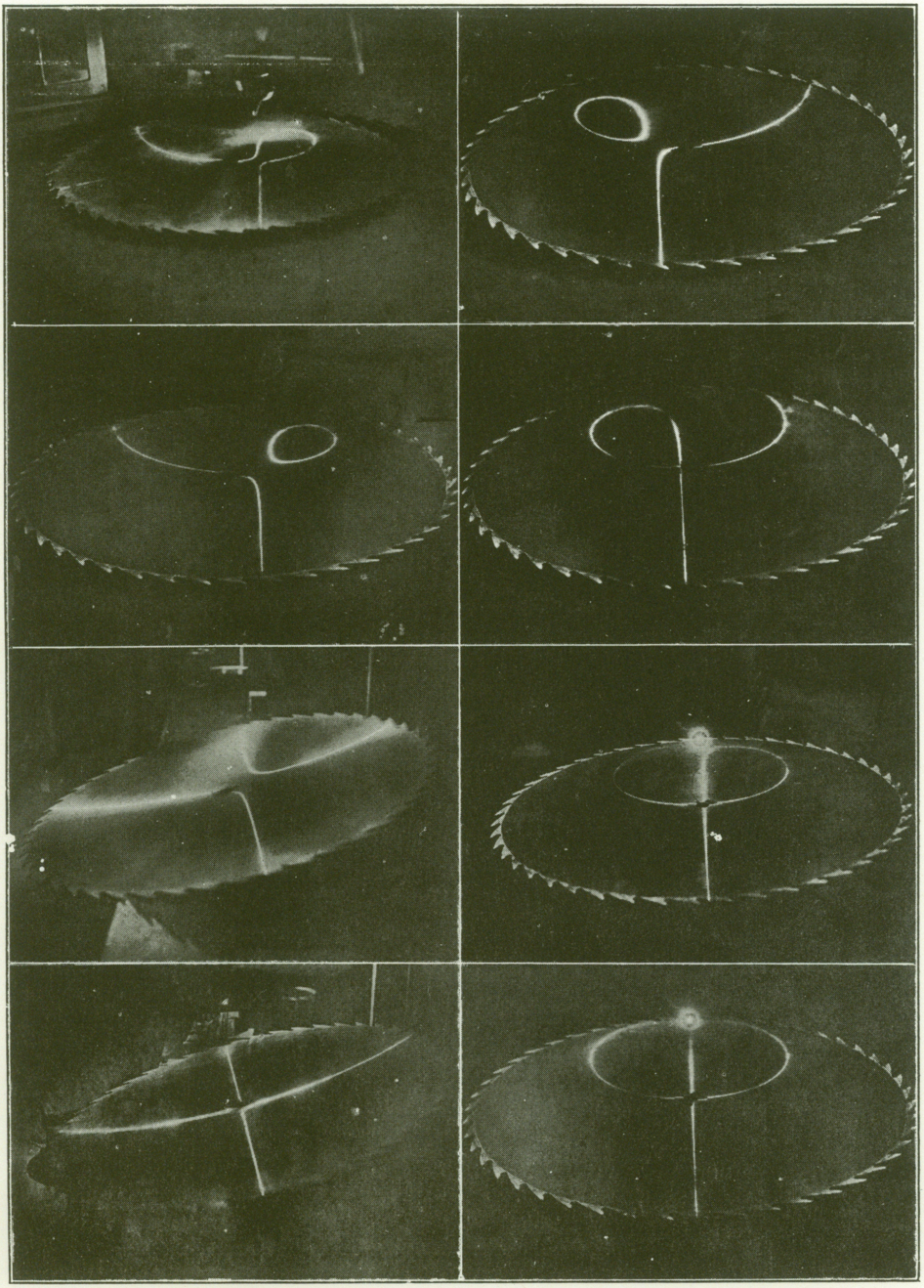

FIG. 1.

[W. H. Roever: Brilliant Points.]

BRILLIANT POINTS OF CURVES AND SURFACES*

BY

WILLIAM H. ROEVER

INTRODUCTION.

The geometrical investigations of this paper may best be introduced by a concrete optical illustration. We select the following one.

When a circular saw, which has been polished with emery while rotating in a lathe, is illuminated by a strong light—as for instance an electric arc—an observer will see in the saw a well defined curve of light. Photographs of this curve are shown in Fig. 1. The surface of the saw, with the circular scratches it has received by the particles of emery during the process of polishing, may be regarded as a corrugated surface of revolution. The curve of light is the assemblage of images of the electric arc in the corrugations of this reflecting surface. Neighboring images are very near together and therefore the assemblage of images resembles a continuous curve of light. In optics the points of light which have just been called images are called *brilliant points.* These points may be defined geometrically as follows:

Definition a. A point P is said to be a brilliant point of a surface t with respect to two points P_1 and P_2,† when the segments of right lines PP_1 and PP_2 lie in one plane through the normal to t at P and the angle P_1PP_2 is bisected *internally* by this normal. See Fig. 2.

In order to determine the geometric nature of the curve of light, let us make the following assumptions:

1) The corrugated surface which approximates the surface of the saw, may itself be approximated by tori ‡ of small radius.

2) A brilliant point of a tubular surface ‡ (in particular of a torus) of small radius may be replaced by the point which it approaches as the radius approaches zero.

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* Presented to the Society February 29, 1908. Received for publication February 13, 1908.

† P_1 and P_2 correspond to the point source of light and the eye of the observer respectively.

‡ A tubular surface is the envelope of a sphere of constant radius whose center describes a curve. The radius of the sphere is called the *radius* of the tubular surface, and the space curve its *axis.* A torus is a tubular surface with a circular axis.

In order to state the relation which defines the limit point of the second assumption, let us make

Definition b. A point P is said to be a brilliant point of the curve c with respect to the two points P_1 and P_2, when the segments of right lines PP_1 and PP_2 make *supplementary* angles with one of the directions of the tangent to c at P. See Fig. 2.

It is not difficult to convince one's self that *a brilliant point of a tubular surface of small radius approaches a brilliant point of the axis as the radius approaches zero.* The problem under consideration is thus reduced to

PROBLEM A. *Find the locus of the brilliant points of a family of concentric circles with respect to the points P_1 and P_2, which do not necessarily lie in the common plane of the circles.*

In the course of a solution of this problem it is desirable to consider also a point which is defined by replacing the word "supplementary" by the word

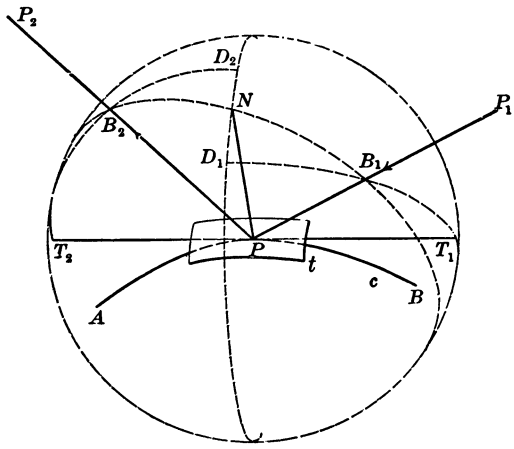


FIG. 2.

"equal" in Definition b. Such a point we will call a *virtual* brilliant point of the *curve* c with respect to P_1 and P_2.* Similarly, if we replace the word "internally" by the word "externally" in Definition a, we define what we will call a *virtual* brilliant point of the *surface* t with respect to P_1 and P_2. Let us call these two definitions the complements of Definitions b and a respectively.

In Fig. 3, which shows on the opposite sides of AB the two planes of descriptive geometry, the heavy full line represents the locus of brilliant points of

* When PP_1 and PP_2 are each perpendicular to the tangent to c at P, this definition and Definition b overlap. To be more precise, we apply the classification of § 1 to this case by putting $X_i = x - a_i$, $Y_i = y - b_i$, $Z_i = z - c_i$ $(i = 1, 2)$, where a_i, b_i, c_i are the coördinates of P_i $(i = 1, 2)$.

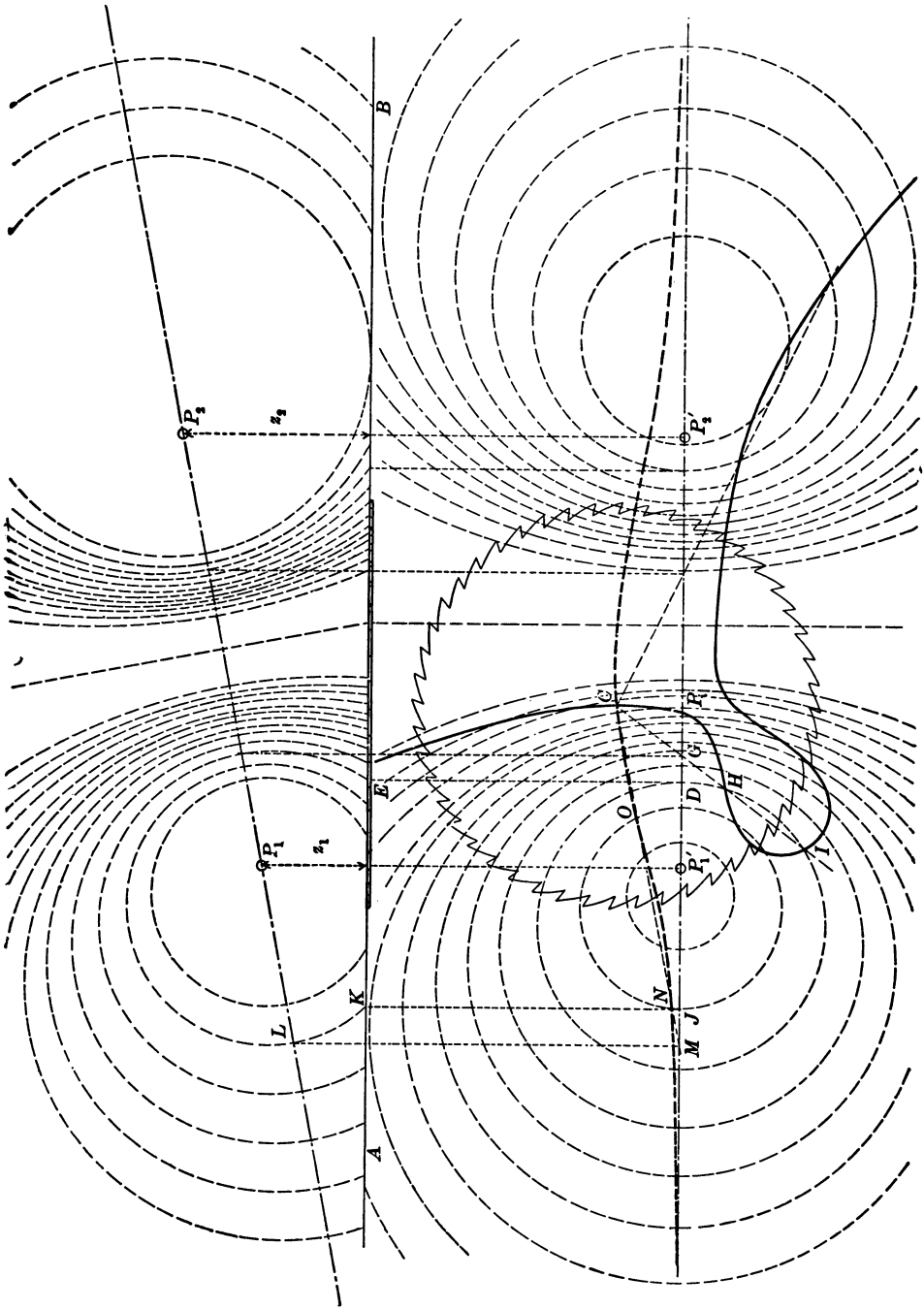


FIG. 3.

Problem A, and the heavy dashed line the locus of virtual brilliant points. The photographs in Fig. 1 do not show the locus of virtual brilliant points.*

While a curve or surface usually has only a finite number of brilliant points, there are exceptional cases. Every point of a $\left\{\begin{matrix}\text{prolate spheroid}\\ \text{biparted hyperboloid of revolution}\end{matrix}\right\}$ is a $\left\{\begin{matrix}\text{brilliant point}\\ \text{virtual brilliant point}\end{matrix}\right\}$ with respect to its foci. If an ellipse and hyperbola lie in perpendicular planes and are so placed that the vertices of each are the foci of the other, every point of either curve is a brilliant point or a virtual brilliant point with respect to any two points of the other.

Problem A may be generalized in several ways. One of the generalizations is the following:

PROBLEM B. *Find the locus of the brilliant points and virtual brilliant points of a two-parameter family of curves with respect to two points* P_1 *and* P_2.†

The term *congruence* is used to signify a two-parameter family of curves. All the straight lines which pass through the point P_1 (or P_2) form a rectilinear congruence. Problem B may be generalized by replacing the rectilinear congruences corresponding to P_1 and P_2 by two general curvilinear congruences.

In Part I (§§ 1–5) this generalized problem is stated, solved, and discussed. In Part II (§§ 6–8) the generalized problem of brilliant points of surfaces is discussed. In Part III (§§ 9, 10) the conditions are discussed under which some of the results of Parts I and II may be given special geometrical and optical interpretations.

I. BRILLIANT POINTS OF CURVES.

§ 1. *Definitions and classification.*

A congruence may be represented by the system of differential equations

$$\left\|\begin{matrix} dx & dy & dz \\ X & Y & Z \end{matrix}\right\| = 0, \ddagger \tag{1}$$

in which X, Y, Z: a) are single valued, continuous real functions of the rectangular Cartesian coördinates x, y, z; b) have first partial derivatives which are continuous; c) do not all vanish at the same point. Let Σ denote the region

* It is possible to find a family of curves such that if the surface of the saw were scratched along these instead of along the concentric circles formed by the emery particles during the process of polishing, the locus of the brilliant points would be the curve which for the circles is the locus of virtual brilliant points, and the locus of the virtual brilliant points would be the curve which for the circles is the locus of brilliant points. This is a consequence of the corollary to Theorem 7.

† For a solution of this problem, without derivation, together with some applications see a paper by the author, Annals of Mathematics, ser. 2, vol. 3 (1902), p. 113.

‡ This expresses that each of the three determinants of the matrix vanishes.

in which these conditions are satisfied. Through each point of Σ there passes one, and only one,* curve of the congruence (1).†

The two systems of differential equations

$$\left\| \begin{matrix} dx & dy & dz \\ X_i & Y_i & Z_i \end{matrix} \right\| = 0 \qquad (i=1,2), \tag{2}$$

in which X_i, Y_i, Z_i satisfy conditions $a)$, $b)$, $c)$ in Σ, represent two other congruences.

Definition 1. Let c be a curve with a continuously turning tangent which lies in Σ. A point P on c is said to be a brilliant point of c with respect to the two congruences (2), when the curves of the two congruences (2) which pass through P make equal angles with c at P.

In order to make a classification we will assign to each of the congruences (2) a *direction of propagation.* For this purpose we define $\bar{X}_i$, $\bar{Y}_i$, $\bar{Z}_i$ by the equations

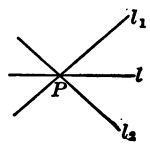
$$\frac{\bar{X}_i}{X_i} = \frac{\bar{Y}_i}{Y_i} = \frac{\bar{Z}_i}{Z_i} = \frac{1}{+\sqrt{X_i^2 + Y_i^2 + Z_i^2}} \qquad (i=1,2). \tag{3}$$

The directions of propagation for the two congruences (2) *shall be determined by the directional cosines* $\bar{X}_1$, $\bar{Y}_1$, $\bar{Z}_1$ *and* $-\bar{X}_2$, $-\bar{Y}_2$, $-\bar{Z}_2$ *respectively.*

Consider first the following special cases. In describing these, let l, l_1, l_2 represent respectively the tangents at P to the curve c and the curves of the two congruences (2) which pass through P.

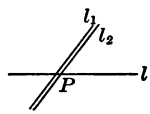
CASE I. The tangents l, l_1, l_2 lie in a plane:

1) l_1 and l_2 do not coincide;

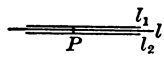


2) l_1 and l_2 coincide,

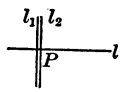
a) the coincident tangents are oblique to l,



b) the coincident tangents coincide with l,



c) the coincident tangents are perpendicular to l.

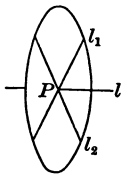


* The uniqueness follows from Condition b), which insures the LIPSCHITZ conditions.

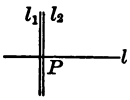
† For the purposes of this paper it is only necessary to know that a solution of (1) exists in the immediate neighborhood of a point of Σ.

CASE II. The tangents l_1 and l_2 are each perpendicular to the tangent l:

1) l_1 and l_2 do not coincide,



2) l_1 and l_2 coincide.*



A brilliant point which does not belong to either of these cases is said to belong to the general case. Brilliant points which belong to the general case or to the special case (I, 1) are called *actual* or *virtual* according as the directions of propagation at P of the two congruences (2) lie on the *same side* or on *opposite sides* of the normal plane to c at P. For convenience of reference let us make the following definitions:

Definition 2. Brilliant points which belong to the special case I are called *extra* brilliant points.

Definition 3. Brilliant points which belong to the special case II are called *exceptional* brilliant points.

Definition 4. Brilliant points which belong to the special cases I and II, with the exception of those which belong to the sub-case (I, 2, a), are called *extreme* brilliant points.

§ 2. *Conditions for brilliant points.*

Let us now find the condition that a point P of the region Σ should be a brilliant point of the curve of the congruence (1) which passes through it with respect to the two congruences (2).

The directions of propagation at P of the congruences (2) make with one of the directions of the curve of the congruence (1) which passes through P angles whose cosines are proportional to

$$X\overline{X}_i + Y\overline{Y}_i + Z\overline{Z}_i \qquad (i=1,2).$$

Therefore the condition for a brilliant point is

$$X\overline{X}_1 + Y\overline{Y}_1 + Z\overline{Z}_1 = \pm(X\overline{X}_2 + Y\overline{Y}_2 + Z\overline{Z}_2), \tag{4}$$

with the sign $-$ for the actual, and the sign $+$ for the virtual brilliant point. This condition may be written as follows:

* The sub-cases (I, 2, c) and (II, 2) are identical.

$$(5)\quad \begin{aligned}&[X(\bar{X}_1-\bar{X}_2)+Y(\bar{Y}_1-\bar{Y}_2)+Z(\bar{Z}_1-\bar{Z}_2)]\\ &\qquad\times[X(\bar{X}_1+\bar{X}_2)+Y(\bar{Y}_1+\bar{Y}_2)+Z(\bar{Z}_1+\bar{Z}_2)]=0.\end{aligned}$$

The two factors in (5) do not vanish together unless

$$(6)\quad X:Y:Z=\begin{vmatrix}\bar{Y}_1-\bar{Y}_2 & \bar{Z}_1-\bar{Z}_2\\ \bar{Y}_1+\bar{Y}_2 & \bar{Z}_1+\bar{Z}_2\end{vmatrix}:\begin{vmatrix}\bar{Z}_1-\bar{Z}_2 & \bar{X}_1-\bar{X}_2\\ \bar{Z}_1+\bar{Z}_2 & \bar{X}_1+\bar{X}_2\end{vmatrix}:\begin{vmatrix}\bar{X}_1-\bar{X}_2 & \bar{Y}_1-\bar{Y}_2\\ \bar{X}_1+\bar{X}_2 & \bar{Y}_1+\bar{Y}_2\end{vmatrix},$$

but these equations are the conditions for the special cases after (I, 1).

Let the product of the left member of equation (5) and the non-vanishing expression

$$(X_1^2+Y_1^2+Z_1^2)\cdot(X_2^2+Y_2^2+Z_2^2)$$

be denoted by Ω. We shall find it convenient to express Ω in other forms. For this purpose let

$$U=\begin{vmatrix}Y_1 & Z_1\\ Y_2 & Z_2\end{vmatrix},\qquad V=\begin{vmatrix}Z_1 & X_1\\ Z_2 & X_2\end{vmatrix},\qquad W=\begin{vmatrix}X_1 & Y_1\\ X_2 & Y_2\end{vmatrix},$$

$$U_1=\begin{vmatrix}Y_2 & Z_2\\ Y & Z\end{vmatrix},\qquad V_1=\begin{vmatrix}Z_2 & X_2\\ Z & X\end{vmatrix},\qquad W_1=\begin{vmatrix}X_2 & Y_2\\ X & Y\end{vmatrix},$$

$$U_2=\begin{vmatrix}Y & Z\\ Y_1 & Z_1\end{vmatrix},\qquad V_2=\begin{vmatrix}Z & X\\ Z_1 & X_1\end{vmatrix},\qquad W_2=\begin{vmatrix}X & Y\\ X_1 & Y_1\end{vmatrix},$$

$$R=\begin{vmatrix}V & W\\ Y & Z\end{vmatrix},\qquad S=\begin{vmatrix}W & U\\ Z & X\end{vmatrix},\qquad T=\begin{vmatrix}U & V\\ X & Y\end{vmatrix},$$

$$t_i=XX_i+YY_i+ZZ_i\qquad (i=1,2).$$

Then

$$(7)\qquad \Omega=\begin{vmatrix}t_1^2 & X_1^2+Y_1^2+Z_1^2\\ t_2^2 & X_2^2+Y_2^2+Z_2^2\end{vmatrix}.$$

The right member of (7) may be written

$$\frac{\begin{vmatrix}t_1^2 & (R^2+S^2+T^2)(X_1^2+Y_1^2+Z_1^2)-(U^2+V^2+W^2)t_1^2\\ t_2^2 & (R^2+S^2+T^2)(X_2^2+Y_2^2+Z_2^2)-(U^2+V^2+W^2)t_2^2\end{vmatrix}}{R^2+S^2+T^2},$$

which, by means of the identities

$$(8)\quad \begin{aligned}&(R^2+S^2+T^2)(X_i^2+Y_i^2+Z_i^2)-(U^2+V^2+W^2)t_i^2\\ &\qquad=(RX_i+SY_i+TZ_i)^2\quad (i=1,2),^*\end{aligned}$$

* See footnote to formulas (25).

may be written

$$\frac{\begin{vmatrix} t_1 & RX_1 + SY_1 + TZ_1 \\ t_2 & RX_2 + SY_2 + TZ_2 \end{vmatrix} \cdot \begin{vmatrix} t_1 & -(RX_1 + SY_1 + TZ_1) \\ t_2 & RX_2 + SY_2 + TZ_2 \end{vmatrix}}{R^2 + S^2 + T^2}.$$

Hence, by the identity

$$R^2 + S^2 + T^2 = \begin{vmatrix} t_1 & RX_1 + SY_1 + TZ_1 \\ t_2 & RX_2 + SY_2 + TZ_2 \end{vmatrix}, * \tag{9}$$

$$\Omega = t_1(RX_2 + SY_2 + TZ_2) + t_2(RX_1 + SY_1 + TZ_1). \tag{10}$$

Since

$$\begin{cases} RX_1 + SY_1 + TZ_1 = UU_2 + VV_2 + WW_2, \dagger \\ RX_2 + SY_2 + TZ_1 = -[UU_1 + VV_1 + WW_1], \dagger \end{cases} \tag{11}$$

we also have, from (10),

$$\Omega = \begin{vmatrix} UU_2 + VV_2 + WW_2 & XX_1 + YY_1 + ZZ_1 \\ UU_1 + VV_1 + WW_1 & XX_2 + YY_2 + ZZ_2 \end{vmatrix}. \ddagger \tag{12}$$

The proportion (6) may be written

$$X : Y : Z = U : V : W,$$

or

$$R = S = T = 0. \tag{13}$$

The foregoing results may be expressed as follows:

THEOREM 1. *The necessary and sufficient condition that a point P of the region Σ should be a brilliant point of the curve of the congruence* (1) *which passes through it, with respect to the two congruences* (2), *is that $\Omega = 0$ for P. Furthermore, when $R^2 + S^2 + T^2 > 0$, P is actual or virtual according as it causes the second or the first of the factors in equation* (5) *to vanish; and when $R = S = T = 0$, P belongs to the special cases after* (I, 1).

* See footnote to formula (27).

† Different expansions of the determinants

$$\begin{vmatrix} U & V & W \\ X & Y & Z \\ X_i & Y_i & Z_i \end{vmatrix} \qquad (i = 1, 2).$$

‡ If, in the left member of (12), we put

$$X_i = x - a_i, \qquad Y_i = y - b_i, \qquad Z_i = z - c_i \qquad (i = 1, 2),$$

$$X = \begin{vmatrix} \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \end{vmatrix}, \quad Y = \begin{vmatrix} \frac{\partial F_1}{\partial z} & \frac{\partial F_1}{\partial x} \\ \frac{\partial F_2}{\partial z} & \frac{\partial F_2}{\partial x} \end{vmatrix}, \quad Z = \begin{vmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{vmatrix},$$

we obtain the left member of equation (b), p. 117, of the paper in the Annals.

§ 3. *Conditions for special cases.*

We will now take up a discussion of the special cases referred to at the end of § 1. This discussion will prove of fundamental importance when we come to the subject of brilliant points of surfaces.

Extra brilliant points. The necessary and sufficient condition that the three tangents l, l_1, l_2 should lie in a plane is

$$\Delta \equiv \begin{vmatrix} X & Y & Z \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix} = 0. \tag{14}$$

Therefore the conditions for an extra brilliant point are

$$\Omega = 0, \qquad \Delta = 0. \tag{15}$$

These conditions may be expressed in another form by means of the identity

$$\begin{aligned} &\left[\begin{vmatrix} Y & Z \\ \bar{Y}_1+\bar{Y}_2 & \bar{Z}_1+\bar{Z}_2 \end{vmatrix}^2 + \begin{vmatrix} Z & X \\ \bar{Z}_1+\bar{Z}_2 & \bar{X}_1+\bar{X}_2 \end{vmatrix}^2 + \begin{vmatrix} X & Y \\ \bar{X}+\bar{X}_2 & \bar{Y}+\bar{Y}_2 \end{vmatrix}^2\right] \\ &\times\left[\begin{vmatrix} Y & Z \\ \bar{Y}_1-\bar{Y}_2 & \bar{Z}_1-\bar{Z}_2 \end{vmatrix}^2 + \begin{vmatrix} Z & X \\ \bar{Z}_1-\bar{Z}_2 & \bar{X}_1-\bar{X}_2 \end{vmatrix}^2 + \begin{vmatrix} X & Y \\ \bar{X}_1-\bar{X}_2 & \bar{Y}_1-\bar{Y}_2 \end{vmatrix}^2\right] \\ &= \frac{\Omega^2 + 4(X^2+Y^2+Z^2)(X_1^2+Y_1^2+Z_1^2)(X_2^2+Y_2^2+Z_2^2)\Delta^2}{(X_1^2+Y_1^2+Z_1^2)^2\cdot(X_2^2+Y_2^2+Z_2^2)^2},^* \end{aligned} \tag{16}$$

from which it follows that, when the equations (15) are satisfied, either

$$\left\| \begin{matrix} X & Y & Z \\ \bar{X}_1+\bar{X}_2 & \bar{Y}_1+\bar{Y}_2 & \bar{Z}_1+\bar{Z}_2 \end{matrix} \right\| = 0, \tag{17}$$

or

$$\left\| \begin{matrix} X & Y & Z \\ \bar{X}_1-\bar{X}_2 & \bar{Y}_1-\bar{Y}_2 & \bar{Z}_1-\bar{Z}_2 \end{matrix} \right\| = 0, \tag{18}$$

and conversely.

The system of equations (17) [or (18)] might be obtained as the solution of the pair of equations consisting of (14) and the equation obtained by equating to zero the first (or the second) factor of the left member of (5). In getting (17) and (18) in this way it is assumed that the two sets of equations

$$\bar{X}_1 - \bar{X}_2 = \bar{Y}_1 - \bar{Y}_2 = \bar{Z}_1 - \bar{Z}_2 = 0, \quad \bar{X}_1 + \bar{X}_2 = \bar{Y}_1 + \bar{Y}_2 = \bar{Z}_1 + \bar{Z}_2 = 0, \tag{19}$$

are *not* satisfied, i. e., that $U^2 + V^2 + W^2 > 0$.† But when the first (or second) of the sets of equations (19) is satisfied, equations (18) [or (17)] are satisfied.

* See footnote to formula (28).

† $$[(\bar{X}_1-\bar{X}_2)^2+(\bar{Y}_1-\bar{Y}_2)^2+(\bar{Z}_1-\bar{Z}_2)^2]\cdot[(\bar{X}_1+\bar{X}_2)^2+(\bar{Y}_1+\bar{Y}_2)^2+(\bar{Z}_1+\bar{Z}_2)^2] = 4\frac{U^2+V^2+W^2}{(X_1^2+Y_1^2+Z_1^2)(X_2^2+Y_2^2+Z_2^2)}$$

identically. See footnote to formulas (21).

The foregoing results may be expressed as follows:

THEOREM 2. *The necessary and sufficient conditions that a point P of the region Σ should be an extra brilliant point of the curve of the congruence* (1) *which passes through it, with respect to the two congruences* (2), *may be expressed in either of the following two ways: a) the system of equations* (15) *is satisfied by P, b) either the system of equations* (17) *or the system of equations* (18) *is satisfied by P. Furthermore, when $U^2+V^2+W^2>0$, P belongs to case* (I, 1), *being an actual extra brilliant point when it satisfies* (18) *and a virtual extra brilliant point when it satisfies* (17); *and when $U=V=W=0$, P belongs to case* (I, 2).*

Exceptional Brilliant Points. The following theorem is evident:

THEOREM 3. *The necessary and sufficient conditions that a point P of the region Σ should be an exceptional brilliant point of the curve of the congruence* (1) *which passes through it, with respect to the two congruences* (2), *are that $t_i=0$, $(i=1,2)$ for P. Furthermore, when $U^2+V^2+W^2>0$, P belongs to case* (II, 1); *and when $U=V=W=0$, P belongs to case* (II, 2).

Extreme Brilliant Points. The following theorem will now be proved:

THEOREM 4. *The necessary and sufficient conditions that a point P of the region Σ should be an extreme brilliant point of the curve of the congruence* (1) *which passes through it, with respect to the two congruences* (2), *is that P satisfies the three equations*

$$\begin{vmatrix} t_1 & U_2 \\ t_2 & U_1 \end{vmatrix} = 0, \qquad \begin{vmatrix} t_1 & V_2 \\ t_2 & V_1 \end{vmatrix} = 0, \qquad \begin{vmatrix} t_1 & W_2 \\ t_2 & W_1 \end{vmatrix} = 0.\dagger$$

Corollary. When the second of the congruences (2) coincides with the first throughout Σ, these equations become

$$t_1U_1=0, \qquad t_1V_1=0, \qquad t_1W_1=0.$$

Therefore either

$$t_1=0 \qquad \text{or} \qquad U_1=V_1=W_1=0.$$

---

* For special cases (I, 2, *b*) and (I, 2, *c*) the conditions are

$$U=V=W=U_i=V_i=W_i=0 \qquad (i=1,2),$$

and

$$U=V=W=t_i=0 \qquad (i=1,2)$$

respectively. The last of these are given in Theorem 3, since the cases (I, 2, *c*) and (II, 2) are identical.

† Any one of these equations is dependent on the other two, since the left members satisfy the identity

$$X\begin{vmatrix} t_1 & U_2 \\ t_2 & U_1 \end{vmatrix} + Y\begin{vmatrix} t_1 & V_2 \\ t_2 & V_1 \end{vmatrix} + Z\begin{vmatrix} t_1 & W_2 \\ t_2 & W_1 \end{vmatrix} = 0$$

in virtue of the identities

$$XU_i+YV_i+ZW_i=0 \qquad =1,2).$$

The curves of the two congruences under consideration which pass through a point P, are perpendicular when P satisfies the equation $t=0$, and tangent when P satisfies the system of equations $U_1=V_1=W_1=0$.

In order to prove Theorem 4 it will be necessary to establish a few more identities.

From the well known identity

$$(20)\qquad \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 + \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2 + \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 = (a_1^2+b_1^2+c_1^2)\cdot(a_2^2+b_2^2+c_2^2) - (a_1a_2+b_1b_2+c_1c_1)^2,$$

immediately follow the identities

$$(21)\qquad \begin{cases} U_1^2+V_1^2+W_1^2=(X^2+Y^2+Z^2)(X_2^2+Y_2^2+Z_2^2)-t_2^2, \\ U_2^2+V_2^2+W_2^2=(X^2+Y^2+Z^2)(X_1^2+Y_1^2+Z_1^2)-t_1^2,^* \end{cases}$$

$$(22)\qquad R^2+S^2+T^2=(X^2+Y^2+Z^2)(U^2+V^2+W^2)-\Delta^2.$$

From (20) and the well known identities†

$$(23)\qquad \begin{aligned} &\begin{vmatrix} V_1 & W_1 \\ V_2 & W_2 \end{vmatrix} = X\Delta, && \begin{vmatrix} W_1 & U_1 \\ W_2 & U_2 \end{vmatrix} = Y\Delta, && \begin{vmatrix} U_1 & V_1 \\ U_2 & V_2 \end{vmatrix} = Z\Delta, \\ &\begin{vmatrix} V_2 & W_2 \\ V & W \end{vmatrix} = X_1\Delta, && \begin{vmatrix} W_2 & U_2 \\ W & U \end{vmatrix} = Y_1\Delta, && \begin{vmatrix} U_2 & V_2 \\ U & V \end{vmatrix} = Z_1\Delta, \\ &\begin{vmatrix} V & W \\ V_1 & W_1 \end{vmatrix} = X_2\Delta, && \begin{vmatrix} W & U \\ W_1 & U_1 \end{vmatrix} = Y_2\Delta, && \begin{vmatrix} U & V \\ U_1 & V_1 \end{vmatrix} = Z_2\Delta, \end{aligned}$$

immediately follow the identities

$$(24)\qquad [U_1U_2+V_1V_2+W_1W_2]^2=(U_1^2+V_1^2+W_1^2)(U_2^2+V_2^2+W_2^2) - (X^2+Y^2+Z^2)\Delta^2,$$

$$(25)\qquad \begin{aligned} [UU_1+VV_1+WW_1]^2&=(U^2+V^2+W^2)(U_1^2+V_1^2+W_1^2) - (X_2^2+Y_2^2+Z_2^2)\Delta^2,‡ \\ [UU_2+VV_2+WW_2]^2&=(U^2+V^2+W^2)(U_2^2+V_2^2+W_2^2) - (X_1^2+Y_1^2+Z_1^2)\Delta^2.‡ \end{aligned}$$

---

* Similarly $U^2+V^2+W^2=(X_1^2+Y_1^2+Z_1^2)(X_2^2+Y_2^2+Z_2^2)-(X_1X_2+Y_1Y_2+Z_1Z_2)^2$, which enables one to get the identity in the footnote to equations (19).

† From the theory of determinants.

‡ By (11) the left members become respectively

$$(RX_i+SY_i+TZ_i)^2 \qquad (i=2,1),$$

and by (21) and (22) the right members become respectively

$$(R^2+S^2+T^2)(X_i^2+Y_i^2+Z_i^2)-t_i^2(U^2+V^2+W^2) \qquad i=2,1),$$

whence follows (8).

The identities

$$
\begin{aligned}
t_1 U_1 + t_2 U_2 &= \Delta X - (X^2 + Y^2 + Z^2) U, \\
t_1 V_1 + t_2 V_2 &= \Delta Y - (X^2 + Y^2 + Z^2) V, \qquad (26) \\
t_1 W_1 + t_2 W_2 &= \Delta Z - (X^2 + Y^2 + Z^2) W,
\end{aligned}
$$

are easily established. From them and (22) is obtained the identity

$$
\begin{aligned}
(t_1 U_1 + t_2 U_2)^2 + (t_1 V_1 + t_2 V_2)^2 + (t_1 W_1 + t_2 W_2)^2 \\
\doteq (X^2 + Y^2 + Z^2)(R^2 + S^2 + T^2).^*
\end{aligned} \qquad (27)
$$

The product of the left member of (27) and the expression

$$(t_1 U_1 - t_2 U_2)^2 + (t_1 V_1 - t_2 V_2)^2 + (t_1 W_1 - t_2 W_2)^2$$

is

$$[t_1^2(U_1^2 + V_1^2 + W_1^2) + t_2^2(U_2^2 + V_2^2 + W_2^2)]^2 - 4t_1^2 t_2^2 [U_1 U_2 + V_1 V_2 + W_1 W_2]^2$$

which by means of (24) becomes

$$[t_1^2(U_1^2 + V_1^2 + W_1^2) - t_2^2(U_2^2 + V_2^2 + W_2^2)]^2 + 4t_1^2 t_2^2 (X^2 + Y^2 + Z^2)\Delta^2.$$

By means of (21) and (7) the first term of this expression is $[(X^2 + Y^2 + Z^2)\Omega]^2$ and therefore we have the identity

$$
\begin{aligned}
[R^2 + S^2 + T^2] \cdot \left\{ \begin{vmatrix} t_1 & U_2 \\ t_2 & U_1 \end{vmatrix}^2 + \begin{vmatrix} t_1 & V_2 \\ t_2 & V_1 \end{vmatrix}^2 + \begin{vmatrix} t_1 & W_2 \\ t_2 & W_1 \end{vmatrix}^2 \right\} \\
= (X^2 + Y^2 + Z^2)\Omega^2 + 4t_1^2 t_2^2 \Delta^2 \dagger.
\end{aligned} \qquad (28)
$$

---

* Multiplying identities (26) by U, V, W and adding, we obtain

$$\begin{vmatrix} t_1 & -(UU_2 + VV_2 + WW_2) \\ t_2 & UU_1 + VV_1 + WW_1 \end{vmatrix} = \Delta^2 - (X^2 + Y^2 + Z^2)(U^2 + V^2 + W^2),$$

which by (11) and (22), becomes identity (9).

† In a similar manner identity (16) is obtained:

$$\begin{vmatrix} Y & Z \\ \bar{Y}_1 \pm \bar{Y}_2 & \bar{Z}_1 \pm \bar{Z}_2 \end{vmatrix} = \frac{U_2}{\sqrt{X_1^2 + Y_1^2 + Z_1^2}} \mp \frac{U_1}{\sqrt{X_2^2 + Y_2^2 + Z_2^2}},$$

$$\begin{vmatrix} Z & X \\ \bar{Z}_1 \pm \bar{Z}_2 & \bar{X}_1 \pm \bar{X}_2 \end{vmatrix} = \frac{V_2}{\sqrt{X_1^2 + Y_1^2 + Z_1^2}} \mp \frac{V_1}{\sqrt{X_2^2 + Y_2^2 + Z_2^2}},$$

$$\begin{vmatrix} X & Y \\ \bar{X}_1 \pm \bar{X}_2 & \bar{Y}_1 \pm \bar{Y}_2 \end{vmatrix} = \frac{W_2}{\sqrt{X_1^2 + Y_1^2 + Z_1^2}} \mp \frac{W_1}{\sqrt{X_2^2 + Y_2^2 + Z_2^2}},$$

where the upper signs are read together and the lower are read together. The product of the

From identity (28) the following three lemmas are deduced:

Lemma I. If the point P of the region Σ satisfies the two equations

$$\Omega = 0, \qquad \Delta = 0, \tag{29}$$

and the inequality

$$U^2 + V^2 + W^2 > 0, \tag{30}$$

or if it satisfies the two equations

$$t_1 = 0, \qquad t_2 = 0, \tag{31}$$

it then also satisfies the three equations

$$\begin{vmatrix} t_1 & U_2 \\ t_2 & U_1 \end{vmatrix} = 0, \qquad \begin{vmatrix} t_1 & V_2 \\ t_2 & V_1 \end{vmatrix} = 0, \qquad \begin{vmatrix} t_1 & W_2 \\ t_2 & W_1 \end{vmatrix} = 0. \tag{32}$$

Lemma II. If the point P of the region Σ satisfies the three equations (32) and the inequality (30), it then also satisfies either *a*) the two equations (29) and the inequality $t_1 t_2 \neq 0$, or *b*) the two equations (31) and the inequality $\Delta \neq 0$.

Lemma III. If the point P of the region Σ satisfies the three equations

$$U = V = W = 0, \tag{33}$$

it does *not* satisfy the three equations (32), *except* when either

$$U = V = W = U_i = V_i = W_i = 0 \qquad (i = 1, 2), \tag{34}$$

or

$$U = V = W = t_i = 0 \qquad (i = 1, 2). \tag{35}$$

The proofs follow:

When $\Delta = 0$ and $U^2 + V^2 + W^2 > 0$, it follows from (22) that $R^2 + S^2 + T^2 > 0$. Therefore it follows from (28) that, when equations (29) and inequality (30) are satisfied, equations (32) are satisfied. When $t_1 = t_2 = 0$, equations (32) are evidently satisfied. Hence Lemma I is proved.

---

sums of the squares for each reading of sign is

$$\left[\frac{U_2^2 + V_2^2 + W_2^2}{X_1^2 + Y_1^2 + Z_1^2} + \frac{U_1^2 + V_1^2 + W_1^2}{X_2^2 + Y_2^2 + Z_2^2}\right]^2 - 4\,\frac{[U_2U_1 + V_2V_1 + W_2W_1]^2}{(X_1^2 + Y_1^2 + Z_1^2)(X_2^2 + Y_2^2 + Z_2^2)},$$

which by means of (24) becomes

$$\left[\frac{U_2^2 + V_2^2 + W_2^2}{X_1^2 + Y_1^2 + Z_1^2} - \frac{U_1^2 + V_1^2 + W_1^2}{X_2^2 + Y_2^2 + Z_2^2}\right]^2 + 4\,\frac{(X^2 + Y^2 + Z^2)\Delta^2}{(X_1^2 + Y_1^2 + Z_1^2)(X_2^2 + Y_2^2 + Z_2^2)}.$$

By means of (21) and (7) the first term of this expression is

$$\frac{\Omega^2}{[(X_1^2 + Y_1^2 + Z_1^2)(X_2^2 + Y_2^2 + Z_2^2)]^2}$$

and therefore we have identity (16).

In order to prove Lemma II it is first necessary to show that when equations (32) are satisfied, one of the two expressions t_1, t_2 cannot vanish unless the other vanishes also. For let us suppose either

$$t_1 = 0 \qquad \text{or} \qquad t_2 = 0.$$

Then equations (32) become either

$$t_2 U_2 = t_2 V_2 = t_2 W_2 = 0 \qquad \text{or} \qquad t_1 U_1 = t_1 V_1 = t_1 W_1 = 0$$

respectively. But by (21)

$$U_2^2 + V_2^2 + W_2^2 > 0 \qquad \text{or} \qquad U_1^2 + V_1^2 + W_1^2 > 0.$$

Therefore

$$t_2 = 0 \qquad \text{or} \qquad t_1 = 0.$$

Now let us suppose equations (32) to be satisfied. Then by (28) the equations

$$\Omega = 0 \qquad \text{and} \qquad t_1 t_2 \Delta = 0$$

are satisfied. Therefore, by the first part of this proof, either

$$\Omega = 0 \qquad \text{and} \qquad \Delta = 0$$

or

$$t_1 = 0 \qquad \text{and} \qquad t_2 = 0.*$$

In order to show that the system of equations $t_i = 0$ $(i = 1, 2)$, and the equation $\Delta = 0$ cannot be simultaneously satisfied when $U^2 + V^2 + W^2 > 0$, let us solve the system of equations

$$t_i = X_i X + Y_i Y + Z_i Z = 0 \qquad (i = 1, 2).$$

The solution is

$$X : Y : Z = U : V : W.$$

When $\Delta = 0$ and $U^2 + V^2 + W^2 > 0$, it follows from (22) that $R^2 + S^2 + T^2 > 0$. But from the solution, $R = S = T = 0$. Hence a contradiction, and therefore $t_1 \neq 0$, $t_2 \neq 0$.† When $t_i = 0$ $(i = 1, 2)$ and $U^2 + V^2 + W^2 > 0$, it follows that $R = S = T = 0$, whence by (22), $\Delta \neq 0$. Hence Lemma II is proved.

When $U = V = W = 0$, $\Delta = 0$ also, and therefore by (26),

$$t_1 U_1 + t_2 U_2 = t_1 V_1 + t_2 V_2 = t_1 W_1 + t_2 W_2 = 0.$$

Then the left members of equations (32) become either $2t_1 U_1$, $2t_1 V_1$, $2t_1 W_1$ or $-2t_2 U_2$, $-2t_2 V_2$, $-2t_2 W_2$. By (21), t_1 and $U_2^2 + V_2^2 + W_2^2$ cannot vanish together, and t_2 and $U_1^2 + V_1^2 + W_1^2$ cannot vanish together. Hence under the hypothesis $U = V = W = 0$, the equations (32) are not satisfied unless either $t_i = 0$ $(i = 1, 2)$ or $U_i = V_i = W_i = 0$ $(i = 1, 2)$.

From Lemmas I, II, III follows Theorem 4.

* Evidently $\Omega = 0$ when $t_1 = 0$ and $t_2 = 0$.

† By first part of proof, either t_1 and t_2 both vanish or neither vanishes.

§ 4. *Invariance of the condition for a brilliant point.*

It is evident that the expression Ω is invariant for any point transformation which preserves angles. An example of such a transformation is inversion with respect to a sphere.

It is not in this sense however that the term *invariance* is used in this article. It will be found that if each of the three congruences (1), (2) be replaced by certain other congruences (each involving an arbitrary function),

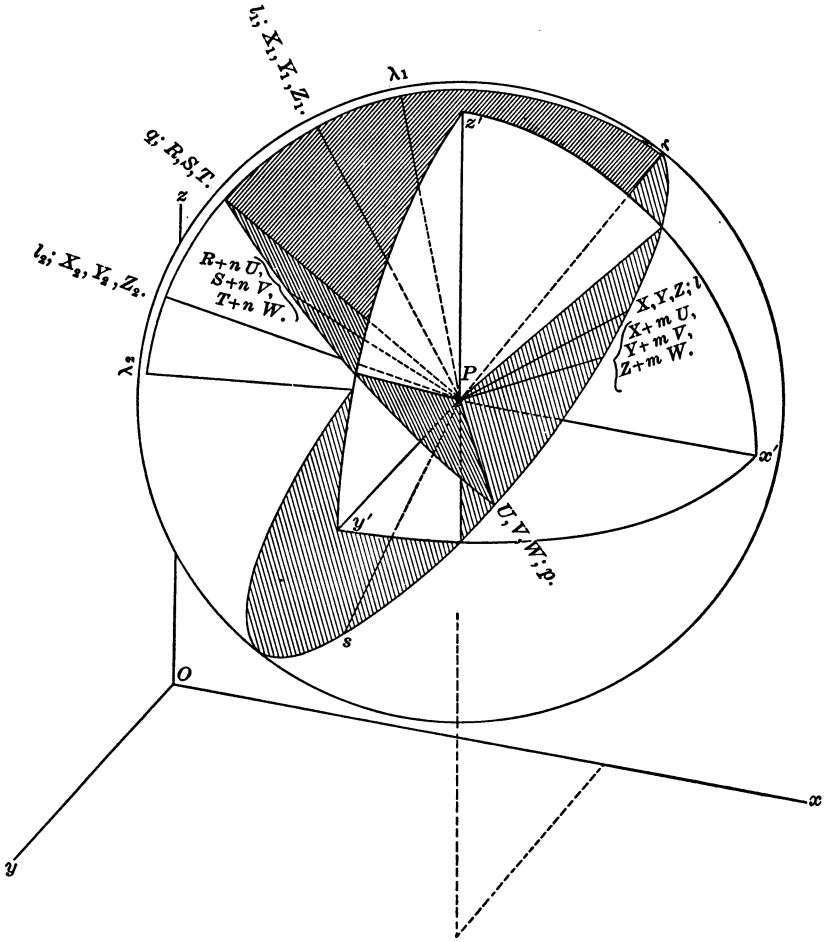


FIG. 4.

the expression Ω formed from the new congruences differs from that formed from the old ones by only a multiplicative factor.

Change of congruence (1) **alone.** In Fig. 4, P represents a point of the region Σ, and the right lines l and l_i $(i = 1, 2)$ represent tangents at P to the curves of the congruences (1) and (2) which pass through P. If l_1 and l_2 are distinct,

i. e., if $U^2 + V^2 + W^2 > 0$, the directional cosines of the bisectors of the angles formed by l_1 and l_2 are proportional to

$$\bar{X}_1 - \bar{X}_2,\ \bar{Y}_1 - \bar{Y}_2,\ \bar{Z}_1 - \bar{Z}_2, \tag{36}$$

and

$$\bar{X}_1 + \bar{X}_2,\ \bar{Y}_1 + \bar{Y}_2,\ \bar{Z}_1 + \bar{Z}_2. \tag{37}$$

Let π_1 and π_2 denote the planes which pass through P and are perpendicular respectively to these bisectors. Equation (5) is the condition that l should lie in one of the planes π_1, π_2. The perpendicular planes π_1 and π_2 intersect in the right line p, whose directional cosines are proportional to U, V, W. If l does not coincide with p, i. e., if $R^2 + S^2 + T^2 > 0$, the point P is an actual brilliant point or a virtual brilliant point according as l lies in π_2 or π_1. The expressions R, S, T are proportional to the directional cosines of the right line q which passes through P and is perpendicular to l and p. The three lines q, l_1, l_2 lie in one plane through P.

Let us now consider all the right lines which pass through P and lie either in the plane of l and p, or in the plane of p and q. The directional cosines of those which lie in the plane of l and p are proportional to

$$X + mU,\ Y + mV,\ Z + mW, \tag{38}$$

and of those which lie in the plane of p and q are proportional to

$$R + nU,\ S + nV,\ T + nW, \tag{39}$$

where m and n are parameters which may assume all real values. Each of these sets of lines excludes p. The planes of the two sets are perpendicular and intersect in p. The planes π_1 and π_2 are also perpendicular and intersect in p. Therefore, if l lies in $\pi_1(\pi_2)$ all the lines of the first set also lie in $\pi_1(\pi_2)$, and all those of the second set lie in $\pi_2(\pi_1)$; and if l lies in neither π_1 nor π_2 none of the lines of either set lie in π_1 or π_2. Let c denote the curve of congruence (1) which passes through P and therefore has l for a tangent at P, and γ and δ, curves which pass through P and have at P directional cosines proportional respectively to the expressions (38) and (39). Then follows from what has just been said: *If P is an actual (a virtual) brilliant point of c with respect to the two congruences* (2), *it is also an actual (a virtual) brilliant point of γ, and a virtual (an actual) brilliant point of δ, with respect to the two congruences* (2); *and if P is not a brilliant point of c it is not a brilliant point of γ or δ.*

In the portion of the region Σ for which $R^2 + S^2 + T^2 > 0$, let m and n in the expressions (38) and (39) now be regarded as functions of x, y, z which *a*) are single valued and continuous, *b*) have first partial derivatives which are con-

tinuous. Then each of the sets of functions (38), (39) satisfies in this region the conditions which the set of functions X, Y, Z satisfies in Σ. Therefore, for every choice of the functions m and n, each of the two systems of differential equations

$$(40) \qquad \left\| \begin{matrix} dx & dy & dz \\ X+mU & Y+mV & Z+mW \end{matrix} \right\| = 0,$$

$$(41) \qquad \left\| \begin{matrix} dx & dy & dz \\ R+nU & S+nV & T+nW \end{matrix} \right\| = 0,$$

represents a congruence. From the italicized statement above follows

THEOREM 5. *If, in the portion of the region Σ for which $R^2+S^2+T^2>0$, P $\{{}^{is}_{is\ not}\}$ a brilliant point of the curve of congruence* (1) *which passes through it with respect to the two congruences* (2), *then it* $\{{}^{is}_{is\ not}\}$ *a brilliant point of the curves of congruences* (40) *and* (41) *which pass through it with respect to the two congruences* (2), *for every choice of the functions m and n. Furthermore, if P is actual* (*virtual*) *for congruence* (1) *it is actual* (*virtual*) *for congruence* (40) *and virtual* (*actual*) *for congruence* (41).

Since the vanishing of Ω is the condition for a brilliant point of congruence (1) with respect to congruences (2) (see Theorem 1), Theorem 5 suggests that the substitution of the expressions (38) [or (39)] for X, Y, Z in Ω, results in an expression which differs from Ω by, at most, a factor which does not vanish when $R^2+S^2+T^2>0$. The expressions

$$XX_i+YY_i+ZZ_i \qquad (i=1,2)$$

are unaltered by the first substitution and changed into the expressions

$$RX_i+SY_i+TZ_i \qquad (i=1,2)$$

by the second substitution, since $UX_i+VY_i+WZ_i=0$ $(i=1,2)$. The expressions

$$RX_i+SY_i+TZ_i \qquad (i=1,2)$$

are unaltered by the first substitution and changed into the expressions

$$-(U^2+V^2+W^2)(XX_i+YY_i+ZZ_i) \qquad (i=1,2)$$

by the second substitution.* Therefore by (10) the expression Ω is unaltered

* Since

$$-\begin{vmatrix} R & U & X_i \\ S & V & Y_i \\ T & W & Z_i \end{vmatrix} = -(U^2+V^2+W^2)(XX+YY_i+ZZ_i).$$

See the next footnote.

by the first substitution and changed into the expression

$$-(U^2+V^2+W^2)\Omega \tag{42}$$

by the second substitution. From this analytic fact Theorem 5 might have been deduced.

The following are interesting corollaries of Theorem 5:

Corollary 1. When the function m is

$$\frac{-\Delta}{U^2+V^2+W^2},^*$$

every brilliant point of congruence (40) with respect to the congruences (2) is an extra brilliant point.

Corollary 2. When the function n is the constant zero, every brilliant point of the congruence (41) with respect to the congruences (2) is an extra brilliant point.

For, in the first case the new form of Δ is

$$\begin{vmatrix} X+mU & Y+mV & Z+mW \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix} = \Delta + m(U^2+V^2+W^2),$$

which, for the given value of m, equals zero; and in the second case the new form if Δ is

$$\begin{vmatrix} R & S & T \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix} = RU+SV+TW = \begin{vmatrix} U & V & W \\ X & Y & Z \\ U & V & W \end{vmatrix} = 0.$$

Change of congruences (2) alone. Let us now consider the two right lines which pass through P (Fig. 4) and whose directional cosines are proportional to the expressions

$$f_1\overline{X}_1+f_2\overline{X}_2,\ f_1\overline{Y}_1+f_2\overline{Y}_2,\ f_1\overline{Z}_1+f_2\overline{Z}_2, \tag{43}$$

and

$$f_2\overline{X}_1+f_1\overline{X}_2,\ f_2\overline{Y}_1+f_1\overline{Y}_2,\ f_2\overline{Z}_1+f_1\overline{Z}_2, \tag{44}$$

in which f_1 and f_2 may have any real values not both zero. Let these lines be denoted by λ_1 and λ_2 respectively. They lie in the plane of l_1 and l_2. If l_1

* When m has this form, the expressions (38) are proportional to the directional cosines of the right line r which passes through P and is perpendicular to both p and q (see Fig. 4). For

$$\begin{vmatrix} S & V \\ T & W \end{vmatrix} = (U^2+V^2+W^2)X-\Delta U, \qquad \begin{vmatrix} T & W \\ R & U \end{vmatrix} = (U^2+V^2+W^2)Y-\Delta V, \qquad \begin{vmatrix} R & U \\ S & V \end{vmatrix} = (U^2+V^2+W^2)Z-\Delta W.$$

and l_2 are distinct and $f_1^2 - f_2^2 \neq 0$, λ_1 and λ_2 are distinct. In this case the directional cosines of the bisectors of the angles formed by λ_1 and λ_2 are proportional to the expressions

$$(45) \qquad (f_1 - f_2)(\overline{X}_1 - \overline{X}_2),\ (f_1 - f_2)(\overline{Y}_1 - \overline{Y}_2),\ (f_1 - f_2)(\overline{Z}_1 - \overline{Z}_2),$$

and

$$(46) \qquad (f_1 + f_2)(\overline{X}_1 + \overline{X}_2),\ (f_1 + f_2)(\overline{Y}_1 + \overline{Y}_2),\ (f_1 + f_2)(\overline{Z}_1 + \overline{Z}_2).$$

Comparing (45) and (46) with (36) and (37) respectively, we see that the bisectors of the angles formed by λ_1 and λ_2 are the same as those of the angles formed by l_1 and l_2. *Therefore the planes π_1 and π_2 are also perpendicular to the bisectors of the angles formed by λ_1 and λ_2.*

In the portion of the region Σ for which $U^2 + V^2 + W^2 > 0$, let f_1 and f_2 of the expressions (43) and (44) now be regarded as functions of x, y, z which *a*) are single valued and continuous, *b*) have first partial derivatives which are continuous, *c*) do not both vanish at the same point. Then each of the sets of functions (43), (44) satisfies in this region the conditions which the set of functions X, Y, Z satisfies in Σ. Therefore, for every choice of the functions f_1 and f_2, each of the two systems of differential equations

$$(47) \qquad \begin{Vmatrix} dx & dy & dz \\ f_1\overline{X}_1 + f_2\overline{X}_2 & f_1\overline{Y}_1 + f_2\overline{Y}_2 & f_1\overline{Z}_1 + f_2\overline{Z}_2 \end{Vmatrix} = 0,$$

$$(48) \qquad \begin{Vmatrix} dx & dy & dz \\ f_2\overline{X}_1 + f_1\overline{X}_2 & f_2\overline{Y}_1 + f_1\overline{Y}_2 & f_2\overline{Z}_1 + f_1\overline{Z}_2 \end{Vmatrix} = 0$$

represents a congruence.* From the last italicized statement follows

THEOREM 6. *If, in the portion of the region Σ for which $U^2 + V^2 + W^2 > 0$ and $f_1^2 - f_2^2 \neq 0$, P $\left\{\begin{smallmatrix} is \\ is\ not \end{smallmatrix}\right\}$ a brilliant point of the curve of congruence* (1) *which passes through it with respect to the two congruences* (2), *then it* $\left\{\begin{smallmatrix} is \\ is\ not \end{smallmatrix}\right\}$ *a brilliant point of the same curve with respect to the two congruences* (47) *and* (48), *for every choice of the functions f_1 and f_2.*

Theorem 6 may also be deduced in the following manner. If $U^2 + V^2 + W^2 > 0$, not all three of either of the sets of expressions (36), (37) vanish. The substitution of expressions (43) for $\overline{X}_1$, $\overline{Y}_1$, $\overline{Z}_1$, and expressions (44) for $\overline{X}_2$, $\overline{Y}_2$, $\overline{Z}_2$, changes the expressions (36) and (37) into the expressions (45) and (46) respectively, and therefore Ω into

$$(49) \qquad (f_1^2 - f_2^2)\,\Omega.$$

If $f_1^2 - f_2^2 \neq 0$, (49) vanishes when, and only when, $\Omega = 0$. But by Theorem 1,

* If we regard congruences (2) as lines of force in fields of intensity f_1 and f_2 respectively, then congruence (47) gives the lines of force of the resultant field. By interchanging f_1 and f_2 we get congruence (48) as the lines of force of another field.

$\Omega = 0$ is condition that P should be a brilliant point of (1) with respect to (2). Hence Theorem 6.

Change of all three congruences. Let the directions of propagation of the two congruences (47) and (48) be determined by the senses of the vectors whose projections are the expressions (43) and (44) respectively.

Adopting this convention and uniting Theorems 5 and 6 we have

THEOREM 7. *If, in the portion of the region Σ for which $R^2+S^2+T^2>0$ and $f_1^2-f_2^2 \neq 0$, P $\{^{is}_{is\ not}\}$ a brilliant point of the curve of congruence* (1) *which passes through it with respect to the two congruences* (2), *then it* $\{^{is}_{is\ not}\}$ *a brilliant point of the curves of congruences* (40) *and* (41) *which pass through it with respect to the two congruences* (47) *and* (48), *for every choice of the four functions m, n, f_1, f_2. Furthermore, if P is actual (virtual) for congruence* (1) *with respect to the two congruences* (2), *it is actual (virtual) for congruence* (40) *and virtual (actual) for congruence* (41) *with respect to the two congruences* (47) *and* (48).

Let σ denote a surface composed of a one-parameter family of curves of congruence (1). We will call this family of curves the *first generation of* σ. Let P in Fig. 4 be a point of σ. The tangent plane to σ at P passes through l. This tangent plane cuts the plane of p and q in the right line λ. The right lines λ taken for all the points P of σ determine a second one-parameter family of curves on σ. We will call this family the *second generation of* σ. The second generation of σ is a one-parameter family of curves of congruence (41) for a particular function n. Therefore from Theorem 7 follows the

Corollary. In the portion of Σ for which $R^2+S^2+T^2>0$ and $f_1^2-f_2^2 \neq 0$, the locus of the brilliant points of both the first and second generations of σ with respect to the two congruences (47) and (48), for every choice of the functions f_1 and f_2, is identical with that of the first generation of σ with respect to the two congruences (2). Furthermore, the portion of this locus which is composed of actual (virtual) brilliant points for the first generation of σ, is composed of virtual (actual) brilliant points for the second generation of σ.*

§ 5. *An illustrative example.*

For the three congruences (1), (2) let us take three families of radiating right lines. There will be no loss of generality in taking the "radiants" of these at the points $O=(0,0,0)$ and $P_i=(a_i,0,c)$ $(i=1,2)$. We shall assume

$$a_1-a_2 \neq 0 \qquad \text{and} \qquad c \neq 0.$$

Then

$$X=x, \qquad Y=y, \qquad Z=z,$$

$$X_i=x-a_i, \qquad Y_i=y, \qquad Z_i=z-c, \qquad (i=1,2).$$

* The theorem stated on p. 121 of the paper in the Annals of Mathematics is a very special case of this corollary.

Thence follow

$$U = 0, \qquad U_1 = cy, \qquad U_2 = -cy,$$

$$V = (a_1 - a_2)(z - c), \quad V_1 = a_2 z - cx, \quad V_2 = -a_1 z + cx,$$

$$W = -(a_1 - a_2)y, \qquad W_1 = -a_2 y, \qquad W_2 = a_1 y,$$

$$t_i = x^2 + y^2 + z^2 - cz - a_i x \qquad (i = 1, 2),$$

$$R = (a_1 - a_2)(y^2 + z^2 - cz), \quad S = (a_1 - a_2)(-xy), \quad T = (a_1 - a_2)(-xz + cx),$$

$$\Omega = (a_1 - a_2) \cdot [-cx(z - c)\rho + (y^2 + z^2 - cz)\sigma],$$

$$\Delta = -(a_1 - a_2)cy,$$

$$\begin{vmatrix} t_1 & U_2 \\ t_2 & U_1 \end{vmatrix} = cy\rho, \qquad \begin{vmatrix} t_1 & V_2 \\ t_2 & V_1 \end{vmatrix} = -cx\rho + z\sigma, \qquad \begin{vmatrix} t_1 & W_2 \\ t_2 & W_1 \end{vmatrix} = -y\sigma,$$

in which

$$\rho = 2(x^2 + y^2 + z^2 - cz) - (a_1 + a_2)x,$$

$$\sigma = (a_1 + a_2)(x^2 + y^2 + z^2 - cz) - 2a_1 a_2 x,$$

and the determinant of the coefficients is $(a_1 - a_2)^2 \neq 0$. The region Σ includes all space except the three points O, P_1, P_2.

From the form of Ω it follows that the locus of brilliant points of the congruence of right lines of radiant O, with respect to the two points P_1, P_2, has the equation

$$(\alpha) \qquad \begin{aligned} &-cx(z - c) \cdot [2(x^2 + y^2 + z^2 - cz) - (a_1 + a_2)x] \\ &\qquad + (y^2 + z^2 - cz) \cdot [(a_1 + a_2)(x^2 + y^2 + z^2 - cz) - 2a_1 a_2 x] = 0.^* \end{aligned}$$

The three points O, P_1, P_2, which are excluded from Σ, satisfy equation (α). The section of the surface (α) by the plane $y = 0$ is the locus of extra brilliant points (since $\Delta = 0$ when $y = 0$). It consists of the right line

$$(\beta) \qquad y = 0, \qquad z - c = 0,$$

and the cubic

$$(\gamma) \quad \begin{cases} y = 0, \\ -2c(x^3 + xz^2) + (a_1 + a_2)(x^2 z + z^3) + c(a_1 + a_2)(x^2 - z^2) + 2(c^2 - a_1 a_2)xz = 0. \end{cases}$$

In Fig. 5 the right line passing through P_1 and P_2 and the curve $AOQP_2OP_1B$ represent this locus. The curve (γ) is the locus of extra

* A method of constructing this surface is given below. A property of the surface is given in § 10.

brilliant points belonging to case (I, 1) (since for it $U^2 + V^2 + W^2 > 0$), and the line (β) is the locus of those belonging to case (I, 2) (since for it $U = V = W = 0$). The section of the surface (α) by the plane $x = 0$ is the locus of the exceptional brilliant points (since for it $t_1 = t_2 = 0$). It is the circle

$$(\delta) \qquad x = 0, \qquad y^2 + z^2 - cz = 0.$$

This circle is a double line on the surface. It is the curve $OCQD$ in Fig. 5.

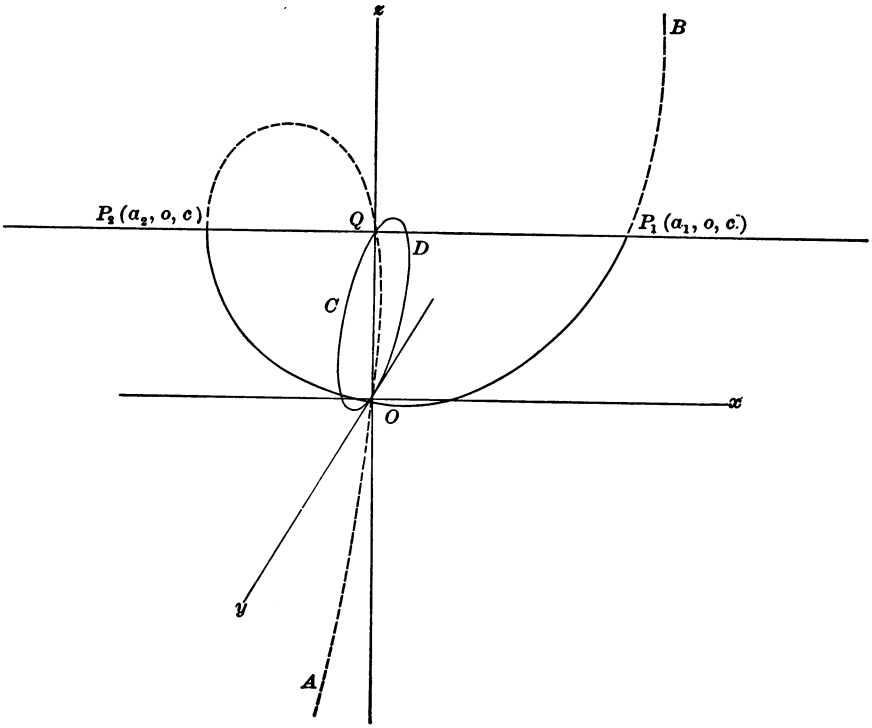


FIG. 5.

The locus of the extreme brilliant points is given by the three equations

$$cy\rho = 0, \qquad -cx\rho + z\sigma = 0, \qquad -y\sigma = 0.$$

1) If $y \neq 0$, it follows from these that $\rho = 0$ and $\sigma = 0$, and therefore that

$$x = 0, \qquad x^2 + y^2 + z^2 - cz = 0.$$

But these are the equations of the circle (δ).

2) If $y = 0$, the equation $-cx\rho + z\sigma = 0$ must be satisfied without ρ and σ necessarily being zero. But these equations are those of the cubic (γ). Therefore the locus of the extreme brilliant points consists of the cubic (γ) and the circle (δ).

For this example the systems of differential equations (40) and (41) are

$$(\epsilon)\qquad \left\| \begin{matrix} dx & dy & dz \\ x & y+(a_1-a_2)(z-c)\cdot m(x,y,z) & z-(a_1-a_2)y\cdot m(x,y,z) \end{matrix} \right\| = 0,$$

and

$$(\zeta)\qquad \left\| \begin{matrix} dx & dy & dz \\ y^2+z^2-cz & -xy+(z-c)\cdot n(x,y,z) & -x(z-c)-y\cdot n(x,y,z) \end{matrix} \right\| = 0,$$

respectively. $R^2+S^2+T^2>0$ for all space except the right line (β) and the circle (δ).

1) Let us suppose the function m to be of the form

$$\frac{cy}{(a_1-a_2)\left[(z-c)^2+y^2\right]}$$

(see Theorem 5, Cor. 1). Then (ϵ) becomes

$$(\eta)\qquad \left\| \begin{matrix} dx & dy & dz \\ x & y+\dfrac{cy(z-c)}{(z-c)^2+y^2} & z-\dfrac{cy^2}{(z-c)^2+y^2} \end{matrix} \right\| = 0.$$

The solution of (η) may be written in the form

$$(\theta)\qquad y=\kappa_1(z-c), \qquad \kappa_3 x = z - c\frac{\kappa_1^2}{1+\kappa_1^2},$$

in which κ_1 and κ_3 are the constants of integration. Equations (θ) represent the two-parameter family of right lines of which a one-parameter family lies in every plane through the right line (β), and all the right lines in any one of these planes pass through the point where this plane is pierced by the circle (δ).

2) Let the function n in (ζ) be identically zero. Then (ζ) becomes

$$(\iota)\qquad \left\| \begin{matrix} dx & dy & dz \\ y^2+z^2-cz & -xy & -x(z-c) \end{matrix} \right\| = 0.$$

The solution of (ι) may be written in the form

$$(\kappa)\qquad y=\kappa_1(z-c), \qquad x^2+y^2+z^2=\kappa_2,$$

in which κ_1 and κ_2 are the constants of integration. Equations (κ) represent the two-parameter family of circles of which a one-parameter family lies in every plane through the right line (β), and all the circles in any one of these planes have as common center the point where this plane is pierced by the circle (δ).

3) Let the function m in (ϵ) be

$$\frac{x^2+y^2+z^2}{c(a_1-a_2)\cdot y} \qquad \left(\text{of the form } \frac{X^2+Y^2+Z^2}{-\Delta}\right).$$

Then (ϵ) becomes

$$(\lambda) \qquad \left\| \begin{matrix} dx & dy & dz \\ x & y + \dfrac{(x^2+y^2+z^2)(z-c)}{cy} & z - \dfrac{x^2+y^2+z^2}{c} \end{matrix} \right\| = 0.$$

The solution of (λ) may be written in the form

$$(\mu) \qquad x^2+y^2+z^2 = \kappa_2, \qquad \kappa_4 x + (x^2+y^2+z^2-cz) = 0,$$

in which κ_2 and κ_4 are the constants of integration. Equations (μ) represent the two-parameter family of circles which are the intersections of the spheres having the origin as center with the spheres which pass through the circle (δ).

By Theorem 5 the surface (α) is also the locus of the brilliant points of each of the two-parameter family of curves (θ), (κ), (μ) with respect to the two points P_1, P_2. By Corollaries 1 and 2 of Theorem 5, each of the brilliant points of the right lines (θ) and the circles (κ) is an extra brilliant point with respect to P_1 and P_2.

Construction of the surface (α). The locus of the brilliant points of a plane one-parameter family of radiating right lines with respect to two points P_1 and P_2, in the common plane of these lines, is easily constructed.* In Fig. 5 the cubic $AOQP_2OP_1B$ and the right line P_1P_2 is this locus for the right lines which pass through O and lie in the plane xOz, with respect to P_1 and P_2. This locus is the section of the surface (α) by the plane $y=0$. In order to get the section by any other plane through P_1P_2, as for instance the plane which passes through the point C of the circle $OCQD$, construct in this plane the locus of brilliant points of the right lines which pass through C, with respect to P_1 and P_2. This locus consists of the straight line P_1P_2 and a cubic curve of the same nature as $AOQP_2OP_1B$, with O replaced by C. This follows from the fact that the surface (α) is also the locus of the brilliant points of the two-parameter family of right lines (θ).

According to the convention in § 1, the direction of propagation of the congruence of right lines which pass through P_1, is *from* P_1, and that of those which pass through P_2, is *toward* P_2. Therefore the part of the curve $AOQP_2OP_1B$ (Fig. 5) which is drawn full is the locus of the actual brilliant points of the right lines which pass through O and lie in the plane xOz, and the dotted part is that of the virtual brilliant points. Similarly on the other sections of the surface (α) by the planes through P_1P_2, there are full and dotted parts which have the same significance.

* See page 128 of the paper in the Annals of Mathematics to which reference has already been made.

For this example the system of differential equations (47) and (48) are

$$(\nu)\qquad \left\|\begin{matrix} dx & dy & dz \\ f_1\dfrac{x-a_1}{r_1}+f_2\dfrac{x-a_2}{r_2} & f_1\dfrac{y}{r_1}+f_2\dfrac{y}{r_2} & f_1\dfrac{z-c}{r_1}+f_2\dfrac{z-c}{r_2}\end{matrix}\right\|=0,$$

and

$$(\xi)\qquad \left\|\begin{matrix} dx & dy & dz \\ f_2\dfrac{x-a_1}{r_1}+f_1\dfrac{x-a_2}{r_2} & f_2\dfrac{y}{r_1}+f_1\dfrac{y}{r_2} & f_2\dfrac{z-c}{r_1}+f_1\dfrac{z-c}{r_2}\end{matrix}\right\|=0,$$

respectively, where

$$r_i=+\sqrt{(x-a_i)^2+y^2+(z-c)^2}\qquad (i=1,2).$$

It is easy to see that the integral curves of both (ν) and (ξ) lie in the one-parameter family of planes

$$(o)\qquad y=\kappa_1(z-c).$$

By means of this relation, r_i becomes

$$+\sqrt{(x-a_i)^2+\left(1+\frac{1}{\kappa_1^2}\right)y^2}\qquad (i=1,2).$$

Then the differential equation for determining the integral curves of (ν) in any one of the planes (o) is

$$(\pi)\qquad \left|\begin{matrix} dx & d\eta \\ f_1\dfrac{x-a_1}{\sqrt{(x-a_1)^2+\eta^2}}+f_2\dfrac{x-a_2}{\sqrt{(x-a_2)^2+\eta^2}} & f_1\dfrac{\eta}{\sqrt{(x-a_1)^2+\eta^2}}+f_2\dfrac{\eta}{\sqrt{(x-a_2)^2+\eta^2}}\end{matrix}\right|=0,$$

where

$$\eta=\sqrt{1+\frac{1}{\kappa_1^2}}\,y$$

and represents the perpendicular distance from the right line P_1P_2. If now we put

$$(\rho)\qquad \cos\omega_i=\frac{x-a_i}{\sqrt{(x-a_i)^2+\eta^2}},\qquad \sin\omega_i=\frac{\eta}{\sqrt{(x-a_i)^2+\eta^2}},$$

equation (π) becomes

$$(\sigma)\qquad \begin{vmatrix} d\eta & f_1\sin\omega_1+f_2\sin\omega_2 \\ dx & f_1\cos\omega_1+f_2\cos\omega_2\end{vmatrix}=0.$$

From equations (ρ) we find

$$(\tau)\qquad x-\frac{a_1+a_2}{2}=\frac{a_1-a_2}{2}\cdot\frac{\sin(\omega_1+\omega_2)}{\sin(\omega_1-\omega_2)},\qquad \eta=\frac{a_1-a_2}{2}\cdot\frac{2\sin\omega_1\sin\omega_2}{\sin(\omega_1-\omega_2)}.$$

If we make these substitutions in (σ), we get the differential equation

$$(\nu) \qquad \frac{a_1 - a_2}{\sin(\omega_1 - \omega_2)}\left[f_1 \sin\omega_2\, d\omega_1 + f_2 \sin\omega_1\, d\omega_2\right] = 0,$$

or

$$(\nu_1) \qquad f_1 r_1 d\omega_1 + f_2 r_2 d\omega_2 = 0,$$

since

$$r_1 = (a_1 - a_2)\frac{\sin\omega_2}{\sin(\omega_1 - \omega_2)}, \qquad r_2 = (a_1 - a_2)\frac{\sin\omega_1}{\sin(\omega_1 - \omega_2)}.$$

To get the corresponding equation for (ξ), interchange f_1 and f_2 in (ν_1).

Let us now find the solutions of (ν) and (ξ) when

$$(\phi) \qquad f_1 = \frac{M_1}{r_1} \quad \text{and} \quad f_2 = \frac{M_2}{r_2},$$

where M_1 and M_2 are constants. Then f_1 and f_2 are independent of κ_1. For these functions, the solution of equation (ν_1) is

$$(\chi) \qquad M_1\omega_1 + M_2\omega_2 = K,$$

where K is the constant of integration. If we interchange f_1 and f_2 in (ν), we get the equation

$$M_2 \csc^2\omega_1\, d\omega_1 + M_1 \csc^2\omega_2\, d\omega_2 = 0,$$

whence

$$(\psi) \qquad M_2 \operatorname{ctn}\omega_1 + M_1 \operatorname{ctn}\omega_2 = C,$$

where C is the constant of integration. From (ρ)

$$\operatorname{ctn}\omega_i = \frac{x - a_i}{\eta} \qquad (i = 1, 2),$$

and therefore (ψ) becomes

$$(\psi_1) \qquad C\eta = (M_1 + M_2)x - (M_1 a_2 + M_2 a_1).$$

Therefore for the functions (ϕ) the solution of (ν) is given by the equations (χ) and (o), and that of (ξ) by the equations (ψ) and (o). That is:

1) For the functions (ϕ), the integral curves of (ν) are the lines of force of a field due to two centers of force situated at P_1 and P_2, the intensity of the force due to each center being proportional to the reciprocal of the distance from that center.*

2) For the functions (ϕ), the integral curves of (ξ) are the right lines which pass through the point O' on P_1P_2, at which the force of the above field is zero.

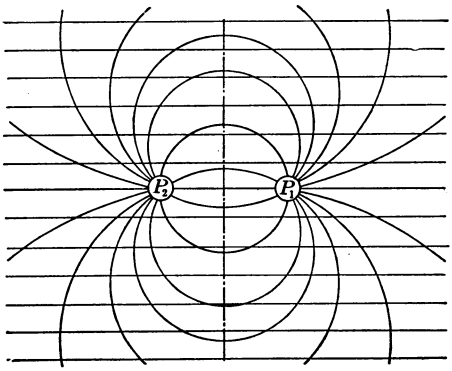
*For a method of making these curves physically evident see a note by the author: Bulletin of American Mathematical Society, vol. 12 (1906), p. 425.

For the functions (ϕ), $f_1^2 - f_2^2 \neq 0$ everywhere except on the sphere

$$(\omega) \qquad M_1^2 r_2^2 - M_2^2 r_1^2 = 0 .$$

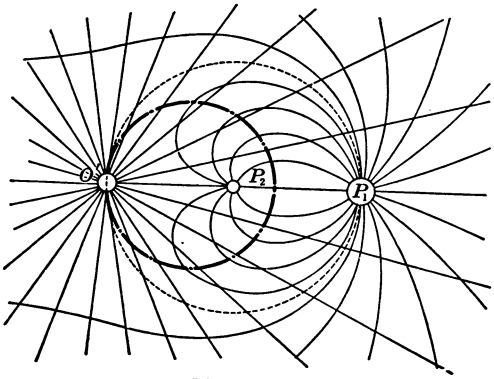
This sphere has P_1 and P_2 as a pair of conjugate points, and passes through the point O'.

Figs. 6, 7, 8, 9 show the integral curves of (ν) and (ξ) which lie in a plane



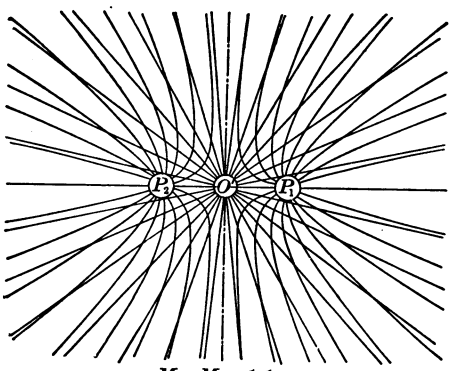
$M_1 : M_2 = 1:-1.$

FIG. 6.



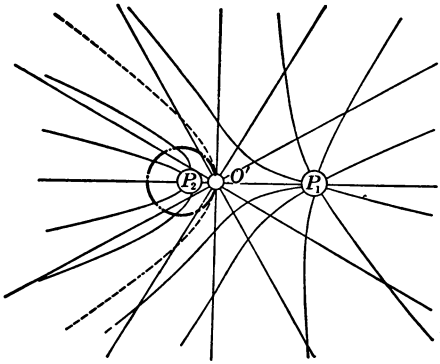
$M_1 : M_2 = 2:-1.$

FIG. 8.



$M_1 : M_2 = 1:1.$

FIG. 7.



$M_1 : M_2 = 4:1.$

FIG. 9.

through $P_1 P_2$ for the particular cases in which M_1 and M_2 of the particular functions (ϕ) have the values indicated. The dotted-and-dashed curve is a section of the sphere (ω) by a plane through $P_1 P_2$.

By Theorem 7, the surface (α) is also the locus of the brilliant points of the right lines through O, and of each of the congruences (θ), (κ), and (μ), with respect to each of the pairs of congruences represented in section in Figs. 6, 7, 8, 9. The sphere shown in section by the dotted-and-dashed curve is excluded

from the region Σ. The direction of propagation of the lines of force being *from* P_1, and that of the right lines through O' being *toward* O', the distribution of the actual and virtual brilliant points is the same with respect to these congruences as with respect to P_1 and P_2.

II. Brilliant Points of Surfaces.

§ 6. *Definitions, classification and conditions.*

As a generalization of Definition a and its complement (see Introduction) we have the following:

Definition 5. Let t be a surface with a continuously turning tangent plane which lies in Σ. A point P on t is said to be a brilliant point of t with respect to the two congruences (2) when the directions of propagation of the two congruences (2) at P lie in one plane through the normal to t at P and the angle formed at P by these two directions is bisected either externally or internally by this normal.*

This definition does not include the cases in which the directions of propagation of the two congruences (2) at P are equal or opposite along any right line through P, except the normal and a tangent line.

If, in the figures of § 1, l_1 and l_2 are tangents at P to the curves of the two congruences (2) which pass through P and l is the normal to t at P, then all the possibilities of Definition 5 are shown in the first, third and fourth figures. Of the two following definitions, the first includes, besides the possibilities of Definition 5, those shown in the second figure; and the second includes, besides the possibilities of Definition 5, those shown in the fifth figure.

Definitions 6. Let t be a surface with a continuously turning tangent plane which lies in Σ. A point P on t is said to be a $\left\{\begin{smallmatrix}\textit{first}\\ \textit{second}\end{smallmatrix}\right\}$ *extended* brilliant point of t with respect to the two congruences (2), when it is an $\left\{\begin{smallmatrix}\textit{extra}\\ \textit{extreme}\end{smallmatrix}\right\}$ brilliant point of a curve c which cuts the surface t orthogonally at P, with respect to the two congruences (2).

The point P is said to be an actual or a virtual brilliant point of t according as it is a virtual or an actual brilliant point of c.

A one-parameter family of surfaces may be represented by the equation

$$F(x, y, z) = C, \tag{50}$$

in which C represents the parameter. Let us assume that the first partial derivatives of F, namely F_x, F_y, F_z, satisfy in Σ the same conditions which

* If we apply this definition to the case in which $X_i = x - a_i$, $Y_i = y - b_i$, $Z_i = z - c_i$, where a_i, b_i, c_i are the coördinates of P_i $(i = 1, 2)$, we get Definition a and its complement.

X, Y, Z satisfy. Then the system of differential equations

$$\left\| \begin{matrix} dx & dy & dz \\ F_x & F_y & F_z \end{matrix} \right\| = 0 \tag{51}$$

represents the congruence of curves which cut the surfaces (50) orthogonally.

From Definitions 6 and Theorems 2 and 4 the following theorems are immediately deduced:

THEOREM 8. *The necessary and sufficient conditions that a point P of the region Σ should be a first extended brilliant point of that one of the surfaces* (50) *which passes through P, with respect to the two congruences* (2), *may be expressed in either of the following two ways: a) the point P satisfies the two equations*

$$\underline{\Omega} = 0, \qquad \underline{\Delta} = 0, \tag{52}$$

where $\underline{\Omega}$ and $\underline{\Delta}$ are the expressions obtained from Ω and Δ respectively by replacing X, Y, Z by F_x, F_y, F_z respectively; b) the point P satisfies either the system of equations

$$\left\| \begin{matrix} F_x & F_y & F_z \\ \overline{X}_1 + \overline{X}_2 & \overline{Y}_1 + \overline{Y}_2 & \overline{Z}_1 + \overline{Z}_2 \end{matrix} \right\| = 0, \tag{53}$$

or the system of equations

$$\left\| \begin{matrix} F_x & F_y & F_z \\ \overline{X}_1 - \overline{X}_2 & \overline{Y}_1 - \overline{Y}_2 & \overline{Z}_1 - \overline{Z}_2 \end{matrix} \right\| = 0. \tag{54}$$

Furthermore, when $U^2 + V^2 + W^2 > 0$, P is an actual or a virtual brilliant according as it satisfies (53) *or* (54).

THEOREM 9. *The necessary and sufficient conditions that a point P of the region Σ should be a second extended brilliant point of that one of the surfaces* (50) *which passes through P, with respect to the two congruences* (2), *is that P satisfies the three equations*

$$\begin{vmatrix} \underline{t}_1 & \underline{U}_2 \\ \underline{t}_2 & \underline{U}_1 \end{vmatrix} = 0, \qquad \begin{vmatrix} \underline{t}_1 & \underline{V}_2 \\ \underline{t}_2 & \underline{V}_1 \end{vmatrix} = 0, \qquad \begin{vmatrix} \underline{t}_1 & \underline{W}_2 \\ \underline{t}_2 & \underline{W}_1 \end{vmatrix} = 0, \tag{55}$$

*where $\underline{t}_i$, $\underline{U}_i$, $\underline{V}_i$, $\underline{W}_i$ are the expressions obtained from t_i, U_i, V_i, W_i $(i = 1, 2)$ respectively by replacing X, Y, Z by F_x, F_y, F_z respectively.**

We may now easily deduce

THEOREM 10. *The necessary and sufficient conditions that a point P of the region Σ should be a brilliant point of that one of the surfaces* (50) *which*

* See corollary and footnote to Theorem 4.

passes through P, *with respect to the two congruences* (2), *may be expressed in either of the following two ways:*

a) The point P *satisfies either the equations* (52) [*or* (53) *or* (54)] *and the inequality* $U^2+V^2+W^2>0$, *or one of the systems of equations*

$$U=V=W=\underline{U}_i=\underline{V}_i=\underline{W}_i=0 \quad (i=1,2), \qquad U=V=W=\underline{t}_i=0 \quad (i=1,2).$$

b) The point P *satisfies either the equations* (55) *and the inequality* $\underline{t}_1\cdot\underline{t}_2\neq 0$, *or the system of equations* $U=V=W=\underline{t}_1=\underline{t}_2=0$.

§ 7. *Invariance of the conditions for brilliant points.*

In § 4 it was shown that the expression Ω is changed into the expression (49) by the substitution of the expressions (43) for $\overline{X}_1$, $\overline{Y}_1$, $\overline{Z}_1$ and the expressions (44) for $\overline{X}_2$, $\overline{Y}_2$, $\overline{Z}_2$. The same substitution changes the expression Δ into the expression

$$\kappa(f_1^2-f_2^2)\Delta,$$

where κ is the non-vanishing expression $1/\sqrt{(X_1^2+Y_1^2+Z_1^2)(X_2^2+Y_2^2+Z_2^2)}$. As the analogon of Theorem 6 we therefore have

THEOREM 11. *If, in the portion of the region* Σ *for which* $U^2+V^2+W^2>0$ *and* $f_1^2-f_2^2\neq 0$, P $\left\{\begin{smallmatrix}is\\ is\ not\end{smallmatrix}\right\}$ *a brilliant point (second extended brilliant point) of that one of the surfaces* (50) *which passes through* P, *with respect to the two congruences* (2), *then it* $\left\{\begin{smallmatrix}is\\ is\ not\end{smallmatrix}\right\}$ *a brilliant point (second extended brilliant point) of the same surface with respect to the two congruences* (47) *and* (48), *for every choice of the functions* f_1 *and* f_2.

§ 8. *An example.*

For the one-parameter family of surfaces (50) let us take the concentric spheres

$$(\varpi) \qquad x^2+y^2+z^2=C,$$

of which the common center is the origin $O=(0,0,0)$. The orthogonal congruence (51) is the two-parameter family of right lines which pass through O. The loci of the extra and extreme brilliant points of this congruence, with respect to the two points P_1 and P_2, were found in § 5. The locus of the extra brilliant points consists of the right line (β) and the cubic (γ). By Theorem 8 these curves are also the locus of the first extended brilliant points of the spheres (ϖ) with respect to P_1 and P_2. The locus of the extreme brilliant points of the right lines through O consists of the cubic (γ) and the circle (δ). By Theorem 9 these curves are also the locus of the second extended brilliant points of the spheres (ϖ) with respect to P_1 and P_2. By Theorem 10 the cubic (γ) is

the locus of the brilliant points of the spheres (ϖ) with respect to P_1 and P_2. The portion of the cubic (γ) which is drawn dotted in Fig. 5 is the locus of the actual, and that which is drawn full, the locus of the virtual brilliant points of the spheres (ϖ).

By Theorem 11, in the portion of space which excludes the right line (β) and the locus $f_1^2 - f_2^2 = 0$, the curve (γ) and the pair of curves (γ), (δ) are the loci of the brilliant points and second extended brilliant points respectively of the spheres (ϖ), with respect to the two congruences (ν) and (ξ), for every choice of the functions f_1 and f_2. Four particular cases of the pairs of congruences (ν), (ξ) are represented in Figs. 6, 7, 8, 9, which have already been described.

III. Special Properties.

§ 9. *Brilliant points as points of contact.*

In the portion of the region Σ for which $U^2 + V^2 + W^2 > 0$, each of the three systems of differential equations

$$(56)\qquad \left\|\begin{matrix} dx & dy & dz \\ \bar{X}_1 - \bar{X}_2 & \bar{Y}_1 - \bar{Y}_2 & \bar{Z}_1 - \bar{Z}_2 \end{matrix}\right\| = 0, \qquad \left\|\begin{matrix} dx & dy & dz \\ \bar{X}_1 + \bar{X}_2 & \bar{Y}_1 + \bar{Y}_2 & \bar{Z}_1 + \bar{Z}_2 \end{matrix}\right\| = 0,$$

$$(57)\qquad \left\|\begin{matrix} dx & dy & dz \\ U & V & W \end{matrix}\right\| = 0,$$

represents a congruence.

Let us suppose that both of the congruences (56) are *normal congruences* * and that the corresponding families of orthogonal surfaces are represented by the equations

$$(58)\qquad F_1(x, y, z) = C_1 \qquad \text{and} \qquad F_2(x, y, z) = C_2$$

respectively. The planes π_1 and π_2 defined in § 4 are the tangent planes at $P = (\xi, \eta, \zeta)$ to the surfaces $F_1(x, y, z) = F_1(\xi, \eta, \zeta)$ and $F_2(x, y, z) = F_2(\xi, \eta, \zeta)$ respectively, these surfaces being the members of the families (58) which pass through P. From this fact flows the following theorem about curves:

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* A congruence

$$\left\|\begin{matrix} dx & dy & dz \\ X & Y & Z \end{matrix}\right\| = 0$$

is a *normal congruence* when there exists a one-parameter family of surfaces which cut orthogonally the curves of the congruence. The necessary and sufficient condition for this is that the equation

$$X\left(\frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}\right) + Y\left(\frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}\right) + Z\left(\frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}\right) = 0$$

is identically satisfied.

THEOREM 12. *In the portion of the region Σ for which $U^2+V^2+W^2>0$, a point $P=(\xi,\eta,\zeta)$ of a curve c is, or is not, a brilliant point of c with respect to the two congruences* (2), *according as c is, or is not, tangent at P to at least one of the two surfaces $F_i(x,y,z)=F_i(\xi,\eta,\zeta)$, $(i=1,2)$. Furthermore if c is tangent at P to only one of the surfaces $F_i(x,y,z)=F_i(\xi,\eta,\zeta)$, $(i=1,2)$, P is actual or virtual according as c is tangent to $F_2(x,y,z)=F_2(\xi,\eta,\zeta)$ or $F_1(x,y,z)=F_1(\xi,\eta,\zeta)$; and if c is tangent at P to both of these surfaces, P is an exceptional brilliant point.*

Let us now suppose that the congruence (57) is also normal and that the corresponding family of orthogonal surfaces is represented by the equation

$$\Psi(x,y,z)=C. \tag{59}$$

Then we have the following theorem about surfaces:

THEOREM 13. *In the portion of the region Σ for which $U^2+V^2+W^2>0$, a point $P=(\xi,\eta,\zeta)$ of a surface t is, or is not,* $\left\{\begin{smallmatrix}\textit{a brilliant point}\\ \textit{a second extended brilliant point}\end{smallmatrix}\right\}$ *of t with respect to the two congruences* (2), *according as t is, or is not, tangent at P to one of the* $\left\{\begin{smallmatrix}\textit{two}\\ \textit{three}\end{smallmatrix}\right\}$ *surfaces* $\left\{\begin{smallmatrix}F_i(x,y,z)=F_i(\xi,\eta,\zeta),\ (i=1,2)\\ F_i(x,y,z)=F_i(\xi,\eta,\zeta),\ (i=1,2),\ \Psi(x,y,z)=\Psi(\xi,\eta,\zeta)\end{smallmatrix}\right\}$. *Furthermore P is actual or virtual according as t is tangent to $F_2(x,y,z)=F_2(\xi,\eta,\zeta)$ or $F_1(x,y,z)=F_1(\xi,\eta,\zeta)$.*

REMARK. *All the facts stated in Theorems* 12 *and* 13 *are true, not only with respect to the two congruences* (2), *but also with respect to the two congruences* (47), (48), *for every choice of the functions f_1 and f_2, provided that the portion of Σ in which $f_1^2-f_2^2=0$ be also excluded.*

In the examples of §§ 5 and 8 the two one-parameter families of surfaces (58) are the ellipsoids and hyperboloids of revolution of which P_1 and P_2 are the common foci and P_1P_2 is the common axis, and the family (59) is the family of planes which pass through P_1P_2. These three families of surfaces form a triply orthogonal system. This is always true of the three families (58), (59), since the planes π_1 and π_2 are perpendicular and intersect in p.

§ 10. *Optical interpretations.*

In the example of § 5, the points P_1 and P_2 may be regarded as a point source of light and the eye of an observer respectively, and the right lines which pass through O, as reflecting wires of small gauge. The observer will then see as a bright surface the locus of the actual brilliant points. If the circles (κ) were reflecting wires, the observer would see as a bright surface the locus of the virtual brilliant points of the right lines which pass through O.

In general the curves of congruence (1) may be regarded as reflecting wires of small gauge, as may also the curves of the congruences (40) and (41). Under certain conditions the curves of congruences (2) may be regarded as rays of light.

The conditions under which the curves of a congruence

$$(60)\qquad \frac{dx}{ds}=\bar{X},\qquad \frac{dy}{ds}=\bar{Y},\qquad \frac{dz}{ds}=\bar{Z},$$

may be regarded as rays of light in a medium of variable index of refraction $n(x, y, z)$ *are those under which the system of partial differential equations**

$$(61)\qquad \begin{aligned}(\bar{X}_1^2-1)\frac{\partial n}{\partial x}+\bar{X}\cdot\bar{Y}\cdot\frac{\partial n}{\partial y}+\bar{X}\cdot\bar{Z}\cdot\frac{\partial n}{\partial z}&=-n\left(\frac{\partial\bar{X}}{\partial x}\bar{X}+\frac{\partial\bar{X}}{\partial y}\bar{Y}+\frac{\partial\bar{X}}{\partial z}\bar{Z}\right)\\ \bar{Y}\bar{X}\frac{\partial n}{\partial x}+(\bar{Y}^2-1)\frac{\partial n}{\partial y}+\bar{Y}\cdot\bar{Z}\cdot\frac{\partial n}{\partial z}&=-n\left(\frac{\partial\bar{Y}}{\partial x}\bar{X}+\frac{\partial\bar{Y}}{\partial y}\bar{Y}+\frac{\partial\bar{Y}}{\partial z}\bar{Z}\right),\\ \bar{Z}\bar{X}\frac{\partial n}{\partial x}+\bar{Z}\cdot\bar{Y}\cdot\frac{\partial n}{\partial y}+(\bar{Z}^2-1)\frac{\partial n}{\partial z}&=-n\left(\frac{\partial\bar{Z}}{\partial x}\bar{X}+\frac{\partial\bar{Z}}{\partial y}\bar{Y}+\frac{\partial\bar{Z}}{\partial z}\bar{Z}\right).\end{aligned}$$

have a common solution.

The system of equations (61) is merely another form of the system of equations

$$(62)\qquad \frac{d}{ds}\left(n\frac{dx}{ds}\right)-\frac{\partial n}{\partial x}=0,\qquad \frac{d}{ds}\left(n\frac{dy}{ds}\right)-\frac{\partial n}{\partial y}=0,\qquad \frac{d}{ds}\left(n\frac{dz}{ds}\right)-\frac{\partial n}{\partial z}=0,$$

which is obtained by minimizing the integral

$$I=\int_{(A)}^{(B)} n(x, y, z)\,ds.\dagger$$

When, in particular, congruence (60) is rectilinear, the coefficients of n in the right members of equations (61) are identically zero. When this is the case, we have the

THEOREM 14. *The necessary and sufficient condition that the lines of a rectilinear congruence may be regarded as rays of light, is that the congruence be a normal congruence.*

In order to prove this, let us denote by u, v, w the determinants of the matrix formed from the coefficients of left members of the last two of the equations (61). They are found to be:

$$u=\bar{X}\cdot\bar{X},\qquad v=\bar{X}\cdot\bar{Y},\qquad w=\bar{X}\cdot\bar{Z}.$$

* These three equations reduce to two, as is verified by multiplying in succession by $\bar{X}$, $\bar{Y}$, $\bar{Z}$, adding, and making use of the relations

$$\bar{X}^2+\bar{Y}^2+\bar{Z}^2=1,\qquad \bar{X}\frac{\partial}{\partial\xi}\bar{X}+\bar{Y}\frac{\partial}{\partial\xi}\bar{Y}+\bar{Z}\frac{\partial}{\partial\xi}\bar{Z}=0\qquad (\xi=x, y, z).$$

† It is assumed that n is positive for all values of x, y, z. See APPELL, *Traité de mécanique rationnelle*, vol. 1, §§ 146, 150; A. WINKELMANN, *Handbuch der Physik* (1893), vol. 2_1, p. 344.

Therefore by a well-known theorem * and identity,† the necessary and sufficient condition that there exists a function n which will make the left members of the last two of equations (61) vanish, is that $\overline{X}^2 D$ vanishes identically, where

$$D = \overline{X}\left(\frac{\partial \overline{Y}}{\partial z} - \frac{\partial \overline{Z}}{\partial y}\right) + \overline{Y}\left(\frac{\partial \overline{Z}}{\partial x} - \frac{\partial \overline{X}}{\partial z}\right) + \overline{Z}\left(\frac{\partial \overline{X}}{\partial y} - \frac{\partial \overline{Y}}{\partial x}\right).$$

For the other two pairs of the left members of equations (61) we find $\overline{Y}^2 \cdot D$ and $\overline{Z}^2 \cdot D$ instead of $\overline{X}^2 \cdot D$. Since not all three of the expressions $\overline{X}, \overline{Y}, \overline{Z}$ vanish at the same point, it follows that $D \equiv 0$. But this is the condition that congruence (60) be normal. Hence the theorem.

By means of a theorem due to LEVI-CIVITA ‡ a congruence which is normal and rectilinear may be refracted (or in particular reflected) into any other congruence which is normal and rectilinear. Therefore any normal rectilinear congruence may be refracted (or reflected) into a two-parameter family of radiating right lines. Therefore if the congruences (2) are normal and rectilinear, it is possible to find surfaces at which they may be refracted or reflected into families of radiating right lines. At the radiants of these two families, a point source of light and the eye of an observer may be placed. The curves of congruence (1) being regarded as reflecting wires of small gauge, the observer whose eye is at one of the radiants will see as a bright surface the locus of the actual brilliant points due to the source of light at the other radiant.

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* THEOREM. *The necessary and sufficient condition that the two independent linear partial differential equations*

$$X_1(x, y, z)\frac{\partial f}{\partial x} + Y_1(x, y, z)\frac{\partial f}{\partial y} + Z_1(x, y, z)\frac{\partial f}{\partial z} = 0,$$

$$X_2(x, y, z)\frac{\partial f}{\partial x} + Y_2(x, y, z)\frac{\partial f}{\partial y} + Z_2(x, y, z)\frac{\partial f}{\partial z} = 0$$

have a common solution is that the equation

$$U\left(\frac{\partial V}{\partial z} - \frac{\partial W}{\partial y}\right) + V\left(\frac{\partial W}{\partial x} - \frac{\partial U}{\partial z}\right) + W\left(\frac{\partial U}{\partial y} - \frac{\partial V}{\partial x}\right) = 0,$$

n which

$$U = \begin{vmatrix} Y_1 & Z_1 \\ Y_2 & Z_2 \end{vmatrix}, \qquad V = \begin{vmatrix} Z_1 & X_1 \\ Z_2 & X_2 \end{vmatrix}, \qquad W = \begin{vmatrix} X_1 & Y_1 \\ X_2 & Y_2 \end{vmatrix},$$

entically satisfied

† Identity:

$$\Theta X\left(\frac{\partial(\Theta Y)}{\partial z} - \frac{\partial(\Theta Z)}{\partial y}\right) + \Theta Y\left(\frac{\partial(\Theta Z)}{\partial x} - \frac{\partial(\Theta X)}{\partial z}\right) + \Theta Z\left(\frac{\partial(\Theta X)}{\partial y} - \frac{\partial(\Theta Y)}{\partial x}\right)$$
$$= \Theta^2\left[X\left(\frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}\right) + Y\left(\frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}\right) + Z\left(\frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}\right)\right].$$

‡ THEOREM. *Two rectilinear congruences (both normal or both non-normal) are always obtainable one from the other by means of a finite number of refractions. More precisely, for the normal congruences one refraction suffices, for the others there occur in general two. The indices of refraction may be taken at pleasure, in particular equal to* -1, *which corresponds to reflection.*

See R. Accademia dei Lincei, *Rendiconti*, ser. 5, vol. 9^1 (1900), pp. 185 and 237.

Another kind of optical interpretation may also be given to the example of § 5 and, under certain broad restrictions, to the general problem of § 2. It is evident from Theorem 12 that the surface (α) of § 5 may be regarded as the locus of the lines of shade of the ellipsoids and hyperboloids of revolution of which P_1 and P_2 are the common foci, with regard to the rays of light which emanate from a point source at O. The portion of the surface (α) which is the locus of the actual brilliant points is the locus of the lines of shade of the ellipsoids, and that which is the locus of the virtual brilliant points is the locus of the lines of shade of the hyperboloids. Thus we deduce

A property of surface (α). The surface (α), § 5, has a generation of conics, the conic-generator being the intersection of the quadric surface of revolution of which P_1 and P_2 are the foci with the polar plane of this quadric surface with respect to the origin O. These conic-generators are in general not circles.

If the congruences (56) are normal, surfaces exist for the general problem of § 2 which play the rôle of the ellipsoids and hyperboloids in the example of § 5. These are the surfaces (58). The locus of the brilliant points of congruence (1) with respect to the two congruences (2) may then be regarded as the locus of the lines of shade of the surfaces (58) with regard to the congruence (1), or any of the congruences (40) or (41), for every choice of the functions m and n.

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