NON-DESARGUESIAN AND NON-PASCALIAN GEOMETRIES*†

BY

O. VEBLEN AND J. H. MACLAGAN-WEDDERBURN

§ 1. The object of this paper is to discuss certain non-pascalian and non-desarguesian geometries. Beside their intrinsic interest, these geometries are useful in forming independence proofs for systems of axioms for geometry.

In a previous paper ‡ it has been shown how to construct all finite projective geometries of three or more dimensions and all two dimensional projective geometries in which the Desargues theorem (cf. third footnote) is valid. The present paper supplements the other by showing the existence of non-desarguesian finite geometries. It is to be noted that, since every finite linear associative algebra in which every number possesses an inverse is commutative, § therefore every non-pascalian finite geometry is necessarily non-desarguesian. Moreover, as has been shown by HESSENBERG,|| every non-desarguesian geometry is also non-pascalian, whether finite or infinite.

A *projective plane geometry* is a set of elements called points, finite or infinite in number, subject to the following conditions:

1. If A and B are any two points, there is (α) one and (β) only one set of points called a line and containing both A and B.

2. If a and b are any two lines, there is (α) one and (β) only one ¶ point contained in both a and b.

3. Each line contains at least three points.

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* Presented to the Society (Chicago), April 22, 1905. Received for publication, December 6, 1906.

† As originally read, the paper called attention to the fact that HILBERT's example of a non-pascalian geometry in his *Festschrift* did not satisfy his order axioms. The error however is not repeated in later editions of his book. This paper also contained a discussion of the non-pascalian geometry obtained by letting the coördinates of a point be quaternions. As such geometries are discussed in the *Abstrakte Geometrie* of K. T. VAHLEN, Leipzig, 1905, this part of the paper is omitted.

‡ VEBLEN and BUSSEY, *Finite projective geometries*, Transactions, vol. 7 (1906), p. 271. The reader is referred to this paper for references to the literature and discussion of the general problem of finite geometries. Condition VII of that paper is the "Pascal Theorem" and the paragraph following VII contains the "Desargues Theorem."

§ J. H. MACLAGAN-WEDDERBURN, *A theorem on finite algebras*, Transactions, vol. 6 (1905), p. 349.

|| G. HESSENBERG, *Beweis des Desarguesschen Satzes aus dem Pascalschen*, Mathematische Annalen, vol. 61 (1905), p. 161.

¶ The condition $2(\beta)$ is of course logically a consequence of $2(\alpha)$ and $1(\beta)$.

The geometry is non-desarguesian or non-pascalian if the theorems of DESARGUES or of PASCAL, respectively, fail to be valid.

If the number of points is finite it is at once evident that every line contains the same number of points and that if $n+1$ is the number of points on a line, the total number of points is n^2+n+1.

Conversely, if the total number of points is n^2+n+1 and the number of points on each line is $n+1$ and every two points lie on one and only one line, it follows that every two lines have one and only one point in common. For, suppose the two lines a, b had no point in common. A point of a would determine with b $n+1$ lines giving thus $n(n+1)+1$ points and, counting in the points of a, we should have at least n^2+2n points instead of n^2+n+1.

By a dual argument, if the number of lines is n^2+n+1 and the number of lines through each point is $n+1$ and every two lines have one and only one point in common, it follows that every two points determine one and only one line.

§ 2. Analytic geometries may be constructed so as to satisfy the given conditions if the coördinates are numbers of a number-system subject only to the following conditions:

(1) The numbers form a commutative group under the operation of addition: that is, for any two numbers a and b there is a unique number c such that $a+b=c$ and a unique number d such that $a+d=b=d+a$, and there is a number 0 such that, for every a, $a+0=a=0+a$; further, addition is associative and commutative,* i. e. $(a+b)+c=a+(b+c)$ and $b+a=a+b$.

(2) For any two numbers a and b there is a unique number c such that $ab=c$ and if $a \neq 0$ unique numbers d, d' such that $da=b$ and $ad'=b$; also $0a=0=a0$ for every a.

(3) Finally there must be some sort of a distributive law. We shall use the following: $c(a+b)=ca+cb$. We shall not use for the present † the other distributive law $(a+b)c=ac+bc$, nor the associative and commutative laws of multiplication, nor shall we assume that there exists an identity element with respect to multiplication.

A *point* is now defined as one of the systems of three coördinates

$$(\alpha) \qquad x,\ y,\ \phi,$$

$$(\beta) \qquad x,\ \phi,\ 0,$$

$$(\gamma) \qquad \phi,\ 0,\ 0,$$

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* The commutativity of addition is not an independent postulate. Cf. H. HANKEL, *Theorie der complexen Zahlensysteme* (Leipzig, 1867), page 32; E. V. HUNTINGTON, Annals of Mathematics, vol. 8 (1906), page 25.

† Cf. however the last sentence of this §.

where ϕ is not equal to 0 and is the same for all points. There are $n^2 + n + 1$ points in all if there is a finite number, n, of elements in the number-system.

A *line* is defined as all points which satisfy an equation of one of the forms

$$x\psi + yb + zc = 0, \tag{1}$$

$$y\psi + zc = 0, \tag{2}$$

$$z\psi = 0, \tag{3}$$

where ψ is not equal to 0 and is the same for all lines. There are $n^2 + n + 1$ lines altogether in the finite case. Further there are, in that case, $n + 1$ points on a line. For (3) is satisfied by points of types (β) and (γ): in case $c = 0$ (2) is satisfied by the n points of type (α) in which $y = 0$ and by (γ): in case $c \neq 0$ in (2) if $z = 0$, then $y = 0$ and hence (γ) is one of the points, while if $z = \phi$, y must be equal to the solution of $y\psi = -\phi c$ and x may have any of n values. The argument is similar for an equation of type (1). A similar argument will show that in the finite case the number of lines through a point is $n + 1$.

To show that our points and lines form a projective plane geometry it is necessary to prove that any two lines have in common one and only one point. This requires the consideration of several cases.

(a) Two lines of type (1):

$$x\psi + yb + zc = 0,$$

$$x\psi + yb' + zc' = 0.$$

The coördinates of a point of intersection must satisfy the condition

$$y(b - b') + z(c - c') = 0.$$

In case $c = c'$, b cannot be the same as b' (else the lines would be identical) and hence $y = 0$; the lines then have in common only the point $(\alpha, 0, \phi)$ where α is the unique solution of $x\psi = -\phi c$. In case $c \neq c'$ and $b = b'$, we have $z = 0$, $y = \phi$ and x is the solution of $x\psi = -\phi b$. In case $c \neq c'$, $b \neq b'$ we have $z = \phi$ and hence y is the solution, say α, of $y(b - b') = -\phi(c - c')$ and x is the solution of $x\psi = -(\alpha b + \phi c)$. On account of the commutative law of addition the values of x, y, and z thus found must satisfy the given equations.

(b) Lines of types (1) and (2):

$$x\psi + yb + zc' = 0,$$

$$y\psi + zc' = 0.$$

The coördinate z cannot be zero, else y and x would also be zero. Hence $z = \phi$, y is the solution, α, of $y\psi = -\phi c'$, and x the solution of $x\psi = -(\alpha b + \phi c')$.

(c) Lines of types (1) and (3):

$$x\psi + yb + zc = 0,$$

$$z\psi = 0.$$

The coördinate y cannot be zero since then x and z would both be zero. Hence $y = \phi$ and x is the solution of $x\psi = -\phi b$.

(d) Two lines of type (2):

$$y\psi + zc = 0,$$

$$y\psi + zc' = 0.$$

Any solution must satisfy the condition $z(c - c') = 0$ or $z = 0$. Hence $y = 0$ and $x = \phi$.

(e) Lines of types (2) and (3):

$$y\psi + zc = 0,$$

$$z\psi = 0.$$

The only solution is $(\phi, 0, 0)$.

This, according to the last sentence of § 1, completes the proof that all finite number systems of the type described above lead to finite plane geometries. If the number system is not finite it is necessary to prove algebraically that every two points determine a unique line. If the other distributive law holds also or if multiplication is commutative this can be done by the analysis used above.

§ 3. In order to construct a particular non-desarguesian geometry it is necessary only to adduce a particular number-system. The first class of systems which we shall use is due to L. E. Dickson.* It is such that the number of elements in each system is finite and such that multiplication is associative but not commutative and the distributive law holds only in the form $x(y + z) = xy + xz$. Each system possesses an identity which in the analysis above may be taken as the common value of ϕ and ψ. These systems are discussed on pages 5 to 22 of Dickson's paper. A characteristic system of this type is the following. There are p^{2m} elements, (a, b), where a and b are marks of a field $GF[p^m]$, p being an odd prime. These elements obey the following laws of combination:

$$(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$$

$$(a_1, b_1) \times (a_2, b_2) = (a_1 a_2 + \epsilon\nu b_1 b_2, b_1 a_2 + \epsilon a_1 b_2)$$

where ν is a fixed not-square in the $GF[p^m]$, and $\epsilon = \pm 1$ according as $a_1^2 - \nu b_1^2$ is a square or not-square, i. e., $\epsilon = (a_1^2 - \nu b_1^2)^{(p^m-1)/2}$.

* L. E. Dickson, *On finite algebras*, Göttinger Nachrichten, 1905, pp. 358–394.

The simplest special case is where $p^m = 3$. The algebra may then be thought of as having two units 1 and j and composed of all numbers of the form $x + yj$, x and y being integers modulo 3 and $j^2 = -1$. The system* is thus composed of nine numbers $0, 1, 2, j, 2j, 1+j, 1+2j, 2+j, 2+2j$.

It is not difficult (as the reader may verify analytically) to find cases which show that the geometry founded on this number-system is non-desarguesian and non-pascalian. Instead of doing this, however, we shall now show how to get another geometry based on the same number-system. The new geometry is so constructed as to be capable of being explicitly exhibited in a compact form.

Since the number-system is associative and distributive [in the sense, $a(b+c) = ab + ac$] the point (x, y, z) may also be represented by $(\kappa x, \kappa y, \kappa z)$, $\kappa \neq 0$. With this understanding a linear transformation changes points into points and, if their coefficients are scalar, two transformations may be combined in the usual way. They do not in general change lines into lines.†

Consider now the following transformation of period 13:

$$A: \qquad x = x' + z', \qquad y = 2x', \qquad z = 2y' + 2z'.$$

Let $A_0, B_0, \cdots, G_0$ denote the points $(2, 0, 1)$, $(2+2j, j, 1)$, $(2+j, 2j, 1)$, $(1+2j, 1+j, 1)$, $(j, 2+2j, 1)$, $(1+j, 1+2j, 1)$, $(2j, 2+j, 1)$ and let $A_k, B_k, \cdots, G_k$ $(k < 13)$ denote the points derived from these by transforming them by the kth power of A. In this notation, seven of the lines of our original geometry are:

$$x + y + z = 0: \; A_0 A_1 A_3 A_9 B_0 C_0 D_0 E_0 F_0 G_0,$$
$$x + yj + z = 0: \; A_0 B_1 B_8 D_3 D_{11} E_2 E_5 E_6 G_7 G_9,$$
$$x + y(2j) + z = 0: \; A_0 C_1 C_8 E_7 E_9 F_3 F_{11} G_2 G_5 G_6,$$

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* In this special case the distributive law has a very simple form; namely

$$(a + bj)j = -aj - bj^2 = b - aj$$

where neither a nor b is zero. Though the commutative law is not valid it is always true either that $xy = yx$ or that $xy = -yx$.

For the convenience of the reader in verifying computations, we subjoin the multiplication table:

	2	j	$2j$	$1+j$	$1+2j$	$2+j$	$2+2j$
2	1	$2j$	j	$2+2j$	$2+j$	$1+2j$	$1+j$
j	$2j$	2	1	$2+j$	$1+j$	$2+2j$	$1+2j$
$2j$	j	1	2	$1+2j$	$2+2j$	$1+j$	$2+j$
$1+j$	$2+2j$	$1+2j$	$2+j$	2	$2j$	j	1
$1+2j$	$2+j$	$2+2j$	$1+j$	j	2	1	$2j$
$2+j$	$1+2j$	$1+j$	$2+2j$	$2j$	1	2	j
$2+2j$	$1+j$	$2+j$	$1+2j$	1	j	$2j$	2

† For example, the transformation A changes $(1, 0, 0)$, $(0, j, 1)$ $(j, j, 1)$ which lie on $yj + z = 0$ into $(1, 2, 0)$ $(1, 0, 2+2j)$, $(1+j, 2j, 2+2j)$ of which the first two, but not the third, lie on $x(1+j) + y(1+j) + z = 0$.

$$
\begin{aligned}
x + y(1+j) + z = 0 &: \ A_0 B_7 B_9 D_1 D_8 F_2 F_5 F_6 G_3 G_{11}, \\
x + y(2+2j) + z = 0 &: \ A_0 B_2 B_5 B_6 C_3 C_{11} E_1 E_8 F_7 F_9, \\
x + y(1+2j) + z = 0 &: \ A_0 C_7 C_9 D_2 D_5 D_6 E_3 E_{11} F_1 F_8, \\
x + y(2+j) + z = 0 &: \ A_0 B_3 B_{11} C_2 C_5 C_6 D_7 D_9 G_1 G_8 .
\end{aligned}
$$

A and its powers give 84 other heptads of points which may be obtained by adding successively the integers $1, 2, \cdots, 12$ to the subscripts of the points of each of the above lines, the resultant subscripts being, of course, reduced modulo 13. *The ninety-one heptads so obtained are the lines of a new non-desarguesian and non-pascalian geometry.*

It is not difficult to verify *a posteriori* that this scheme does define a geometry. By trial we see that if two lines intersect in more than one point, then neither of these points can be of the form A_k or B_k. Further, the scheme is invariant under the following transitive substitution group:

$$
1,\ (BDG)(CEF),\ (BGD)(CFE),
$$
$$
(BC)(DF)(EG),\ (BE)(CD)(FG),\ (BF)(CG)(DE).
$$

Hence the same is true of the remaining letters. The other axioms may be tested in the same way. That the geometry is non-desarguesian and non-pascalian is proved by examples which are sufficiently indicated in the diagrams below.

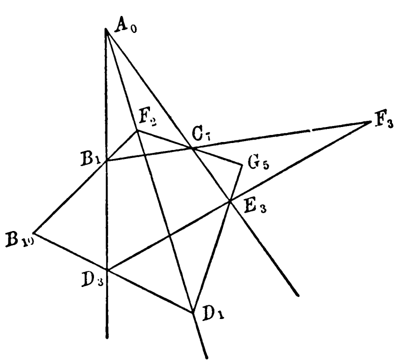

FIG. 1.

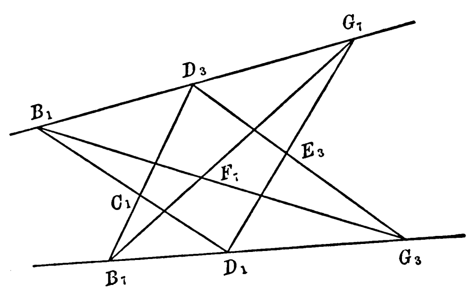

FIG. 2.

More interesting, however, than this tactical verification is the analytical proof that the 91 heptads derived by A and its powers from the lines B are the lines of a plane geometry. As in § 2, it is only necessary to show that any two heptads have one and only one point in common. It is a direct consequence of definition that a heptad is the set of points satisfying the equation,

$$
x(a_{11} + a_{31}) + y(a_{12} + a_{32}) + z(a_{13} + a_{33}) + (xa_{21} + ya_{22} + za_{23})\beta = 0,
$$

where β is any one of the seven numbers of the system and a_{rs} are the coefficients of any one of the thirteen powers of A. This equation may be abbreviated as follows:

(1) $$xa + yb + zc + (xa' + yb' + zc')\beta = 0.$$

Consider the intersection of this with

(2) $$x + y\alpha + z = 0.$$

In (1) and (2), α and β are the only coefficients which are not necessarily scalar. From (1) and (2) we deduce

(3) $$y(2a\alpha + b) + z(2a + c) + \{y(2a'\alpha + b') + z(2a' + c')\}\beta = 0,$$

and shall now show that there is only one point whose coördinates satisfy (2) and (3) simultaneously. There are three cases.

(i) $2a' + c' = 0$. In this case (3) becomes:

(4) $$y\{2a\alpha + b + (2a'\alpha + b')\beta\} + z(2a + c) = 0.$$

If the coefficients in (4) do not vanish identically, (2) and (4) have evidently one and only one point in common. On forming the powers* of the transformation A we find that $2a + c$ and $2a' + c'$ cannot vanish simultaneously except for the identical transformation, and in this case the vanishing of the coefficient of y leads to $\alpha = \beta$. Hence except in this trivial case (4) is not an identity.

(ii) $2a + c = 0$. In this case

$$y\{(2a\alpha + b)\beta^{-1} + 2a'\alpha + b'\} + z(2a' + c') = 0.$$

Hence as in (i) there is one and only one point common to (2) and (3).

(iii) $2a' + c' = (2a + c)d \neq 0$, where d is scalar. Then (3) may be written in the form

$$y\{2(a + 2a'd^{-1})\alpha + b + 2b'd^{-1}\} + \{y(2a'\alpha + b')d^{-1} + z(2a + c)\}(1 + d\beta) = 0,$$

or say

$$ye + (yf + zg)\gamma = 0.$$

This is equivalent to

$$y(e\gamma^{-1} + f) + zg = 0.$$

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* The first twelve powers have the matrices

101	120	201	221	021	122	212	110	001	022	111	020
200	202	210	102	112	012	211	121	220	002	011	222
022	111	020	100	101	120	201	221	021	122	212	110

and the thirteenth power is the identical transformation.

Hence, since $g \neq 0$, (1) and (3) and therefore also (1) and (2) have one and only one point in common.

The point transformation A transforms lines of the new geometry into lines. Hence, if any two lines intersect in more than one point, the lines obtained by transforming these simultaneously by any power of A will also intersect in more than one point. But any pair of lines can, by transformation by a suitable power of A, be transformed into such a pair as has been considered above. Hence any two lines intersect in one and only one point.

§ 4. On pages 381–394, loc. cit.,* L. E. Dickson has given another set of algebras which are available for geometrical purposes. The following multiplication table (cf. (68), loc. cit., p. 394) defines an algebra which satisfies the conditions of § 2 together with the commutative law of multiplication and has an identity element. It therefore leads to a geometry.

	i	j
i	j	$b+\beta i$
j	$b+\beta i$	$-\beta^2-8bi-2\beta j$

Any number of the algebra is of the form $a + di + ej$, where a, d and e are any marks of any field F, not having the modulus 2, in which $x^3 = b + \beta x$ is irreducible. In such an algebra an equation of the form $ax = b$ has a unique solution if $a \neq 0$.

If $b = 2$, $\beta = 0$ and F is the field of rational numbers this table becomes

	i	j
i	j	2
j	2	$-16i$.

Points and lines are defined as above, but as multiplication is commutative the equation of a line is more conveniently written in the form $ax + by + cz = 0$.

That the geometry is non-desarguesian is shown by the following example. Consider the six points $A = (0, 0, 1)$, $A' = (0, i, 1)$, $B = (i, i, 1)$, $B' = (i, j, 1)$, $C = (j, 0, 1)$, $C' = (j, i, 1)$. These points lie in pairs on the lines $x = 0$, $x - iz = 0$, and $x - jz = 0$, which meet in the point $(0, 1, 0)$. The equations of the sides of the triangles ABC and $A'B'C'$ are as follows:

$$AC:\ y = 0, \qquad A'C':\ y - iz = 0,$$

$$AB:\ x - y = 0, \qquad A'B':\ x + (1 + i + j)y/17 - (2 + i + j)z/17 = 0,$$

$$BC:\ x - (1 - i)y - jz = 0, \qquad B'C':\ x + y - (i + j)z = 0.$$

*See also L. E. Dickson, *Linear algebras in which division is always uniquely possible*, these Transactions, vol. 7 (1906), p. 371.

If D, E and F are the intersections of AC with $A'C'$, AB with $A'B'$, and BC with $B'C'$, the coördinates of these points are: $D=(1,0,0)$, $E=(x=y=8(17/8+2i+j)/177, 1)$ and $F=[(-1+i+2j)/3, (1+2i+j)/3, 1]$. The equation of the line DE is $y-8(17/8+2i+j)z/177=0$ and evidently does not contain F. Hence D, E and F are not collinear and the geometry is non-desarguesian.

In the same way by taking the six points $A'=(1,0,1)$, $B'=(i,0,1)$, $C'=(i+j,0,1)$, $A=(1,1,0)$, $B=(i,1,0)$, $C=(j,1,0)$, it can be shown that the geometry is non-pascalian.

The *finite* number-system of the present type with the smallest number of elements is obtained by letting F be the $GF[3]$ and letting $b=1$ and $\beta=1$. This however leads to a geometry with 28 points on a line and $(27)^2+27+1=757$ points in all and has not been exhibited explicitly.

§ 5. The number systems discussed above all have identity elements. The presence of an identity is not necessary for the analysis of § 2, and examples of number systems that lead to geometries, though lacking the identity, are not hard to find. Two simple cases are constructed as follows:

Let e_1 and e_2 be two independent units. The systems under consideration consist of all numbers of the form ae_1+be_2 where a and b are integers reduced modulo 2. Addition is defined by the equation

$$(ae_1+be_2)+(ce_1+de_2)=(a+c)e_1+(b+d)e_2.$$

The two multiplication tables are

$$(i)\quad \begin{array}{c|ccc} & e_1 & e_2 & e_3 \\ \hline e_1 & e_3 & e_2 & e_1 \\ e_2 & e_2 & e_1 & e_3 \\ e_3 & e_1 & e_3 & e_2 \end{array} \qquad\qquad (ii)\quad \begin{array}{c|ccc} & e_1 & e_2 & e_3 \\ \hline e_1 & e_1 & e_2 & e_3 \\ e_2 & e_3 & e_1 & e_2 \\ e_3 & e_2 & e_3 & e_1 \end{array}$$

where $e_3=e_1+e_2$. In (i) multiplication is commutative and in both systems the two-sided distributive law holds. Each system gives rise to a geometry with five points on a line. Both geometries however turn out to be identical with $PG(2, 2^2)$ which is based on the Galois field, $GF(2^2)$. This suggests the following theorem: *There is only one projective plane geometry with five points on a line.*

To prove this consider the following Pappus configuration in which O, A, B, C, A_1, B_1, C_1 are distinct. Consider the lines DE, EF, FD. These lines obviously cannot intersect the line OA in any of the points A, B, C; hence as there are only two other points on OA, D, E, and F must be collinear. The geometry is therefore pascalian. DESARGUES's theorem may be proved in the

same way or it may be deduced from PASCAL's theorem. Now there is only one pascalian geometry for a given finite number of points on a line,* therefore there is only one geometry with five points on a line.

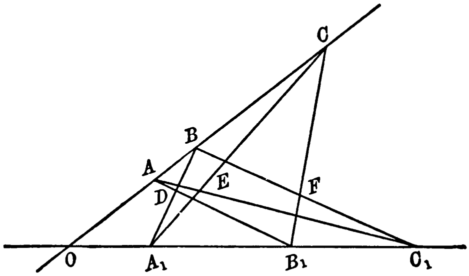

FIG. 3.

Thus a non-desarguesian geometry must have at least six points to the line. A number system which furnishes a geometry with this number of points is the following. Denote the elements by $0, 1, 2, 3, 4$, and let addition be identical with ordinary addition modulo 5. Let multiplication be defined by the table:

	1	2	3	4
1	1	2	3	4
2	4	3	2	1
3	3	1	4	2
4	2	4	1	3

The element, 1, is a left-hand identity. All the conditions of § 2 are satisfied and we may conveniently take $\phi = \psi = 1$. The resulting geometry however turns out to be $PG(2, 5)$. The problem of determining a non-desarguesian geometry of minimum number of points to the line remains unsolved.

* Cf. these transactions, vol. 7 (1906), p. 247 and § 1 above.

[Note added April 22, 1907. Dr. C. R. MACINNES has found by a tactical investigation that the only plane projective geometry of six points to a line is the $PG(2, 5)$ and that there is none at all with seven points to a line. The latter result has also been found by SAFFORD, American Mathematical Monthly, vol. 14 (1907), p. 84.]