xy} \) and by a certain lowest power of \( \hat{yx} \); let \( \alpha \) and \( \beta \) be the exponents of the highest powers of \( \hat{xy} \) and \( \hat{yx} \), respectively, that do not annihilate \( \phi \) (the ranks of \( \phi \) qua \( \hat{xy} \) and \( \hat{yx} \), respectively), so that

\[ \hat{xy}^\alpha \cdot \phi \neq 0, \quad \hat{xy}^{\alpha+1} \cdot \phi \equiv 0, \quad \hat{yx}^\beta \cdot \phi \neq 0, \quad \hat{yx}^{\beta+1} \cdot \phi \equiv 0; \]

then, by the first formula of (18) as applied to \( \hat{xy}^{\alpha+\beta+1} \cdot \hat{yx}^{\beta+1} \cdot \phi \) and \( \hat{yx}^{\alpha+\beta+1} \cdot \hat{xy}^{\alpha+1} \cdot \phi \),
\[ 0 \equiv (\alpha + \beta + 1)^{(\beta + 1)} \cdot (w_x - w_y + \alpha)^{(\beta + 1)} \cdot \phi \]

and

\[ 0 \equiv (\alpha + \beta + 1)^{(\alpha + 1)} \cdot (w_y - w_x + \beta)^{(\alpha + 1)} \cdot \phi ; \]

therefore,

\[ (w_x - w_y + \alpha)^{(\beta + 1)} = 0 \quad \text{and} \quad (w_y - w_x + \beta)^{(\alpha + 1)} = 0 , \]

that is, by (14),

\[ w_x - w_y + \alpha \leq \beta \quad \text{and} \quad w_y - w_x + \beta \leq \alpha , \]

so that

\[ \beta = w_x - w_y + \alpha \quad \text{or} \quad \alpha = w_y - w_x + \beta . \]

If \( \phi \) is an \( xy \)-differentiant, \( \alpha = 0 \) and, therefore, \( \beta = w_x - w_y \), so that

\[ w_y \leq w_x \]

and

A. Every \( xy \)-differentiant is of at least as great a weight in \( x \) as in \( y \).

If \( \phi \) is an \( xy \)-differentiant and \( w_x = w_y \), then \( \beta = 0 \), so that

B. Every \( xy \)-differentiant whose weights in \( x \) and \( y \) are equal is also a \( yx \)-differentiant.

From the third identity of (17) follows that \( \tilde{xz} \cdot \phi \equiv 0 \) if both \( \tilde{xy} \cdot \phi \equiv 0 \) and \( \tilde{yz} \cdot \phi \equiv 0 \); that is,

C. Every \( xy \)- and \( yz \)-differentiant is also an \( xz \)-differentiant (\( xyz \)-differentiant). From this theorem it appears that every chain-differentiant has every simple differentiant character that is defined by two letters of the corresponding chain, taken in the sequence in which they occur.

It is evident that

D. If two given shear-operators are commutative, the result of applying any power of the first to a form that is annihilated by the second is also a form annihilated by that second operator;

that is, the repeated application of one of two commutative shear-operators does not affect the simple differentiant character defined by the other. Similarly, from the second and third formulae of (18), together with theorem D, it appears that,

E. If \( \phi \) is an \( xy \)- and \( xx \)-differentiant, so is also \( \tilde{yz}^k \cdot \phi \); and, if \( \phi \) is an \( xx \)- and \( yz \)-differentiant, so is also \( \tilde{xy}^k \cdot \phi \); provided, in each case, the result is not identically 0.

A polynomial that has all the simple differentiant characters that are defined by pairs of variables taken from a certain set (including both orders of each pair) may be called a complete differentiant relatively to that set of variables. In the paper in vol. 41 of the Annalen to which I have referred, I have proved that every isobaric complete differentiant in all the variables is a covariant (or invariant); but the proof shows also that an isobaric complete differentiant in any set of variables is simply multiplied by a power of the determinant.
of transformation when the variables of that set alone are subjected to any linear transformation.

From (19) follows also that

\[ F. \text{ If } \phi \text{ is a form of weights } v_x \text{ and } v_y \text{ in } x \text{ and } y, \text{ respectively, and of rank } r \text{ qua } xy, \text{ then } v_y - v_x \leq r \text{ (of consequence only if } v_x < v_y). \]

Theorems B and C show that an isobaric chain-differentiant is a complete differentiant in the variables defining the chain provided it is of one and the same weight in those variables (which, by theorem A, requires only that it shall be of the same weight in the first and last variables of the chain).

9. In the papers cited in § 1* I have shown that the operators

\[
\{v; u, \ldots, z, y, x\}
\]

\[
\equiv \sum (-1)^\sigma \frac{(w_x - w_v + v)! \bar{v}x^\sigma \cdot vy^\beta \cdot vz^\gamma \cdots vu^\xi \cdot w \bar{w}^\varphi \cdots xv^\epsilon}{\alpha! \beta! \gamma! \cdots \epsilon! (\sigma + w_x - w_v + v)!}
\]

and

\[
\{x, y, z, \ldots, u; v\}
\]

\[
\equiv \sum (-1)^\sigma \frac{(w_v - w_z + v)! \bar{v}x^\sigma \cdot vy^\beta \cdot vz^\gamma \cdots u\bar{u}^\xi \cdot w \bar{w}^\varphi \cdots xv^\epsilon}{\alpha! \beta! \gamma! \cdots \epsilon! (\sigma + w_v - w_z + v)!}
\]

where \( \nu \) is the number of variables \( x, y, z, \ldots, u \) (without \( v \)), the summations extend to all integral but not negative values of \( \alpha, \beta, \gamma, \ldots, \epsilon \), and \( \sigma = \alpha + \beta + \gamma + \cdots + \epsilon \) for each term, have, when applied to a form \( \phi \) of weights \( v_x, v_y, w_z, \ldots, w_u, w_v \) in \( x, y, z, \ldots, u, v \), respectively (whatever the number of other variables and the weights of \( \phi \) in them may be), these properties:

If \( \phi \) is an \( xy-, \ldots, xu \)-differentiant, \( \{v; u, \ldots, z, y, x\} \cdot \phi \) is an \( xy-, \ldots, xu \)-differentiant of the same type as \( \phi \) (if it is not identically 0), is the most general differentiant of that type and these characters if \( \phi \) is the most general differentiant of its type and the characters stated, and is \( \phi \) itself if that form is an \( xy-, \ldots, xu \)-differentiant.

If \( \phi \) is a \( yx-, \ldots, ux \)-differentiant, \( \{x, y, z, \ldots, u; v\} \cdot \phi \) is a \( yx-, \ldots, ux \)-differentiant of the same type as \( \phi \) (if it is not identically 0), is the most general differentiant of that type and these characters if \( \phi \) is the most general differentiant of its type and the characters stated, and is \( \phi \) itself if that is a \( yx-, \ldots, ux \)-differentiant.

The summation in (21) is naturally limited to values of \( \alpha, \beta, \gamma, \ldots, \epsilon \) such that \( \sigma \leq w_x \), and that in (22) to values of \( \sigma \leq w_z, \beta \leq w_v, \gamma \leq w_x, \ldots, \epsilon \leq w_u \).

In particular, if \( \phi \) is any form of type \( (w_x, w_y) \),†


† If \( \phi \) is a form involving only the coefficients of the quantics (that is, a form of order 0) for which \( w_x = w_y \), this formula is equivalent to that for \( \phi \) given by \textit{Hilbert, Mathematische Annalen}, vol. 36 (1890), p. 523.
(23) \[ \{y; x\} \cdot \phi \equiv \sum_{a=0}^{w_x} (-1)^a \cdot \frac{(w_x - w_y + 1)!}{a! (w_x - w_y + a + 1)} \cdot y^{x^a} \cdot x^{y^a} \cdot \phi \]

is an \(xy\)-differentiant of the same type as \(\phi\) (if it is not identically 0), is the most general \(xy\)-differentiant of that type if \(\phi\) is the most general form of the type, and is \(\phi\) itself if that is an \(xy\)-differentiant.

In applying (21), (22) and (23) to any particular form \(\phi\), it is to be observed that the values of \(w_x, w_y, w_z, \ldots, w_u, w_v\) to be used are the weights of that form.

If \(\phi\) is any form of type \((w_x, w_y, w_z, \ldots, w_u, w_v)\),

(24) \[ [\phi]_1 = \{v; u, \ldots, z, y, x\} \cdot \{u; \ldots, z, y, x\} \cdot \{z; y, x\} \cdot \{y; x\} \cdot \phi \]
is an \(xy\), \(xz\), \ldots, \(xu\), \(xy\)-differentiant of that type, and

(25) \[ [\phi]_2 = \{x, y, z, \ldots, u; v\} \cdot \{x, y, z, \ldots, u\} \cdot \{x, y; z\} \cdot \{x; y\} \cdot \phi \]
is a \(yx\), \(zx\), \ldots, \(ux\), \(yx\)-differentiant of that type.

II. Determination of denumerants in general.

Let \(xy\) be any one of the differentiant characters included in the conditions for the denumerant, let the aggregate of all the other conditions be represented by \(K\), and let \((a, b)\) be any given type (to specify the weights in \(x\) and \(y\) alone). If the shear-operator \(xy\), or differentiant character \(xy\), is so related (or unrelated) to the conditions \(K\) that the form \(xy^k \cdot \phi\) satisfies the conditions \(K\) for every form \(\phi\) that satisfies them and every value of \(k\) for which \(xy^k \cdot \phi\) is not identically 0, we shall say that the operator \(xy\), or the differentiant character \(xy\), does not interfere with the conditions \(K\).

10. If the differentiant character \(xy\) does not interfere with the conditions \(K\), the forms \(xy^r \cdot \phi\) derived from the forms \(\phi\) of a complete system of type \((a, b)\) satisfying the conditions \(K\) and reduced qua \(xy\) by operating on each such form \(\phi\) with the power of \(xy\) whose exponent is the rank of that form \(\phi\) qua \(xy\) are linearly independent \(xy\)-differentiants of the types \((a + r, b - r)\) subordinate to the type \((a, b)\) qua \(xy\), by § 4. Therefore, the number of linearly independent \(xy\)-differentiants of any type \((a + r, b - r)\) subordinate to \((a, b)\) qua \(xy\) is at least as great as the number of forms of rank \(r\) in a complete system of type \((a, b)\) satisfying the conditions \(K\) reduced qua \(xy\); that is, for \(0 \leq r \leq r'\),

(26) \[ N^{xy}_{K}(a, b) \leq N_{K, xy}(a + r, b - r), \]

from which follows

(27) \[ N_{K}(a, b) = \sum_{r=0}^{r'} N^{xy}_{K}(a, b) \leq \sum_{r=0}^{r'} N_{K, xy}(a + r, b - r), \]
where \( r' \) is the greatest value of \( r \) for any possible type \((a + r, b - r)\) subordinate to \((a, b)\) quà \( x'y \).

11. If, for each value of \( r \) from 0 to \( r' \), the greatest value of \( r \) for any possible type subordinate to \((a, b)\) quà \( x'y \), there exists a **determinate** distributive operator \( \omega_r \), such that the forms \( \omega_r \psi_r \), obtained from a complete system of \( xy \)-differentiants \( \psi_r \) of all types \((a + r, b - r)\) subordinate to \((a, b)\) quà \( x'y \) that satisfy the conditions \( K \) are **linearly independent** forms of type \((a, b)\) that satisfy the conditions \( K \), then the number of linearly independent forms of type \((a, b)\) that satisfy the conditions \( K \) is at least as great as the number of linearly independent \( xy \)-differentiants of all types \((a + r, b - r)\) subordinate to \((a, b)\) quà \( x'y \) that satisfy the conditions \( K \) : that is,

\[
\sum_{r=0}^{r'} N_{K, xy}(a + r, b - r) \leq \sum_{r=0}^{r'} N_{K}^x(a, b) = N_{K}(a, b).
\]

12. If the conditions of both § 10 and § 11 are satisfied, we have, by (27) and (28),

\[
\sum_{r=0}^{r'} N_{K, xy}(a + r, b - r) = \sum_{r=0}^{r'} N_{K}^x(a, b) = N_{K}(a, b),
\]

and, because (29) cannot be satisfied unless the sign of equality is to be taken in (26) for every value of \( r \) from 0 to \( r' \), inclusive,

\[
N_{K, xy}(a + r, b - r) = N_{K}^x(a, b) \text{ for } 0 \leq r \leq r'.
\]

In this case, for any particular value of \( r \), the forms \( x'y' \cdot \phi \) of § 10, being, as we have there seen, linearly independent \( xy \)-differentiants of type \((a + r, b - r)\) that satisfy the conditions \( K \), and being, by (30), just equal in number to the number of such linearly independent \( xy \)-differentiants, in themselves constitute a complete system of linearly independent \( xy \)-differentiants of type \((a + r, b - r)\) that satisfy the conditions \( K \) : for any particular value of \( r \), the forms \( \psi_r \) of § 11 are, then, linearly independent linear functions of the forms \( x'y' \cdot \phi \) of § 10, and vice versa, and the former forms may be replaced by the latter, whenever it may be convenient ; that is, when we are considering simply a complete system of linearly independent \( xy \)-differentiants of a type \((a + r, b - r)\) subordinate to \((a, b)\) quà \( x'y \) satisfying the conditions \( K \), we may take these \( xy \)-differentiants to be the forms \( x'y' \cdot \phi \) derived from the forms \( \phi \) of rank \( r \) in a complete system of type \((a, b)\), to which \((a + r, b - r)\) is subordinate quà \( x'y \),

\* By a **determinate** operator \( \omega_r \), I mean one that depends on the type \((a, b)\), the conditions \( K \), the differentiant character \( x'y \), and the value of \( r \), alone, and not at all on the particular form \( \psi_r \) to which it is to be applied.

\* And, therefore, **non-vanishing** forms : for a vanishing form may be regarded as a linear function of any forms, having all its coefficients 0.
satisfying the conditions $K$ and reduced qua $\widetilde{xy}$. Again, in this same case, the forms $\omega_\psi$, of § 11 for all values of $r$, being, as we have there seen, linearly independent forms of type $(a, b)$ that satisfy the conditions $K$, and being, by (29), just equal in number to the number of such linearly independent forms, in themselves constitute a complete system of linearly independent forms of type $(a, b)$ that satisfy the conditions $K$; the forms $\phi$ of § 10 are, then, linearly independent linear functions of the forms $\omega_\psi$, of § 11 for all values of $r$, and vice versa: whenever we are considering simply a complete system of linearly independent forms of type $(a, b)$ that satisfy the conditions $K$, we may, then, take them to be the forms $\omega_\psi$, obtained from complete systems of linearly independent $xy$-differentiants $\psi_r$ of all types $(a + r, b - r)$ subordinate to $(a, b)$ qua $\widetilde{xy}$ that satisfy the conditions $K$: but it is to be observed that we have not shown that the forms $\psi_r$, although they may be replaced by the same number of arbitrary linearly independent linear functions of them for each value of $r$, can be so taken that the forms $\omega_\psi$, for all values of $r$ shall constitute a system reduced qua $\widetilde{xy}$, —however, it would be so if the forms $\omega_\psi$, were of rank $r$ for each value of $r$, as a little consideration will show.

13. If the differentiant character $xy$ does not interfere with the conditions $K$, so that (26) and (27) hold, and if, for each value of $r$ from 0 to $r'$ [the greatest value of $r$ for which $(a + r, b - r)$ is a possible type subordinate to the given type $(a, b)$ qua $\widetilde{xy}$], there exists a determinate distributive operator $\omega_\psi$, such that the form $\omega_\psi$, is a form of type $(a, b)$ that satisfies the conditions $K$ for every $xy$-differentiant $\psi_r$, of type $(a + r, b - r)$ that satisfies the same conditions, and also such that

\[
\widetilde{xy} \cdot \omega_\psi = S_r \cdot \psi_r,
\]

where $S_r$ is a non-vanishing constant, then the forms $\omega_\psi$, obtained from complete systems of $xy$-differentiants of all types $(a + r, b - r)$ subordinate to $(a, b)$ qua $\widetilde{xy}$ that satisfy the conditions $K$ are linearly independent. For, by (31),

\[
\widetilde{xy}^k \cdot \omega_\psi = S_r \cdot \widetilde{xy}^{k - r} \cdot \psi_r = 0 \quad \text{if} \quad r < k,
\]

so that any linear relation between forms $\omega_\psi$, for which the greatest value of $r$ is $k$ would be turned by the operator $\widetilde{xy}^k$ into a linear relation between the forms $\psi_k$ with non-vanishing coefficients, whereas no such relation exists; therefore, such operators $\omega_\psi$, satisfy the conditions of § 11, so that their existence proves (28) and, because (26) and (27) hold, also (29) and (30); then, by § 12, the linearly independent $xy$-differentiants of any type $(a + r, b - r)$ subordinate to the given type $(a, b)$ qua $\widetilde{xy}$ that satisfy the conditions $K$ may be taken to be the forms $\widetilde{xy} \cdot \phi$ derived from the forms $\phi$ of rank $r$ in a complete system of type $(a, b)$ satisfying the conditions $K$ and reduced qua $\widetilde{xy}$.
In this case, for any given value of $k$ from 0 to $r'$, inclusive, the forms $\tilde{xy}^k \cdot \phi$ of type $(a + k, b - k)$ obtained from the forms $\phi$ of rank as great as $k$ are linearly independent, for any linear relation between such of them as are obtained from forms $\phi$ of rank $h$ and less qua $\tilde{xy}$ $(k \leq h \leq r')$ would be turned by the operator $\tilde{xy}^{h-k}$ into a linear relation between the linearly independent forms $\tilde{xy}^k \cdot \phi$ derived from forms $\phi$ of rank as great as $k$ qua $\tilde{xy}$. Again, the types $(a + r, b - r)$ subordinate to $(a, b)$ qua $\tilde{xy}$ for all values of $r$ as great as $k$ are also the types subordinate to $(a + k, b - k)$ qua $\tilde{xy}$, and the $xy$-differentiants $\tilde{xy}^r \cdot \phi \equiv \tilde{xy}^{r-k} \cdot \tilde{xy}^k \cdot \phi$ derived from the forms $\phi$ that are of rank $r$ qua $\tilde{xy}$ are the $xy$-differentiants derived from the forms $\tilde{xy}^k \cdot \phi$ that are of rank $r - k$ qua $\tilde{xy}$, so that the latter forms constitute a system reduced qua $\tilde{xy}$. Finally, the operators $\tilde{xy}^k \cdot \omega_r$ for all values of $r$ as great as $k$ turn the linearly independent $xy$-differentiants $\tilde{xy}^r \cdot \phi$ into linearly independent forms of type $(a + k, b - k)$ and rank $r - k$ qua $\tilde{xy}$ that satisfy the conditions $K$ and for which, by (31),

$$\tilde{xy}^{r-k} \cdot \tilde{xy}^k \cdot \omega_r \cdot \tilde{xy}^r \cdot \phi \equiv \tilde{xy}^r \cdot \omega_r \cdot \tilde{xy}^r \cdot \phi \equiv S_r \cdot \tilde{xy}^r \cdot \phi,$$

which is what (31) becomes when the system of forms $\phi$ is replaced by the system of forms $\tilde{xy}^k \cdot \phi$, so that formulæ (29) and (30) hold also if the type $(a, b)$ is replaced by the type $(a + k, b - k)$ and, of course, the limit $r'$ of $r$ by the limit $r' - k$ of $r - k$. The resultant formulæ are, evidently, equivalent to

$$\sum_{r=k}^{r'} N_{k, xy}(a + r, b - r) = N_k(a + k, b - k)$$

and

$$N_{k, xy}(a + r, b - r) = N_k^{r-k}(a + k, b - k) \quad \text{for} \quad k \leq r \leq r'.$$

On subtracting (32) from (29), member from member, we get, for $1 \leq k \leq r'$,

$$\sum_{r=0}^{k-1} N_{k, xy}(a + r, b - r) = N_k(a, b) - N_k(a + k, b - k) = (1 - \tilde{xy}^k) \cdot N_k(a, b),$$

and, in particular, for $k = 1$,

$$N_{k, xy}(a, b) = N_k(a, b) - N_k(a + 1, b - 1) = (1 - \tilde{xy}) \cdot N_k(a, b),$$

by which (if the conditions for its validity are satisfied) the number of linearly independent forms of given type that satisfy the conditions $K$ and have the additional differentiant character $xy$ is determined when the numbers of linearly independent forms of certain types that satisfy the conditions $K$ are known. In other words (35) is a formula for the addition, in a denumerant, of a differentiant character to other conditions with which it does not interfere.

14. Whatever differentiant and numerical conditions may be given, it seems easy to select one of the differentiant characters that shall not interfere with the
aggregate of the other conditions and, for each of the possible values of \( r \), to find an operator \( \omega_r \) that shall have all the properties stated in § 13, excepting the last, that is, that the constant \( S_r \) shall be different from 0 for each value of \( r \). At least, that is the only difficulty that has presented itself to me in my attempts to determine the denumerants for triple and higher differentiants.

Evidently, the reasoning of §§ 11—13 requires that there shall be forms of type \((a, b)\) that satisfy the conditions \( K \) if there are forms of any type subordinate to \((a, b)\) qua \( xy \) that satisfy them and we shall find that, at least in the cases we consider, the operators \( \omega_r \) that naturally suggest themselves do not satisfy (31) with a non-vanishing constant \( S_r \) for all the values of \( r \) in question unless \( b \leq a \), which, by theorem \( A \) of § 8, is a necessary, though not sufficient, condition that there shall be \( xy \)-differentiants of the type \((a, b)\). In other words, we shall find it impossible to use (35) unless the type \((a, b)\) is such that the condition of theorem \( A \) of § 8 is satisfied by the weights in all the pairs of variables that define the differentiant characters for the denumerant to be determined. But this is really no limitation to the use of (35), because the left member of this formula implies these relations between the weights, or else its value is 0.

15. The numerical conditions included in \( K \) may be of very different kinds, of which it would be impossible to state all that it might be useful to take into account, but we ought to mention one class that we shall have occasion to consider. Numerical conditions satisfied by the derivatives of a system of forms relatively to a given shear-operator may be regarded as conditions satisfied by the forms of the system. In particular, that the rank qua \( yz \) of the derivative qua \( xy \) of any form \( \phi \) of a complete system of type \((a, b)\) satisfying certain other conditions shall lie within given limits is admissible as a numerical condition to restrict the system further, and will be satisfied by the derivatives of the restricted system; the \( xy \)-differentiants \( \psi_r \) of types \((a + r, b - r)\) must then be taken to satisfy the same condition, namely, that their ranks qua \( yz \) shall lie within the given limits. It may be that the conditions \( K \) can be separated into two sets, of which the second set contains only conditions that are satisfied by all forms \( \phi \) of type \((a, b)\) that satisfy the conditions of the first set and by their derivatives qua \( xy \); the conditions of the second set need be specified only for the \( xy \)-differentiants \( \psi_r \) of types \((a + r, b - r)\).

For example, if the differentiant character \( xy \) does not interfere with the conditions \( K \) (representing only the conditions of the first set); if the rank qua \( yz \) of the form \( \omega_0 \cdot \phi \) lies within limits \( l_k \) to \( l'_k \) dependent only on the given type \((a, b)\) and the value of \( k \), inclusive of both limits, for every form of type \((a, b)\) that satisfies the conditions \( K \) and for every value of \( k \) for which \( \omega_0 \cdot \phi \) is not identically 0 (the one condition of the second set); and if, for each value of \( r \)
from 0 to \( r' \) [the greatest value of \( r \) for which \((a + r, b - r)\) is a type subordinate to \((a, b)\) qua \( xy \)], inclusive, there exists a determinate distributive operator \( \omega \), such that the forms \( \omega, \psi \), obtained from the \( xy \)-differentiants \( \psi \), whose ranks qua \( yz \) lie within the limits \( l \) to \( l' \), inclusive, in complete systems of all types \((a + r, b - r)\) subordinate to \((a, b)\) qua \( xy \) satisfying the conditions \( K \) and reduced qua \( yz \) are forms of type \((a, b)\) that satisfy the conditions \( K \) and the identity (31), where \( S \) is a non-vanishing constant; then, by (29),

\[
\sum_{r=0}^{r'} \sum_{p=1}^{l} N_{K,xy}^{\alpha}(a + r, b - r) = N_K(a, b).
\]

### III. Simple differentiants.

16. In this section we consider only forms of a given order in the variables, of given degrees in the coefficients of the several quantics, and of given weights in the several variables excepting two, say \( x \) and \( y \). For any such form of type \((w_x, w_y)\) we have, by (2), \( w_x + w_y = \sigma \), where \( \sigma \) is a constant, that is, is the same number for all the forms considered. It will be convenient to represent \( w_x \) temporarily by a simpler symbol \( a \) (or \( a' \)) : then, \( w_y = \sigma - a \) (or \( \sigma - a' \)).

For \( xy \)-differentiants of a given type \((a, \sigma - a)\) there are no conditions excepting the differentiant character \( xy \) and the types subordinate to \((a, \sigma - a)\) qua \( xy \) are the types \((a + r, \sigma - a - r)\) for which \( 0 \leq r \leq \sigma - a \) (it may even be that \( r \) has a smaller upper limit than \( \sigma - a \)). If \( \psi_{a'} \) is any \( xy \)-differentiant of one of these subordinate types, say \((a', \sigma - a')\) for \( r = a' - a \), where \( a \leq a' \leq \sigma \) and \( \frac{1}{2} \sigma \leq a' \), by (20), then \( xy \cdot \psi_{a'} = \psi_{a'} \cdot \sigma' \) is a form of type \((a, \sigma - a)\) for which, by (18), because \( xy \cdot \psi_{a'} = 0 \),

\[
N_{xy}(a, \sigma - a) = (1 - xy) \cdot N(a, \sigma - a)
\]

or, if we replace \( a \) by \( w_x \) and \( \sigma - a \) by \( w_y \),

\[
N_{xy}(w_x, w_y) = (1 - xy) \cdot N(w_x, w_y) \quad \text{for} \quad w_y \leq w_x.
\]

This is Cayley's formula as extended to a system of any number of quantics in any number of variables.

Formula (37) shows that the number of linearly independent linear equations between the coefficients of the general form \( \phi \) of type \((w_x, w_y)\) implied by the identity \( xy \cdot \phi = 0 \) is just the number of terms in the general form of type \( \overline{xy} \cdot \phi = 0 \), that is, that the linear equations implied by this identity are linearly independent, as Cayley assumed. This proves also that, if \( \phi \) is the general
form of type \((w_x, w_y)\), then \(xy \cdot \phi\) is the general form of type \((w_x + 1, w_y - 1)\) and, therefore, \(xy^k \cdot \phi\) is the most general form of type \((w_x + k, w_y - k)\) for every value of \(k\) for which this type exists.

Incidentally, formula (37) shows that, of the general forms of two types that differ only in the weights in two variables, that has the more terms in which the weights in the two variables in question are the more nearly equal (unless they have the same number of terms, in which case the general forms of all intermediate types,—types subordinate to one and to which the other is subordinate,—have that same number of terms).

Again, formula (37) gives the number of linearly independent covariants or invariants of any system of binary quantics if \(x\) and \(y\) are the only variables and \(w_x = w_y\), by theorems \(A\) and \(B\) of § 8, because every \(xy\)-differentiant for which these conditions are satisfied is a complete differentiant in \(x\) and \(y\) and, therefore, a covariant or invariant.

### IV. Double differentiants.

In §§ 17–22 we shall consider only forms of a given order in the variables, of given degrees in the coefficients of the several quantics, and of given weights in the several variables excepting three, say \(x\), \(y\), and \(z\). The type of such a form shall be regularly denoted by \((w_x, w_y, w_z)\), representing only the weights in \(x\), \(y\), and \(z\), in this sequence, so that, by (2), \(w_x + w_y + w_z = \sigma\), where \(\sigma\) has the same value for all forms considered; but, in the deduction of formulae, it will be more convenient to represent two of these three weights by simple letters, and for this purpose we shall put either \(w_x = a\), \(w_y = b\) and \(w_z = \sigma - a - b\), —or \(w_x = \sigma - b - c\), \(w_y = b\), and \(w_z = c\), —accenting \(a\), \(b\), or \(c\) when we have occasion to represent more than one value of either.

17. If \(x\) is an \(xz\)- and \(yz\)-differentiant of type \((\sigma - b' - c, b', c)\), —where, by theorem \(A\) of § 8, \(c \leq b' \leq \sigma - 2c\), —so that \(xz \cdot \chi \equiv 0\) and \(yz \cdot \chi \equiv 0\), then, by (17) and (18),

\[
\widehat{xz} \cdot \widehat{xy}^k \cdot \chi \equiv \widehat{xy}^k \cdot \widehat{xz} \cdot \chi \equiv 0 \quad \text{and} \quad \widehat{yz} \cdot \widehat{xy}^k \cdot \chi \equiv \widehat{xy}^k \cdot \widehat{yz} \cdot \chi + k \cdot \widehat{xy}^{k-1} \cdot \widehat{xz} \cdot \chi \equiv 0,
\]

so that the differentiant character \(xy\) does not interfere with the differentiant characters \(xz\) and \(yz\). The derivatives of \(xz\)- and \(yz\)-differentiants of the given type \((\sigma - b' - c, b', c)\) qua \(xy\) are, then, \(xzyz\)-differentiants of types \((\sigma - b - c, b, c)\) subordinate to \((\sigma - b' - c, b', c)\) qua \(xy\) for which \(0 \leq c \leq b \leq b' \leq \sigma - b - c\) and \(b \leq \frac{1}{3} (\sigma - c)\), by theorems \(A\) and \(F\) of § 8. If \(\psi'_{b-b}\) is any \(xyz\)-differentiant of type \((\sigma - b - c, b, c)\), —for which, in the notation of §§ 10–13, \(r = b' - b\) and \(0 \leq r \leq b' - c\), —then, by (17) and (18),

\[
\widehat{xz} \cdot \widehat{xy}^{b'-b} \cdot \psi'_{b-b} \equiv \widehat{yx}^{b'-b} \cdot \widehat{xz} \cdot \psi'_{b-b} + (b' - b) \cdot \widehat{xy}^{b'-b-1} \cdot \widehat{yz} \cdot \psi'_{b-b} \equiv 0
\]
and
\[ \frac{\partial}{\partial x} \cdot \frac{\partial}{\partial y} \cdot \psi_{y-b} \equiv \frac{\partial}{\partial x} \cdot \frac{\partial}{\partial z} \cdot \psi_{x-b} \equiv 0, \]
so that \( \frac{\partial}{\partial x} \cdot \frac{\partial}{\partial y} \cdot \psi_{y-b} \) is an \( xz \)- and \( yz \)-differentiant of type \( (\sigma - b' - c, b', c) \), if not identically 0, for which, by (18),
\[ \frac{\partial}{\partial x} \cdot \frac{\partial}{\partial y} \cdot \psi_{y-b} \equiv (b' - b) \cdot (\sigma - 2b - c), \]
where the multiplier of \( \psi_{y-b} \) is a non-vanishing constant if \( b \equiv b' \equiv \sigma - b - c \).
Therefore, if \( 0 \leq c \leq b' \leq \frac{1}{2}(\sigma - c) \),
\[ N_{xz,yz}(\sigma - b' - c, b', c) = \sum_{b=c}^{\sigma - c} N_{xyz}(\sigma - b - c, b, c), \]
by (28), and
\[ N_{xyz}(\sigma - b' - c, b', c) = (1 - \frac{\sigma}{\sigma - c}) \cdot N_{xz,yz}(\sigma - b' - c, b', c), \]
by (35), while, if \( 0 \leq c \leq b \leq b' \leq \frac{1}{2}(\sigma - c) \),
\[ N_{xz,yz}(\sigma - b' - c, b', c) = N_{xyz}(\sigma - b - c, b, c), \]
by (30); or, when \( \sigma - b' - c \) is replaced by \( w_x, b' \) by \( w_y, c \) by \( w_z, b \) by \( b' - r \), if \( w_x \leq w_y \leq w_z \),
\[ N_{xz,yz}(w_x, w_y, w_z) = \sum_{r=0}^{w_x-w_z} N_{xyz}(w_x + r, w_y - r, w_z) \]
and
\[ N_{xyz}(w_x, w_y, w_z) = (1 - \frac{\sigma}{\sigma - c}) \cdot N_{xz,yz}(w_x, w_y, w_z), \]
while, if \( w_x \leq w_y \leq w_z \) and \( 0 \leq r \leq w_y - w_x \),
\[ N_{xz,yz}(w_x, w_y, w_z) = N_{xyz}(w_x + r, w_y - r, w_z). \]
Also, by § 12, the forms \( \frac{\partial}{\partial x} \cdot \frac{\partial}{\partial y} \cdot \psi_{y-b} \) obtained from complete systems of \( xyz \)-differentiants of all types \( (\sigma - b - c, b, c) \) subordinate to the given type \( (\sigma - b' - c, b', c) \) qua \( xy \) constitute a complete system of \( xz \)- and \( yz \)-differentiants of the latter type if \( 0 \leq c \leq b' \leq \frac{1}{2}(\sigma - c) \).

18. If \( \chi \) is an \( xy \)- and \( xx \)-differentiant of type \( (a, b', \sigma - a - b') \), — where, by theorem \( A \) of § 8, \( \sigma - 2a \equiv b' \equiv a \), — so that \( \psi_{xy} \cdot \chi \equiv 0 \) and \( \psi_{xz} \cdot \chi \equiv 0 \), then, by (17) and (18),
\[ \psi_{xy} \cdot \chi \equiv \psi_{xy} \cdot \chi \equiv k \cdot \psi_{xy} \equiv k \cdot \psi_{xy} \cdot \chi \equiv 0 \]
and
\[ \psi_{xz} \cdot \chi \equiv \psi_{xz} \cdot \chi \equiv k \cdot \psi_{xz} \cdot \chi \equiv 0, \]
so that the differentiant character \( yz \) does not interfere with the differentiant characters \( xy \) and \( xx \). The derivatives of \( xy \)- and \( xx \)-differentiants of the given type \( (a, b', \sigma - a - b') \) qua \( \psi_{yz} \) are, then, \( xyz \)-differentiants of types \( (a, b, \sigma - a - b) \) subordinate to \( (a, b', \sigma - a - b') \) qua \( \psi_{yz} \) for which
\( \sigma - a - b \leq b' \leq b \leq a \leq \sigma - b \) and \( \frac{1}{2} (\sigma - a) \leq b \), by theorems \( A \) and \( F \) of § 8. If \( \psi_{b' - y'} \) is any \( xyz \)-differentiant of type \((a, b', \sigma - a - b')\),—for which, in the notation of §§ 10-13, \( r = b - b' \) and \( 0 \leq r \leq a - b' \),—then, by (17) and (18),

\[
\tilde{xy} \cdot \tilde{z}y^{b' - y'} \cdot \psi_{b' - y'}' = \tilde{z}y^{b' - y'} \cdot \tilde{xy} \cdot \psi_{b' - y'}' = 0
\]

and

\[
\tilde{zx} \cdot \tilde{z}y^{b' - y'} \cdot \psi_{b' - y'}' = \tilde{z}y^{b' - y'} \cdot \tilde{zx} \cdot \psi_{b' - y'}' = (b - b') \cdot \tilde{z}y^{b' - y' - 1} \cdot \tilde{xy} \cdot \psi_{b' - y'}' = 0,
\]

so that \( \tilde{z}y^{b' - y'} \cdot \psi_{b' - y'}' \) is an \( xy \)- and \( xz \)-differentiant of type \((a, b', \sigma - a - b')\), if not identically 0, for which, by (18),

\[
\tilde{z}y^{b' - y'} \cdot \tilde{z}y^{b' - y'} \cdot \psi_{b' - y'}' = (b - b')! \cdot (a + 2b - \sigma)^{(b - b')} \cdot \psi_{b - y'},
\]

where the multiplier of \( \psi_{b - y'} \) is a non-vanishing constant if \( \sigma - a - b \leq b' \leq b \). Therefore, if \( \frac{1}{2} (\sigma - a) \leq b' \leq a \leq \sigma - b \),

\[
N_{xy, zy}(a, b', \sigma - a - b') = \sum_{b = b'}^{a} N_{xyz}(a, b, \sigma - a - b),
\]

by (28), and

\[
N_{xyz}(a, b', \sigma - a - b') = (1 - yz) \cdot N_{xy, zy}(a, b', \sigma - a - b'),
\]

by (35), while, if \( \frac{1}{2} (\sigma - a) \leq b' \leq b \leq a \leq \sigma - b \),

\[
N_{xy, zy}^{b' - y'}(a, b', \sigma - a - b') = N_{xyz}(a, b, \sigma - a - b),
\]

by (30), or when \( a \) is replaced by \( w_x \), \( b' \) by \( w_y \), \( \sigma - a - b' \) by \( w_z \), and \( b \) by \( b' + r \), if \( w_z \leq w_y \leq w_x \),

\[(41) \quad N_{xy, zy}(w_x, w_y, w_z) = \sum_{r = 0}^{w_z - w_y} N_{xyz}(w_x, w_y + r, w_z - r) \]

and

\[(42) \quad N_{xyz}(w_x, w_y, w_z) = (1 - yz) \cdot N_{xy, zy}(w_x, w_y, w_z), \]

while, if \( w_z \leq w_y \leq w_x \) and \( 0 \leq r \leq w_z - w_y \),

\[(43) \quad N_{xy, zy}^{r}(w_x, w_y, w_z) = N_{xyz}(w_x, w_y + r, w_z - r). \]

Also, by § 12, the forms \( \tilde{z}y^{b' - y'} \cdot \psi_{b' - y'}' \) obtained from complete systems of \( xyz \)-differentiants of all types \((a, b, \sigma - a - b)\) subordinate to the given type \((a, b', \sigma - a - b')\) qua \( yz \) constitute a complete system of \( xy \)- and \( xz \)-differentiants of the latter type if \( \frac{1}{2} (\sigma - a) \leq b' \leq a \leq \sigma - b \).

19. If \( \phi \) is an \( xy \)-differentiant of type \((a', b', \sigma - a' - b')\), where, by theorem \( A \) of § 8, \( b' \leq a' \leq \sigma - b' \), so that \( \tilde{xy} \cdot \tilde{xz} \cdot \phi \equiv 0 \), then, by (17),

\[
\tilde{xy} \cdot \tilde{xz} \cdot \phi \equiv \tilde{zx} \cdot \tilde{xy} \cdot \phi \equiv 0,
\]

that is, the differentiant character \( xz \) does not interfere with the differentiant
character $xy$. The derivatives of $xy$-differentiants of given type $(a', b', \sigma - a' - b')$ qua $zx$ are, then, $xy$- and $xz$-differentiants of types $(a, b', \sigma - a - b')$ for which $\sigma - a - a' \leq b' \leq a' \leq a \leq \sigma - b'$, by theorems $A$ and $F$ of § 8. If $\chi_{a-a'}$ is any $xy$- and $xz$-differentiant of type $(a, b', \sigma - a - b')$, — for which, in the notation of §§ 10—13, $r = a - a'$ and $0 \leq r \leq \sigma - a' - b'$, — then, by § 9, \{ $y : x$ \} $\cdot \hat{zx}^{a-a'} \cdot \chi_{a-a'}$ is an $xy$-differentiant of type $(a', b', \sigma - a' - b')$, if not identically 0.

Now, by § 18, if $\frac{1}{2}(\sigma - a) \leq b' \leq a \leq \sigma - b'$, we may take a complete system of $xy$- and $xz$-differentiants $\chi_{a-a'}$ of type $(a, b', \sigma - a - b')$ in such manner that any one of them of rank $b - b'$ qua $yz$ can be represented as $\chi_{a-a'} = \hat{yz}^{b-b'} \cdot \psi_{b-b'}$, where $\psi_{b-b'}$ is an $yz$-differentiant of some type $(a, b, \sigma - a - b)$ subordinate to $(a, b', \sigma - a - b')$ qua $yz$ for which $b' \leq b \leq a$ and $b \leq \sigma - a$. Then,

$$\hat{xy} \cdot \chi_{a-a'} = 0, \quad \hat{xz} \cdot \chi_{a-a'} = 0,$$

and, therefore,

$$\hat{xz} \cdot \hat{yz}^{b-b'+a} \cdot \psi_{b-b'} = 0 \quad \text{and} \quad \hat{zx}^{a-a'} \cdot \hat{zx}^{a-a'} \cdot \hat{yz}^{b-b'+a} \cdot \psi_{b-b'} = 0 \quad \text{if} \quad a - a' - a' < b,$$

so that, by (17) and (18),

$$\hat{xy}^a \cdot \hat{xz}^{a-a'} \cdot \chi_{a-a'} = (a - a')^{(a)} \cdot \hat{zx}^{a-a'} \cdot \hat{yz}^{b-b'+a} \cdot \psi_{b-b'},$$

$$\hat{xz}^{a-a'} \cdot \hat{yz}^a \cdot \hat{zx}^{a-a'} \cdot \chi_{a-a'} = (a - a')^{(a)} \cdot \hat{yz}^a \cdot \hat{xz}^{a-a'} \cdot \hat{yz}^{b-b'+a} \cdot \psi_{b-b'},$$

$$\equiv (a - a')! (a - a')^{(a)} \cdot (2a + b' - \sigma - \alpha)^{(a-a'-a)} \cdot \hat{yz}^a \cdot \hat{yz}^{b-b'+a} \cdot \psi_{b-b'},$$

$$\equiv (a - a')! (a - a')^{(a)} \cdot (2a + b' - \sigma - \alpha)^{(a-a'-a)} \cdot (b - b' + \alpha)^{(a)} \times (a + b + b' - \sigma)^{(a)} \cdot \hat{yz}^{b-b'} \cdot \psi_{b-b'},$$

$$\equiv \frac{(a - a')! (a + b + b' - \sigma)!}{(a + a' + b' - \sigma)! (b - b')!} \cdot \frac{(2a + b' - \sigma - \alpha)! (b - b' + \alpha)!}{(a + b + b' - \sigma - \alpha)!} \cdot \chi_{a-a'},$$

and, by (23),

$$\hat{zx}^{a-a'} \cdot \{ y ; x \} \cdot \hat{zx}^{a-a'} \cdot \chi_{a-a'} = S_{a-a'} \cdot \chi_{a-a'},$$

where, by (15)

$$S_{a-a'} = \frac{(a - a')! (a' - b' + 1)! (a + b + b' - \sigma)!}{(b - b')! (a + a' + b' - \sigma)!} \sum_{a=0}^{a-a'} (-1)^a \cdot \frac{(2a + b' - \sigma - \alpha)! (b - b' + \alpha + 1)^{(a'-b+1)}}{(a' - b' + \alpha + 1)^{(a'-b+1)}} \cdot \chi_{a-a'},$$

$$= \frac{(a - a')! (a' - b' + 1)! (a + b + b' - \sigma)!}{(b - b')! (a + a' + b' - \sigma)!} \cdot \frac{\chi_{a-a'}}{S_{a-a'} (a' - b' + \alpha + 1)^{(a'-b+1)}}.$$
which, by (14) and (16), is a non-vanishing constant if \( \sigma - a - b \leq b' \leq b \leq a' \leq a \); so that, at least when these conditions are satisfied, the derivative of the form 
\[
\{y; x\} \cdot \widehat{zz}^{a-a'} \cdot \chi_{a-a'} \text{ quà } \widehat{xx}
\]
is a non-vanishing constant multiple of \( \chi_{a-a'} \). Observe that the condition \( \sigma - a - b \leq b' \) is satisfied if \( \frac{1}{2} (\sigma - a) \leq b' \leq b \).

But, if \( \phi \) is an \( xy \)-differentiant of type \((a', b', \sigma - a' - b')\) and of rank \( a - a' \) quà \( \widehat{xx} \), its derivative quà \( \widehat{xx} \), namely \( \widehat{xx}^{a-a'} \cdot \phi \), is an \( xy \)- and \( xx \)-differentiant of type \((a, b', \sigma - a - b')\) and, say, of rank \( b - b' \) quà \( \widehat{yz} \), so that \( \widehat{yz}^{a-b'} \cdot \widehat{xx}^{a-a'} \cdot \phi \) is a non-vanishing \( xyz \)-differentiant of type \((a, b', \sigma - a - b')\), where, by theorems \( A \) and \( F \) of §8, \( \sigma - a - a' \leq b' \leq a' \leq a \leq a - b, b' \leq b \), and \( a - a - b' \leq b' \); then, by (18), because \( xy \cdot \phi \equiv 0 \),

\[
\widehat{xy}^{a-a'} \cdot \widehat{yz}^{a+b-a'-b} \cdot \phi \equiv (-1)^{a-a'} \cdot (a + b - a' - b')^{(a-a')} \cdot \widehat{yz}^{a-b'} \cdot \widehat{xx}^{a-a'} \cdot \phi \not\equiv 0,
\]
by (14), while

\[
\widehat{xy}^{a-a'+1} \cdot \widehat{yz}^{a+b-a'-b} \cdot \phi \equiv 0;
\]
therefore, \( \widehat{yz}^{a+b-a'-b} \cdot \phi \) is a non-vanishing form of type \((a', a + b - a', \sigma - a - b)\) and rank \( a - a' \) quà \( \widehat{xy} \), so that, by theorem \( F \) of §8, \( a + b - 2a' \leq a - a' \), that is, \( b \leq a' \); the types of \( \phi \) and its successive derivatives quà \( \widehat{xx} \) and quà \( \widehat{yz} \) are, then, so related that \( \sigma - a - b \leq b' \leq b \leq a' \leq a \leq \sigma - b \).

If, then, \( \frac{1}{2} (\sigma - a) \leq b' \leq a' \leq a \leq \sigma - b' \), the \( xy \)- and \( xx \)-differentiant \( \chi_{a-a'} \) of type \((a, b', \sigma - a - b')\) subordinate to \((a', b', \sigma - a' - b')\) quà \( \widehat{xx} \) and of rank \( b - b' \) quà \( \widehat{yz} \) is turned by the operator \( \{y; x\} \cdot \widehat{xx}^{a-a'} \) into an \( xy \)-differentiant of type \((a', b', \sigma - a' - b')\) satisfying the condition (44) with a non-vanishing value of the constant \( S_{a-a'} \) if \( b' \leq b \leq a' \); the condition (44) is equivalent to (31) if, in the latter, \( r \) is replaced by \( a - a', \widehat{xy} \) by \( \widehat{xx}, \psi \), by \( \chi_{a-a'}, \) and \( \omega \), by \{\( y; x\}\} \cdot \widehat{xx}^{a-a'} ; \) and the conditions \( b' \leq b \leq a' \) are satisfied by every \( xy \)-differentiant of type \((a', b', \sigma - a' - b')\) whose derivative quà \( \widehat{xx} \) of type \((a, b', \sigma - a - b')\) is of rank \( b - b' \) quà \( \widehat{yz} \). Therefore, by (36), if \( \frac{1}{2} (\sigma - a') \leq b' \leq a' \leq \sigma - b' \),

\[
\sum_{b=b'}^{a'} \sum_{a=a'}^{\sigma-b} N_{xy, xx}^{a-b'} (a, b', \sigma - a - b') = N_{xy} (a', b', \sigma - a' - b')
\]
or, by (43),

\[
\sum_{b=b'}^{a'} \sum_{a=a'}^{\sigma-b} N_{yz} (a, b, \sigma - a - b) = N_{xy} (a', b', \sigma - a' - b'),
\]
where, however, the upper limit of \( b \) is \( \sigma - a' \) if \( \frac{1}{2} \sigma \leq a' \), because otherwise there could be no summation quà \( a \). Evidently, the summations quà \( r \) and \( \rho \) in (36) may be performed in either order if the limits are properly taken, and here it is more convenient to sum first with respect to \( r (a - a') \) and then with respect to \( \rho (b - b') \).
If \( \frac{1}{2} (\sigma - a') \leq b' \leq a' \leq \sigma - b' - 1 \), (45) holds also if \( a' \) is replaced by \( a' + 1 \), and when we subtract the result of the replacement from (45), member from member, we find

\[
\sum_{b = b'}^{a'} N_{xyz}(a', b, \sigma - a' - b) - \sum_{a = a' + 1}^{\sigma - a' - 1} N_{xyz}(a, a' + 1, \sigma - a - a' - 1) = (1 - xz)N_{xy}(a', b', \sigma - a' - b')
\]

if \( a' \leq \frac{1}{2} \sigma \), but

\[
\sum_{b = b'}^{a'} N_{xyz}(a', b, \sigma - a' - b) = (1 - xz)N_{xy}(a', b', \sigma - a' - b')
\]

if \( \frac{1}{2} \sigma \leq a' \); the special case \( a' = \frac{1}{2} (\sigma - 1) \) requires \( a' \) to be taken as the upper limit of \( b \) before the replacement of \( a' \) by \( a' + 1 \) and \( \sigma - a' - 1 = a' \) afterward, and the result of the subtraction is (46); in the special case \( a' = \frac{1}{2} \sigma \), (46) and (47) are identical; the second sum of (46) vanishes if \( a' = \frac{1}{2} \sigma \) or \( \frac{1}{2} (\sigma - 1) \).

If \( \frac{1}{2} (\sigma - a') \leq b' \leq a' = \sigma - b' \), that is, if \( \frac{1}{2} \sigma \leq a' \) and \( b = \sigma - a' \), (45) gives directly

\[
N_{xyz}(a', \sigma - a', 0) = N_{xy}(a', \sigma - a', 0),
\]

which is consistent with (47) [and with (46) if \( b' = a' = \frac{1}{2} \sigma \)],—because \( xz \cdot N_{x}(a', \sigma - a', 0) = 0 \), by § 5,—and true, because every \( xy \)-differentiant of weight 0 in \( z \) is an \( xz \)-differentiant, by the foot-note to § 5. So that (46) holds if \( \frac{1}{2} (\sigma - a') \leq b' \leq a' \leq \sigma - b' \) and \( a' \leq \frac{1}{2} \sigma \), while (47) holds if \( \frac{1}{2} (\sigma - a') \leq b' \leq a' \leq \sigma - b' \) and \( \frac{1}{2} \sigma \leq a' \).

If \( \frac{1}{2} (\sigma - a') \leq b' < a' \leq \sigma - b' - 1 \), (46) holds for \( a' \leq \frac{1}{2} \sigma \) and (47) for \( \frac{1}{2} \sigma \leq a' \) when \( b' \) is replaced by \( b' + 1 \), the difference of the formulæ before and after replacement, in either case, being

\[
N_{xyz}(a', b', \sigma - a' - b') = (1 - yz)(1 - xz)N_{xy}(a', b', \sigma - a' - b'),
\]

which, because \( yz \), \( xz \) and \( yz \cdot xz \) annihilate \( N_{xy}(a', \sigma - a', 0) \), includes also (48),—that being the case of (49) for which \( a' = \sigma - b' \).

If \( \frac{1}{2} (\sigma - a') \leq b' \leq a' \leq \sigma - b' \), that is, if \( \frac{1}{2} \sigma \leq a' \leq \frac{1}{2} \sigma \) and \( b' = a' \), (45) gives

\[
\sum_{a = a'}^{\sigma - a' - 1} N_{xyz}(a, a', \sigma - a - a') = N_{xy}(a', a', \sigma - 2a')
\]

or, with the replacement of \( N_{xyz}(a, a', \sigma - a - a') \) by its expression from (49) for all values of \( a \) greater than \( a' \) [for which, evidently, (49) holds],

\[
N_{xyz}(a', a', \sigma - 2a') = N_{xy}(a', a', \sigma - 2a') - (1 - yz)\sum_{a = a'}^{\sigma - a' - 1} (1 - xz)N_{xy}(a, a', \sigma - a - a')
\]
\[ N_{xy}(a', a', \sigma - 2a') - N_{xy}(a' + 1, a', \sigma - 2a' - 1) + yz \cdot N_{xy}(a' + 1, a', \sigma - 2a' - 1) = (1 - \overline{xx} + \overline{xx} \cdot \overline{yz}) \cdot N_{xy}(a', a', \sigma - 2a') \]

\[ = (1 - \overline{yz})(1 - \overline{xx}) \cdot N_{xy}(a', a', \sigma - 2a') \]

because \( \overline{yz} \cdot N_{xy}(a', a', \sigma - 2a') = N_{xy}(a', a' + 1, \sigma - 2a' - 1) = 0 \), by theorem \( A \) of § 8, — and this result is what (49) becomes for \( b' = a' \). Therefore, (49) holds for \( \frac{1}{2} (\sigma - a') \leq b' \leq a' \leq \sigma - b' \), that is, for all types of which there can be \( xyz \)-differentiants in accordance with theorem \( A \) of § 8.

On replacing \( N_{xy}(a', b', \sigma - a' - b') \) in (49) by its expression from (37), we have, if \( \frac{1}{2} (\sigma - a') \leq b' \leq a' \leq \sigma - b' \),

\[ N_{xy}(a', b', \sigma - a' - b') = (1 - \overline{yz})(1 - \overline{xx})(1 - \overline{xy}) \cdot N(a', b', \sigma - a' - b') \]

after the restoration of \( w_x, w_y, w_z \) for \( a', b', \sigma - a' - b' \), respectively, we have, if \( w_x \leq w_y \leq w_z \),

\[ \text{(50) } N_{xy}(w_x, w_y, w_z) = (1 - \overline{yz})(1 - \overline{xx})(1 - \overline{xy}) \cdot N(w_x, w_y, w_z). \]

By § 5, \( \overline{xy}, \overline{xx}, \) and \( \overline{yz} \) are commutative and \( \overline{xy} \cdot \overline{yz} = \overline{xx} \); therefore, (50) may be written

\[ \text{(51) } \begin{cases} N_{xy} \cdot (w_x, w_y, w_z) = (1 - \overline{xy} - \overline{yz} + \overline{xx})(1 - \overline{xx}) \cdot N(w_x, w_y, w_z) \\ = [1 - \overline{xy} - \overline{yz} + (\overline{xy} + \overline{yz} - \overline{xx})] \cdot N(w_x, w_y, w_z). \end{cases} \]

In particular, if \( w_y = w_z \),

\[ \overline{yz} \cdot N(w_x, w_x, w_x) = N(w_x, w_x + 1, w_x - 1) = N(w_x + 1, w_x, w_x - 1) = \overline{xx} \cdot N(w_x, w_x, w_x), \]

so that

\[ N_{xy}(w_x, w_x, w_x) = (1 - \overline{xx} + \overline{xx} \cdot \overline{yz}) \cdot N(w_x, w_x, w_x), \]

as we have already seen in the special case of (49) for which \( b' = a' \).

20. In combination with (43), (41), and (37), formula (49) gives

\[ \text{(52) } N_{xy}(w_x, w_y, w_z) = (1 - \overline{yz})(1 - \overline{xx}) \cdot \overline{xy} \cdot N_{xy}(w_x, w_y, w_z) \]

if \( w_z \leq w_y \leq w_x \) and \( 0 \leq r \leq w_x - w_y \), and

\[ N_{xy}(w_x, w_y, w_z) = (1 - \overline{xx}) \cdot \sum_{r=0}^{w_y} (1 - \overline{yz}) \cdot \overline{xy} \cdot N_{xy}(w_x, w_y, w_z) \]

\[ = (1 - \overline{xx}) \left[ \overline{yz} \cdot N_{xy}(w_x, w_y, w_z) - \overline{yz} \cdot N_{xy}(w_x, w_x, w_y) + w_z - w_x \right] \]

*If \( a' = \frac{1}{2} \sigma \), the sum in the first expression for \( N_{xy}(a', a', \sigma - 2a') \) vanishes and we have \( N_{xy}(\frac{1}{2} \sigma, \frac{1}{2} \sigma, 0) = N_{xy}(\frac{1}{2} \sigma, \frac{1}{2} \sigma, 0) \), which is a particular case of (49), — because \( xx \) and \( \overline{yz} \) annihilate \( N_{xy}(\frac{1}{2} \sigma, \frac{1}{2} \sigma, 0) \), by § 5, — and this formula is true, by (48).
\[
(1 - xz) \cdot N_{xy}(w_x, w_y, w_z) + xz \cdot yz \cdot N_{xy}(w_x, w_x, w_y + w_z - w_x),
\]
that is, by (37),

\[
\begin{aligned}
N_{xy, xz}(w_x, w_y, w_z) \\
= (1 - xz) \cdot N_{xy}(w_x, w_y, w_z) + N_{xy}(w_x + 1, w_x + 1, w_y + w_z - w_x - 2) \\
= (1 - xy) \cdot (1 - xz) \cdot N(w_x, w_y, w_z) \\
&+ (1 - xy) \cdot N(w_x + 1, w_x + 1, w_y + w_z - w_x - 2)
\end{aligned}
\]

if \( w_z \leq w_y \leq w_x \). If \( w_y + w_z < w_x + 2 \), the last term of each of the expressions for \( N_{xy, xz}(w_x, w_y, w_z) \) given by (53) vanishes.

Because the sequence of variables is unessential, (52) gives, if \( w_y \leq w_z \leq w_x \) and \( 0 \leq r \leq w_x - w_z \),

\[
N_{xy, xz}^{xy}(w_x, w_y, w_z) = (1 - yz)(1 - xz)(1 - xy) \cdot N(w_x, w_z + r, w_y - r) = N_{xy, xz}^{xy}(w_x, w_y, w_z) = N_{xy, xz}^{xy}(w_x, w_y, w_z),
\]

by (19), from which follows, when \( r \) is replaced by \( r + w_y - w_z \),

\[
N_{xy, xz}^{xy}(w_x, w_y, w_z) = (1 - yz)(1 - xz)(1 - xy) \cdot N(w_x, w_y + r, w_z - r)
\]

if \( w_y \leq w_z \leq w_x \) and \( w_z - w_y \leq r \leq w_x - w_y \), but this formula is identical with (52); therefore, (52) holds if \( w_y \leq w_z \leq w_z \), and \( w_z - w_y \leq r \leq w_x - w_y \), without regard to the relative values of \( w_y \) and \( w_z \) (of course, \( r \) is not negative); that is, (52) holds for all types of which there can be \( xy \)- and \( xz \)-differentiants in accordance with theorem A of § 8 and for all possible ranks of such \( xy \)- and \( xz \)-differentiants qua \( yz \), by theorem F of § 8 and by § 18.

Similarly, (53) gives, if \( w_y \leq w_z \leq w_z \),

\[
N_{xy, xx}(w_x, w_z, w_y)
\]

\[
= (1 - xy)(1 - xz) \cdot N(w_x, w_y, w_z) + (1 - xy) \cdot N(w_x + 1, w_x + 1, w_y + w_z - w_x - 2)
\]

\[
= N_{xy, xx}(w_x, w_y, w_z)
\]

\[
= (1 - xz)(1 - xy) \cdot N(w_x, w_y, w_z) + (1 - xy) \cdot N(w_x + 1, w_x + 1, w_y + w_z - w_x - 2),
\]

which is identical with (53), so that (53) holds if \( w_y \leq w_x \) and \( w_x \leq w_z \), without regard to the relative values of \( w_y \) and \( w_z \); that is, (53) holds for all types of which there can be \( xy \)- and \( xz \)-differentiants in accordance with theorem A of § 8.

21. If \( \phi \) is a \( yz \)-differentiant of type \((\sigma - b' - c', b', c')\), — where, by theorem \( A \) of § 8, \( c' \leq b' \leq \sigma - c' \), — so that \( yz \cdot \phi \equiv 0 \); then, by (17),

\[
\widehat{yz} \cdot \overline{xz}^k \cdot \phi \equiv \overline{xz}^k \cdot \widehat{yz} \cdot \phi \equiv 0,
\]
that is, the differentiant character \( xx \) does not interfere with the differentiant character \( yz \); the derivatives of \( yz \)-differentiants of given type \((\sigma - b' - c', b', c')\) qua \( xx \) are, then, \( xx \)- and \( yz \)-differentiants of types \((\sigma - b' - c, b', c)\) for which \( 0 \leq c \leq c' \leq b' \leq \sigma - c - c' \), by theorems \( A \) and \( F \) of \( \S 8 \); if \( \chi_{e-e} \) is any \( xx \)- and \( yz \)-differentiant of type \((\sigma - b' - c, b', c')\), for which, in the notation of \( \S \S 10-13 \), \( r = c' - c \) and \( 0 \leq r \leq c' \), then, by \( \S 9 \), \( \{z ; y\} \cdot \widetilde{xx}^{e-e} \cdot \chi_{e-e} \) is a \( yz \)-differentiant of type \((\sigma - b' - c', b', c')\), if not identically 0.

Now, by \( \S 17 \), if \( 0 \leq c \leq b' \leq \frac{1}{2} (\sigma - c) \), we may take a complete system of \( xx \)- and \( yz \)-differentiants \( \chi_{e-e} \) of type \((\sigma - b' - c, b', c)\) in such a manner that any one of them of rank \( b' - b \) qua \( xy \) can be represented as \( \chi_{e-e} = \widetilde{yb}'_{b} \cdot \psi_{b-b} \), where \( \psi_{b-b} \) is an \( xy \)-differentiant of some type \((\sigma - b - c, b, c)\) subordinate to \((\sigma - b' - c, b', c)\) qua \( xy \) for which \( c \leq b \leq b' \); then,

\[
\widetilde{xx} \cdot \chi_{e-e} = 0, \quad \widetilde{yz} \cdot \chi_{e-e} = 0,
\]

and, therefore,

\[
\widetilde{xx} \cdot \widetilde{ybz}^{b-b} \cdot \psi_{b-b} = 0 \quad \text{and} \quad \widetilde{xx} \cdot \widetilde{xx}^{c-a} - \widetilde{yz}^{b-b} + a \cdot \psi_{b-b} = 0 \quad \text{if} \quad c' - c < \alpha < \beta,
\]

so that, by (17) and (18),

\[
\begin{align*}
\widetilde{xx}^{a} \cdot \widetilde{xx}^{e-e} \cdot \chi_{e-e} &= (-1)^{a} \cdot (c' - c)^{(a)} \cdot \widetilde{yy}^{c-a} \cdot \widetilde{xx}^{e-e} \cdot \chi_{e-e} \\
&= (-1)^{a} \cdot (c' - c)^{(a)} \cdot \widetilde{xx}^{c-a} \cdot \widetilde{xx}^{\sigma-b} \cdot \psi_{b-b},
\end{align*}
\]

\[
\begin{align*}
\widetilde{xx}^{e-e} \cdot \widetilde{xx}^{e-e} \cdot \chi_{e-e} &= (-1)^{a} \cdot (c' - c)^{(a)} \cdot \widetilde{yy}^{e-e} \cdot \widetilde{xx}^{c-a} \cdot \widetilde{xx}^{e-e} \cdot \chi_{e-e} \\
&= (c' - c) \cdot (c' - c)^{(a)} \cdot (\sigma - b' - 2c - \alpha)^{(c' - c)^{(a)}} \cdot \widetilde{yy}^{e-e} \cdot \widetilde{xx}^{\sigma-b} \cdot \psi_{b-b} \\
&= (c' - c) \cdot (c' - c)^{(a)} \cdot (\sigma - b' - 2c - \alpha)^{(c' - c)^{(a)}} \cdot (b' - b + \alpha)^{(a)} \\
&\quad \times (\sigma - b - b' - c)^{(a)} \cdot \widetilde{yy}^{b-b} \cdot \psi_{b-b}
\end{align*}
\]

and, by (23),

\[
(54) \quad \widetilde{xx}^{e-e} \cdot \{z, y\} \cdot \widetilde{xx}^{e-e} \cdot \chi_{e-e} = S_{e-e} \cdot \chi_{e-e},
\]

where, by (15),

\[
\begin{align*}
S_{e-e} &= \frac{(c' - c) \cdot (b' - c' + 1) \cdot (\sigma - b' - b' - c)}{(b' - b)! \cdot (\sigma - b' - c - c')!} \cdot \sum_{a=0}^{c' - c} (-1)^{a} \cdot \binom{c' - c}{a} \\
&\quad \times \frac{(\sigma - b' - 2c - \alpha)^{(b' - c') + 1}}{(b' - c' + \alpha + 1)^{b' - c' + 1}} \\
&= \frac{(c' - c) \cdot (b' - c' + 1) \cdot (\sigma - b' - b' - c)}{(b' - b)! \cdot (\sigma - b' - c - c')!} \cdot S_{e-e} \cdot \chi_{e-e}.
\end{align*}
\]
which, by (14) and (16), is a non-vanishing constant if \( c < c' \leq b \leq b' \leq \sigma - b - c \); so that, at least when these conditions are satisfied, the derivative of the form 
\[
\{ z ; y \} \cdot \chi_{\sigma - c} \text{ qua } \tilde{xz}
\]
is a non-vanishing constant multiple of \( \chi_{\sigma - c} \). Observe that the condition \( b' \leq \sigma - b - c \) is satisfied if \( b \leq b' \leq \frac{1}{2}(\sigma - c) \).

But if \( \phi \) is a \( yz \)-differentiant of type \((\sigma - b' - c', b', c')\) and of rank \( c' - c \) qua \( \tilde{xy} \), its derivative qua \( \tilde{xy} \), namely \( \tilde{xy}^{b' - b} \cdot \phi \), is an \( xz \)- and \( yz \)-differentiant of type \((\sigma - b' - c, b', c)\) and, say, of rank \( b' - b \) qua \( \tilde{xy} \), so that \( \tilde{xy}^{b' - b} \cdot \tilde{xy}^{\sigma - c} \cdot \phi \) is a non-vanishing \( xyz \)-differentiant of type \((\sigma - b - c, b, c)\) for which, by theorems \( A \) and \( F \) of § 8, \( 0 \leq c \leq c' \leq b' \leq \sigma - c' \), \( c \leq b \leq b' \), and \( b \leq \sigma - b' - c \); then, by (18), because \( \tilde{yz} \chi \phi \equiv 0 \),
\[
\tilde{xy}^{\sigma - c} \cdot \tilde{xy}^{b' - b} \cdot \phi \equiv (b' + c' - b - c)^{c' - c} \cdot \tilde{xy}^{b' - b} \cdot \tilde{xy}^{\sigma - c} \cdot \phi \equiv 0,
\]
by (14), while
\[
\tilde{xy}^{b' - b} \cdot \tilde{xy}^{\sigma - c} \cdot \phi \equiv 0;
\]
therefore, \( \tilde{xy}^{b' - b} \cdot \tilde{xy}^{\sigma - c} \cdot \phi \) is a non-vanishing form of type \((\sigma - b - c, c + c' - c', c')\) and rank \( c' - c \) qua \( \tilde{yz} \), so that, by theorem \( F \) of § 8, \( 2c' - b - c \leq c' - c \), that is, \( c' \leq b \); the types of \( \phi \) and its successive derivatives qua \( \tilde{xz} \) and qua \( \tilde{xy} \) are, then, so related that \( 0 \leq c \leq c' \leq b \leq b' \leq \sigma - b - c \).

If, then, \( 0 \leq c \leq c' \leq b' \leq \frac{1}{2}(\sigma - c') \), the \( xz \)- and \( yz \)-differentiant \( \chi_{\sigma - c} \) of type \((\sigma - b' - c, b', c)\) subordinate to \((\sigma - b' - c', b', c')\) qua \( \tilde{xz} \) and of rank \( b' - b \) qua \( \tilde{xy} \) is turned by the operator \( \{ z ; y \} \cdot \tilde{xx}^{\sigma - c} \) into a \( yz \)-differentiant of type \((\sigma - b' - c', b', c')\) satisfying the condition (54) with a non-vanishing value of the constant \( S_{c' - c} \) if \( c' \leq b' \leq b \): the condition (54) is equivalent to (31) if, in the latter, \( r \) is replaced by \( c' - c \), \( \tilde{xy} \) by \( \tilde{xx} \), \( \chi_{\sigma - c} \) by \( \chi_{\sigma - c} \), and \( \omega_r \) by \( \{ z ; y \} \cdot \tilde{xx}^{\sigma - c} \); and the conditions \( c' \leq b' \leq b \) are satisfied by every \( yz \)-differentiant of type \((\sigma - b' - c', b', c')\) whose derivative qua \( \tilde{xz} \) of type \((\sigma - b' - c, b', c)\) is of rank \( b' - b \) qua \( \tilde{xy} \). Therefore, by (36), if \( 0 \leq c' \leq b' \leq \frac{1}{2}(\sigma - c') \),
\[
\sum_{c=0}^{c'} \sum_{b=c'}^{b'} N_{xz, yz}^{x b} (\sigma - b' - c, b', c) = N_{xz, yz} (\sigma - b' - c', b', c')
\]
or, by (40),
\[
(55) \quad \sum_{c=0}^{c'} \sum_{b=c'}^{b'} N_{xz yz} (\sigma - b - c, b, c) = N_{yz, yz} (\sigma - b' - c', b', c').
\]
If \( 1 \equiv c' \leq b' \leq \frac{1}{2}(\sigma - c') \), (55) holds also when \( c' \) is replaced by \( c' - 1 \), and when we subtract the result of the replacement from (55), member from member, we find
\[
(56) \quad \sum_{b=c'}^{b'} N_{yz} (\sigma - b - c, b, c') - \sum_{c=0}^{c'-1} N_{yz} (\sigma - c - c' + 1, c' - 1, c) = (1 - xxz) \cdot N_{yz} (\sigma - b' - c', b', c').
\]
If \( c' = 0 \), (56) is identical with (55), because \( \overline{\overline{\alpha z}} \cdot N_{yz}(\sigma - b', b', 0) = 0 \) and the second sum in (56) vanishes by the general principles of summation; therefore, (56) holds for \( 0 \leq c' \leq b' \leq \frac{1}{2}(\sigma - c') \).

If \( 0 \leq c' < b' \leq \frac{1}{2}(\sigma - c') \), (56) holds when \( b' - 1 \), the difference of the formul\ae\ of before and after replacement being

\[
(57) \quad N_{xyz}(\sigma - b' - c', b', c') = (1 - \overline{xy})(1 - \overline{xz}) \cdot N_{yz}(\sigma - b' - c', b', c').
\]

If \( 0 \leq c' = b' \leq \frac{1}{2}(\sigma - c') \), that is, if \( 0 \leq c' \leq \frac{1}{2}\sigma \) and \( b' = c' \), (55) gives

\[
\sum_{c=0}^{c'} N_{xyz}(\sigma - c - c', c', c) = N_{yz}(\sigma - 2c', c', c');
\]
or, with the replacement of \( N_{xyz}(\sigma - c - c', c', c) \) by its expression from (57) for all values of \( c \) less than \( c' \) [for which, evidently, (57) holds],

\[
N_{xyz}(\sigma - 2c', c', c') = N_{yz}(\sigma - 2c', c', c') - (1 - \overline{xy}) \cdot \sum_{c=0}^{c'-1} (1 - \overline{xz})
\]

\[
\times N_{yz}(\sigma - c - c', c', c) = N_{yz}(\sigma - 2c' + 1, c', c' - 1) + \overline{xy} \cdot N_{yz}(\sigma - 2c' + 1, c', c' - 1) = (1 - \overline{xy})(1 - \overline{xz}) \cdot N_{yz}(\sigma - 2c', c', c'),
\]

because \( \overline{xy} \cdot N_{yz}(\sigma - 2c', c', c') = N_{yz}(\sigma - 2c' + 1, c' - 1, c') = 0 \), by theorem \( A \) of § 8, — and this result is what (57) becomes for \( b' = c' \).* Therefore, (57) holds for \( 0 \leq c' \leq b' \leq \frac{1}{2}(\sigma - c') \), that is, for all types of which there can be \( xyz \)-differentiants in accordance with theorem \( A \) of § 8.

On replacing \( N_{yz}(\sigma - b' - c', b', c') \) in (57) by its expression from (37), we have, if \( 0 \leq c' \leq b' \leq \frac{1}{2}(\sigma - c') \),

\[
N_{xyz}(\sigma - b' - c', b', c') = (1 - \overline{xy})(1 - \overline{xz})(1 - \overline{yz}) \cdot N(\sigma - b' - c', b', c');
\]
after the restoration of \( w_x, w_y, w_z \) for \( \sigma - b' - c', b', c' \), respectively, we have, if \( w_x \leq w_y \leq w_z \),

\[
(58) \quad N_{xyz}(w_x, w_y, w_z) = (1 - \overline{xy})(1 - \overline{xz})(1 - \overline{yz}) \cdot N(w_x, w_y, w_z),
\]

which is identical with (50) and (51). In particular, if \( w_y = w_z \),

\[
\overline{xy} \cdot N(w_x, w_z, w_z) = N(w_x + 1, w_z - 1, w_z)
\]

\[
= N(w_x + 1, w_z, w_z - 1) = \overline{xy} \cdot N(w_x, w_z, w_z),
\]

so that

\[
N_{xyz}(w_x, w_z, w_z) = (1 - \overline{xz} + \overline{xy} \cdot \overline{xz})(1 - \overline{yz}) \cdot N(w_x, w_z, w_z),
\]
as we have already seen in the special case of (57) for which \( b' = c' \).

*If \( c' = 0 \), the sum in the first expression for \( N_{xyz}(\sigma - 2c', c', c') \) vanishes, but the result is valid, by (55) for \( b' = c' = 0 \).
22. In combination with (40), (38), and (37), formula (57) gives

\[
N_{xx,yy}(w_x, w_y, w_z) = (1 - xy)(1 - \bar{wx}) \cdot N_{yy}(w_x, w_y, w_z)
\]

(59)

\[
= (1 - xy)(1 - \bar{wx}) N_{yy}(w_x, w_y, w_z)
\]

if \(w_z \leq w_y \leq w_x\) and \(0 \leq r \leq w_y - w_z\), and

\[
N_{xx,yy}(w_x, w_y, w_z) = (1 - \bar{wx}) \cdot \sum_{r=0}^{w_z-w_x} (1 - xy) \cdot N_{yy}(w_x + w_y - w_z - 2, w_z - w_x - 1, w_z - 1)
\]

(60)

if \(w_z \leq w_y \leq w_x\). If \(w_z = 0\), the last term in each of the expressions for \(N_{xx,yy}(w_x, w_y, w_z)\) vanishes and we have

\[
N_{xx,yy}(w_x, w_y, 0) = N_{yy}(w_x, w_y, 0) = N(w_x, w_y, 0),
\]

as it should be, because every form of weight 0 in \(z\) is an \(xz\)- and \(yz\)-differentiant, by the footnote to § 5.

Because the sequence of variables is unessential, (59) gives, if \(w_z \leq w_x \leq w_y\) and \(0 \leq r \leq w_y - w_z\),

\[
N_{xx,yy}(w_y, w_x, w_z) = (1 - \bar{wx}) (1 - \bar{wy}) (1 - \bar{wy}) \cdot N(w_y + w_z, w_x - r, w_z) = N_{yy}(w_x, w_y, w_z) = N_{yy}(w_x, w_y, w_z)
\]

by (19), from which follows, when \(r\) is replaced by \(r + w_x - w_z\),

\[
N_{xx,yy}(w_x, w_y, w_z) = (1 - \bar{wx}) (1 - \bar{wy}) (1 - \bar{wy}) \cdot N(w_x + r, w_y - r, w_z)
\]

if \(w_z \leq w_x \leq w_y\) and \(w_y - w_z \leq r \leq w_y - w_z\); but this formula is identical with (59); therefore, (59) holds if \(w_z \leq w_x, w_z \leq w_y\) and \(w_y - w_z \leq r \leq w_y - w_z\), without regard to the relative values of \(w_x\) and \(w_y\) (of course \(r\) is not negative); that is, (59) holds for all types of which there can be \(xz\)- and \(yz\)-differentiants in accordance with theorem \(A\) of § 8 and for all possible ranks of such \(xz\)- and \(yz\)-differentiants qua \(xy\), by theorem \(F\) of § 8 and by § 17.

Similarly, (60) gives, if \(w_z \leq w_x \leq w_y\),

\[
N_{xx,yy}(w_y, w_z, w_z) = (1 - \bar{wx}) (1 - \bar{wy}) \cdot N(w_y, w_z, w_z)
\]

\[
+ (1 - \bar{wy}) \times N(w_y + w_z - w_z - 2, w_z - 1, w_z - 1) = N_{yy,xx}(w_x, w_y, w_z)
\]

(60)

if \(w_z \leq w_x \leq w_y\) and \(w_y - w_z \leq r \leq w_y - w_z\); but this formula is identical with (59); therefore, (59) holds if \(w_z \leq w_x, w_z \leq w_y\) and \(w_y - w_z \leq r \leq w_y - w_z\), without regard to the relative values of \(w_x\) and \(w_y\) (of course \(r\) is not negative); that is, (59) holds for all types of which there can be \(xz\)- and \(yz\)-differentiants in accordance with theorem \(A\) of § 8 and for all possible ranks of such \(xz\)- and \(yz\)-differentiants qua \(xy\), by theorem \(F\) of § 8 and by § 17.
which is identical with (60); therefore (60) holds if \( w_z \leq w_x \) and \( w_z \leq w_y \), without regard to the relative values of \( w_x \) and \( w_y \), that is, for all types of which there can be \( xxz \)- and \( yxz \)-differentiants in accordance with theorem \( A \) of § 8. Formulae (50) or (58), (53) and (60) express the denumerants of all kinds of double differentiants whose defining pairs involve only three of the variables.

23. The expression for the denumerant of double differentiants of a given type whose two defining pairs involve four different variables is found in the same manner in which that for simple differentiants was found in § 16. We consider only forms of a given order in the variables, of given degrees in the coefficients of the several quantics, and of given weights in each of the variables excepting four, say \( x, y, z, \) and \( s \); in representing the type of such a form, we shall write the weights in these four variables alone, following the sequence of the variables as just given, thus: \((w_x, w_y, w_z, w_s)\), so that, by (2), \( w_z + w_y + w_z + w_s \) has the same value for all forms of our system. In determining the expression for the denumerant \( N_{xy, zs}(w_x, w_y, w_z, w_s) \), we shall confine our attention to the \( zs \)-differentiants that are of given weights \( w_z \) and \( w_s \) in \( z \) and \( s \), respectively, — where, by theorem \( A \) of § 8, \( w_z \leq w_s \), — and, in writing the type of such a \( zs \)-differentiant, shall express the weights in \( x \) and \( y \) alone, in this sequence, as \((w_x, w_y)\). If four weights are written in the type, it is to be understood that they are the weights in \( x, y, z, \) and \( s \), respectively: but if only two weights are written, they are the weights in \( x \) and \( y \), respectively, and the weights in \( z \) and \( s \) are constant for all forms whose types are so written. In the latter representation of the type, we shall replace \( w_x \) by \( a \) (or \( a' \)) and \( w_y \) by \( \sigma - a \) (or \( \sigma - a' \)), where \( \sigma \) has the same value for all the forms considered.

If \( \phi \) is a \( zs \)-differentiant of given type \((a, \sigma - a)\), \( \bar{z}s \cdot \phi \equiv 0 \) and, by (17), \( \bar{z}s \cdot \bar{z}y^k \cdot \phi \equiv \bar{z}y^k \cdot \bar{z}s \cdot \phi \equiv 0 \), so that the differentiant character \( xy \) does not interfere with the differentiant character \( zs \). The types subordinate to \((a, \sigma - a)\) qua \( \bar{z}y \) are types \((a', \sigma - a')\) for which \( a \equiv a' \equiv \sigma \) (in the notation of § 10, the \( zs \) character is the only condition \( K \) and \( r = a' - a \)). If \( \psi_{a'-a} \) is an \( xy \)- and \( zs \)-differentiant of type \((a', \sigma - a')\) subordinate to \((a, \sigma - a)\) qua \( \bar{z}y \), where \( a \equiv a' \equiv \sigma \) and \( \frac{1}{2} \sigma \leq a' \), by theorem \( A \) of § 8, then \( zy^{a'-a} \cdot \psi_{a'-a} \) is a form of type \((a, \sigma - a)\) for which \( \bar{z}s \cdot zy^{a'-a} \cdot \psi_{a'-a} \equiv zy^{a'-a} \cdot \bar{z}s \cdot \psi_{a'-a} \equiv 0 \), by (17), and

\[
\bar{z}s \cdot zy^{a'-a} \cdot \psi_{a'-a} \equiv (a' - a) \cdot (2a' - \sigma)^{(a'-a)} \cdot \psi_{a'-a},
\]

by (18), that is, \( \bar{z}s \cdot zy^{a'-a} \cdot \psi_{a'-a} \) is a \( zs \)-differentiant that satisfies condition (31) \([ r = a' - a, \omega_r = zy^{a'-a}, \text{ and } S_r = (a' - a) \cdot (2a' - \sigma)^{(a'-a)} \] if \( 0 \equiv a \equiv a' \equiv \sigma \) and \( \sigma - a \equiv a' \). Therefore, by (35), if \( \frac{1}{2} \sigma \equiv a \equiv \sigma \),

\[
N_{xy, zs}(a, \sigma - a) = (1 - \bar{z}y) \cdot N_{zs}(a, \sigma - a);
\]

that is, on restoring \( w_x \) for \( a \) and \( w_y \) for \( \sigma - a \), writing the weights in \( z \) and \( s \),
and expressing the denumerator of \(zs\)-differentiants by (37), we have, if \(w_y \leq w_z\) and \(w_z \leq w_x\),

\[
N_{xy,zs}(w_x, w_y, w_z, w_z) = (1 - xy)(1 - zs) \cdot N(w_x, w_y, w_z, w_z).
\]

Formula (61) shows that, if \(\phi\) is the most general form of any type of which there can be \(xy\)- and \(zs\)-differentiants in accordance with theorem \(A\) of § 8, the linear equations between the multipliers of the terms of \(\phi\) that are implied in the identities \(\widetilde{xy} \cdot \phi = 0\) and \(\widetilde{zs} \cdot \phi = 0\) are linearly independent in this sense that, when \(\phi\) shall have been determined as the most general form of its type \((\cdots)\) that satisfies the identity \(\widetilde{zs} \cdot \phi = 0\), so that the number of independent arbitrary multipliers of its terms is \(N_{zs}(\cdots)\), the number of linearly independent linear equations between these multipliers that are implied in the identity \(\widetilde{xy} \cdot \phi = 0\) is \(\widetilde{xy} \cdot N_{zs}(\cdots)\).

It is evident that the method of this section can be employed to determine the denumerator of forms of a proper type, in accordance with theorem \(A\) of § 8, that have any multiple differentiant character that can be compounded of two or more simple or multiple characters of which no two have a common variable in their defining pairs when we know the general formulæ for the denenumers of forms having the several component characters. For example, if \(H\) and \(K\) represent two simple or multiple differentiant characters such that no variable in the pairs that define \(H\) occurs in the pairs that define \(K\), if we know that \(N_H(\cdots) = P \cdot N(\cdots)\) for every type \((\cdots)\) of which there can be \(H\)-differentiants and that \(N_K(\cdots) = Q \cdot N(\cdots)\) for every type \((\cdots)\) of which there can be \(K\)-differentiants, in accordance with theorem \(A\) of § 8, where \(P\) and \(Q\) are certain type-operators, then, if \((\cdots)\) is any type of which there can be \(H\)- and \(K\)-differentiants, we shall have \(N_{H,K}(\cdots) = PQ \cdot N(\cdots)\).

24. If \(w_x = w_y = w_z = w\), we have, by (50) and (51),

\[
N_{xyz}(w, w, w) = N(w, w, w) - 2N(w+1, w, w-1) + N(w+2, w-1, w-1) + N(w+1, w+1, w-2) - N(w+2, w, w-2).
\]

This is the culminating formula for denumerants of double differentiants. It gives, if \(x, y,\) and \(z\) are the only variables, the number of linearly independent invariants or covariants (according as the order is or is not 0) of any possible type \((w, w, w)\) that belong to any system of ternary quantics; namely, as we have stated in § 8, every homogeneous and isobaric \(xyz\)-differentiant of equal weights in \(x\) and \(z\) and (therefore) \(y\) is a complete differentiant in \(x, y\) and \(z\) and, therefore, an invariant or covariant if \(x, y,\) and \(z\) are the only variables; and, conversely, every invariant or covariant in \(x, y,\) and \(z\) is a homogeneous isobaric \(xyz\)-differentiant of equal weights in \(x, y,\) and \(z\). What Cayley's formula does for the invariants and covariants of a binary quantic and Sylves-
ter's extension of it for the invariants and covariants of any system of binary quantics, this formula (62) does for the invariants and covariants of any system of ternary quantics.

25. Just as (30) follows from (26) and (29) so from theorem D and the equation preceding (45), together with (52) and (53), follows, if \( w_z \leq w_y \leq w_x \) and \( 0 \leq r \leq w_z \),

\[
N_{xy}^{xz}(w_x, w_y, w_z) = \sum_{k=0}^{w_z} N_{xy}^{xz,k}(w_x + r, w_y, w_z - r)
\]

\[
= (1 - xz) \cdot \sum_{k=0}^{w_z} (1 - yz) \cdot yz^k \cdot N_{xy}(w_x + r, w_y, w_z - r)
\]

\[
= (1 - xz) \cdot [N_{xy}(w_x + r, w_y, w_z - r) - N_{xy}(w_x + r, w_z + 1, w_y + w_z - w_x - r - 1)]
\]

(63) \[
N_{xy,zz}(w_x + r, w_y, w_z - r) - N_{xy,zz}(w_x + r, w_z + 1, w_y + w_z - w_x - r - 1).
\]

But the condition \( \frac{1}{2}(\sigma - \alpha') \leq \beta' \) for the formula preceding (45) might have been replaced by \( \sigma - \alpha - \beta \leq \beta' \), as was previously stated, so that the first expression given above for \( N_{xy}^{xz}(w_x, w_y, w_z) \) holds also if \( w_y \leq w_z \leq w_x \) and \( 0 \leq r \leq w_z \); excepting that,

if \( w_y \leq w_z \leq w_x \) and \( 0 \leq r \leq w_z - w_y \), the lower limit of \( k \) is \( w_z - w_y - r \) and the first term of the developed expression is \( (1 - xz) \cdot N_{xy}(w_x + r, w_z - r, w_y) \), which, by (37), is the same as \( (1 - xz) \cdot N_{xy}(w_x + r, w_y, w_z - r) \), as before, and

if \( w_y \leq w_x, w_z \leq w_x, \) and \( w_y + w_z - w_x \leq r \leq w_x \), the upper limit of \( k \) is \( w_z - r \) and the second term of the developed expression vanishes, together with the second term of (63). Therefore, (63) holds if \( w_y \leq w_x, w_z \leq w_x, \) and \( 0 \leq r \leq w_z \), that is, for every type of which there can be \( xy- \) and \( xz- \) differentiants in accordance with theorem A and for every possible rank qua \( xz \) of such differentiants.

Formula (63) shows that the number of linearly independent \( xy- \) and \( xz- \) differentiants of any type \( (w_x + r, w_y, w_z - r) \) subordinate to \( (w_x, w_y, w_z) \) qua \( xz \) that must be annexed to those derivable qua \( xz \) from \( xy \)-differentiants of type \( (w_x, w_y, w_z) \) in order to produce a complete system is

\[
N_{xy,zz}(w_x + r, w_x + 1, w_y + w_z - w_x - r - 1) \text{ if } w_y \leq w_x \text{ and } w_z \leq w_x.
\]

If \( w_y + w_z - w_x \leq r \leq w_z \), every \( xy- \) and \( xz- \) differentinant of type \( (w_x + r, w_y, w_z - r) \) is derivable qua \( xz \) from the \( xy \)-differentiants of type \( (w_x, w_y, w_z) \), provided \( w_y \leq w_x \) and \( w_z \leq w_x \). In particular, we find (on putting \( r = 1 \) and replacing \( w_x \) by \( w_x - 1 \)) that the number of linearly independent \( xy- \) and \( xz- \) differentiants that must be annexed to those derivable qua \( xz \) from the \( xy \)-differentiants of type \( (w_x - 1, w_y, w_z + 1) \), where \( w_y \leq w_x - 1 \) and \( w_z \leq w_x - 1 \), in order to produce a complete system of \( xy- \) and \( xz- \) differentiants of type \( (w_x, w_y, w_z) \) is
\( N_{xy, zz}(w_x, w_y, w_z, w_y + w_z - w_x) \), which is evidently \( N_{yz}(w_x, w_x, w_y + w_z - w_x) \), because, by theorems B and C of § 8, every \( xy \) - and \( zz \)-differentiant of equal weights in \( x \) and \( y \) is also a \( yz \)- and \( yz \)-differentiant and, therefore, an \( xz \)-differentiant; but no such differentiants have to be annexed if \( w_y + w_z < w_x \).

It is evident that the result of applying \( xx \) to any \( xy \)-differentiant of type \((w_x - 1, w_y, w_z + 1)\), where \( w_y \leq w_x - 1 \), is an \( xy \)-differentiant of type \((w_x, w_y, w_z)\), if not identically 0, and the number of \( xyz \)-differentiants in a complete system of type \((w_x - 1, w_y, w_z + 1)\) reduced qua \( xx \) that are annihilated by \( xx \) is \( N_{xyz}(w_x - 1, w_y, w_z + 1) \), so that the number of linearly independent \( xy \)-differentiants of type \((w_x, w_y, w_z)\) that can be obtained by applying the operator \( xx \) to \( xy \)-differentiants of type \((w_x - 1, w_y, w_z + 1)\) is

\[
N_{xy}(w_x - 1, w_y, w_z + 1) - N_{xy, xx}(w_x - 1, w_y, w_z + 1)
\]

by (53), if \( w_y \leq w_x - 1 \) and \( w_z \leq w_x - 2 \); that is, the number of linearly independent \( xy \)-differentiants of type \((w_x, w_y, w_z)\) that must be annexed to those obtained by applying the operator \( xx \) to \( xy \)-differentiants of type \((w_x - 1, w_y, w_z + 1)\) in order to produce a complete system of \( xy \)-differentiants of type \((w_x, w_y, w_z)\) is \( N_{xy}(w_x, w_y, w_z) - N_{xy, xx}(w_x, w_x, w_y + w_z - w_x) \); but this number is 0 if \( w_y + w_z < w_x \), and then no such differentiants have to be annexed in order to complete the system. If \( w_x - 1 \leq w_x \), there are no \( xx \)-differentiants of type \((w_x - 1, w_y, w_z + 1)\), and the forms obtained by applying \( xx \) to a complete system of \( xy \)-differentiants of that type are linearly independent \( xy \)-differentiants of type \((w_x, w_y, w_z)\), which, however, do not generally constitute a complete system.

Similarly, from the equation preceding (55), together with (59) and (60), follows, if \( w_z \leq w_x, w_z \leq w_y, \) and \( 0 \leq r \leq w_z \),

\[
N_{xz}^{yz}(w_x, w_y, w_z) = \sum_{k=0}^{w_z - w_y} N_{xz, yz}^{yz}(w_x + r, w_y, w_z - r)
\]

\[
= (1 - xx) \cdot \sum_{k=0}^{w_z - w_y} (1 - yy) \cdot N_{xy}^{zz}(w_x + r, w_y, w_z - r)
\]

\[
= (1 - xx) \cdot [N_{yz}(w_x + r, w_y, w_z - r) - N_{yz}(w_x + w_y - w_z + r + 1, w_z - 1, w_x - r)]
\]

(65) \( = N_{xz, yz}(w_x + r, w_y, w_z - r) - N_{xz, yz}(w_x + w_y - w_z + r + 1, w_z - 1, w_z - r) \); so that the number of linearly independent \( xx \)- and \( yz \)-differentiants of type \((w_x + r, w_y, w_z - r)\) that must be annexed to those derivable qua \( xx \) from \( yz \)-differentiants of type \((w_x, w_y, w_z)\) in order to produce a complete system is \( N_{xz, yz}(w_x + w_y - w_z + r + 1, w_z - 1, w_z - r) \). In particular, we find (on putting \( r = 1 \) and replacing \( w_z \) by \( w_z + 1 \)) that the number of linearly inde-
pendent \(zx\)- and \(yz\)-differentiants of type \((w_x, w_y, w_z)\) that must be annexed to those derivable qua \(zx\) from \(yz\)-differentiants of type \((w_x - 1, w_y, w_z + 1)\), where \(w_z + 1 \leq w_x\) and \(w_z + 1 \leq w_y\), in order to produce a complete system of \(zx\)- and \(yz\)-differentiants of type \((w_x, w_y, w_z)\) is \(N_{zx,yz}(w_z + w_y - w_z, w_z, w_z)\), which is also \(N_{xyz}(w_x + w_y - w_z, w_z, w_z)\).

From (60) follows, if \(w_z + 2 = w_x\) and \(w_z + 1 = w_y\),

\[
N_{yz}(w_x - 1, w_y, w_z + 1) - N_{zx,yz}(w_x - 1, w_y, w_z + 1) = N_{yz}(w_x + w_y - w_z, w_z, w_z),
\]

which shows that the number of linearly independent \(yz\)-differentiants of type \((w_x, w_y, w_z)\) that must be annexed to those obtained by applying the operator \(zx\) to \(yz\)-differentiants of type \((w_x - 1, w_y, w_z + 1)\) in order to produce a complete system of \(yz\)-differentiants of type \((w_x, w_y, w_z)\) is \(N_{yz}(w_x + w_y - w_z, w_z, w_z)\).

If \(w_x = w_z + 1\), there are no \(zx\)-differentiants of type \((w_x - 1, w_y, w_z + 1)\) and the forms obtained by applying \(zx\) to a complete system of \(yz\)-differentiants of that type are linearly independent \(yz\)-differentiants of type \((w_x, w_y, w_z)\), which, however, do not generally constitute a complete system.

26. Because we have always excluded denumerants of such types that their values as given by our formulae might come out negative, these formulae may serve to determine the relative values of some denumerants involved in the expressions of others, just as (37) determined the relative numbers of terms in the general forms of certain types.

It follows from (45) that, if \(w_z \leq w_y \leq w_x\),

\[
N_{xy}(w_x, w_y, w_z + 1, w_z - 1) \leq N_{xy}(w_x, w_y, w_z);
\]

that is, of two types of which there can be \(xyz\)-differentiants in accordance with theorem \(A\) of §8 and of which the one is subordinate to the other qua \(yz\), that one has the more linearly independent \(xy\)-differentiants for which the weights in \(y\) and \(z\) are the more nearly equal, unless they have the same number.

It follows from (55) that, if \(w_z \leq w_y \leq w_x\),

\[
N_{yz}(w_x + 1, w_y - 1, w_z) \leq N_{yz}(w_x, w_y, w_z);
\]

or, because the sequence of variables is immaterial, if \(w_y \leq w_x \leq w_z\),

\[
N_{xy}(w_x - 1, w_y, w_z + 1) \leq N_{xy}(w_x, w_y, w_z);
\]

that is, of two types of which there can be \(zxy\)-differentiants in accordance with theorem \(A\) and of which the one is subordinate to the other qua \(zx\) (or \(zx\)), that one has the more linearly independent \(xy\)-differentiants for which the weights in \(x\) and \(z\) are the more nearly equal, unless they have the same number.

It follows from (61) and (37) that, if \(w_y \equiv w_x\) and \(w_z \equiv w_x\),

\[
N_{xy,zz}(w_x, w_y, w_z, w_z) = (1 - zs) \cdot N_{xy}(w_x, w_y, w_z, w_z);
\]
and, therefore,
\[ N_{xy}(w_x, w_y, w_z + 1, w_z - 1) \leq N_{xy}(w_x, w_y, w_z, w_z). \]

Similarly, if \( w_y \leq w_x \) and \( w_z \leq w_y \),
\[ N_{xy}(w_x, w_y, w_z - 1, w_z + 1) \leq N_{xy}(w_x, w_y, w_z, w_z); \]

that is, of two types of which there can be \( xy \)-differentiants in accordance with theorem \( A \) and of which the one is subordinate to the other relatively to some pair of \emph{other} variables than \( x \) and \( y \), that one has the more linearly independent \( xy \)-differentiants for which the weights in the two other variables are the more nearly equal, unless they have the same number.

It follows from (39) that, if \( wz = wy = wx \),
\[ N_{xz, yz}(wx + 1, wy - 1, wz) \leq N_{xz, yz}(wx, wy, wz); \]
and, if \( wz = wx = wy \),
\[ N_{xz, yz}(wz - 1, wy + 1, wz) \leq N_{xz, yz}(wz, wy, wz); \]

that is, of two types of which there can be \( xz \)- and \( yz \)-differentiants and of which the one is subordinate to the other qua \( xy \), that one has the more linearly independent \( xz \)- and \( yz \)-differentiants for which the weights in \( x \) and \( y \) are the more nearly equal, unless they have the same number.

It follows from (42) that if \( wz = wy = wx \),
\[ N_{xy, zz}(w_x, wy + 1, wz - 1) \leq N_{xy, zz}(w_x, wy, wz) \]
and, if \( wy = wz = wx \),
\[ N_{xy, zz}(w_x, w_y - 1, wz + 1) \leq N_{xy, zz}(w_x, w_y, wz); \]

that is, of two types of which there can be \( xy \)- and \( xz \)-differentiants and of which the one is subordinate to the other qua \( yz \), that one has the more linearly independent \( xy \)- and \( xz \)-differentiants for which the weights in \( y \) and \( z \) are the more nearly equal, unless they have the same number.

For the actual numerical calculation of the denumerants considered in this paper nothing is wanting but a formula for the \textit{number of terms} in the \textit{general form} of any given type, that is, a formula for the denumerant \( N(w_x, w_y, w_z, \cdots) \) for \textit{any type} \( (w_x, w_y, w_z, \cdots) \).

In conclusion, I wish to call attention to the fact that the real basis of this whole investigation is \textsc{Sylvester}'s method, described in his \textit{Proof of the hitherto undemonstrated Fundamental Theorem of Invariants} in the \textit{Philosophical Magazine} for March, 1878, by which we pass from (29) through (26) to (30).

To facilitate reference I append here a brief table of contents.
Contents.

Introduction:

- Historical notice .................................. § 1
- Types and characters of differentiants .......... 2
- Analytical conditions for differentiants, — the shear-operator \( \check{yx} \) ....... 3
- Complete and reduced systems, — ranks, derivatives .... 4
- Notation for denumerants, — the type-operator \( xy \) ........ 5
- Numerical symbols and the sum \( S_{p,q}^{m,n} \) .......... 6
- Commutation of shear-operators .................. 7
- Certain properties of differentiants, theorems \( A-F' \) ...... 8
- Operators that produce differentiants, \( \{ y ; x \} \) etc. ........ 9

Determination of denumerants in general .............. 10–15

Simple differentiants and their denumerants .......... 16

Double differentiants and their denumerants:

- Derivation of \( xyz \)-differentiants from \( xz \) - and \( yz \)-differentiants .... 17
- Derivation of \( xyz \)-differentiants from \( xy \) - and \( xz \)-differentiants .... 18
- Derivation of \( xyz \)-differentiants from \( xy \)-differentiants, — final form of denumerants for \( xyz \)-differentiants ........ 19
- Denumerants for \( xy \) - and \( xz \)-differentiants ........ 20
- Derivation of \( xyz \)-differentiants from \( yz \)-differentiants ......... 21
- Denumerants for \( xz \) - and \( yz \)-differentiants ........ 22
- Denumerants for \( xy \) - and \( zs \)-differentiants ........ 23
- Denumerants for ternary co- and invariants ........ 24
- Completion of derived systems .................... 25
- Relative numbers of terms in differentiants of different types . . . . . . . 26

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