PROJECTIVE DIFFERENTIAL GEOMETRY OF CURVED SURFACES*

(THIRD MEMOIR)

BY

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§ 1. *Canonical development of the equation of a ruled surface.*

In the second memoir† it was shown that, in the vicinity of an ordinary point, the equation of any non-ruled surface could be expressed by a series of the form

$$z = xy + \tfrac{1}{6}(x^3 + y^3) + \tfrac{1}{24}(Ix^4 + Jy^4) + \cdots,$$

where I, J and all of the following coefficients are absolute invariants of the surface. The geometrical significance of this development was fully brought out and gave rise to a number of new and important results. For a ruled surface, however, no series of this form exists. It is the purpose of this paper to find a canonical development for this exceptional case and to interpret it geometrically.

We shall assume that the partial differential equations of the surface are given in their canonical form

$$y_{uu} + 2by_v + fy = 0,$$

$$y_{vv} + 2a'y_u + gy = 0,$$

so that the curves $u = \text{const.}$ and $v = \text{const.}$ are the two families of asymptotic curves. The curves $u = \text{const.}$ will be straight lines, if and only if $a' = 0$.‡ This we shall assume from now on, so that our fundamental system of equations is

(1) $$y_{uu} + 2by_v + fy = 0, \qquad y_{vv} + gy = 0.$$

These equations will characterize a ruled surface S whose generators are the curves $u = \text{const.}$ The integrability conditions of this system are §

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† These Transactions, vol. 9 (1908), p. 79.

§ First Memoir. These Transactions, vol. 8 (1907), p. 260.

‡ These Transactions, vol. 8 (1907), p. 247, eq. (34). In the last equation f and g should be interchanged.

$$(2) \qquad g_u = 0, \qquad b_{vv} + f_v = 0, \qquad -f_{vv} + 4gb_v + 2bg_v = 0.$$

They are supposed to be fulfilled so that, in particular, g will be a function of v only. The four fundamental semi-covariants are

$$(3) \qquad y, \qquad z = y_u, \qquad \rho = y_v, \qquad \sigma = y_{uv}.$$

Along a generator u remains constant, so that the line joining P_y to P_ρ is a generator of the surface and we may think of this surface as being the locus of the lines $P_y P_\rho$ which correspond to the various values of u.

For the purpose of our development we shall need the partial derivatives of y up to those of the fifth order. We may use the expressions for the derivatives of the first four orders which were obtained in the second memoir † and simplify them by putting $a' = 0$. This gives

$$(4) \quad y_u = z, \qquad y_v = \rho, \qquad y_{uu} = -fy - 2b\rho, \qquad y_{uv} = \sigma, \qquad y_{vv} = -gy,$$

and

$$(5) \quad \begin{aligned} y_{uuu} &= -f_u y - fz - 2b_u\rho - 2b\sigma, & y_{uuv} &= (2bg - f_v)y - (f + 2b_v)\rho, \\ y_{uvv} &= -gz, & y_{vvv} &= -g_v y - g\rho. \end{aligned}$$

We have further

$$(6) \quad \begin{aligned} &y_{uuuu} = \Sigma\alpha_1 y, \qquad y_{uuuv} = \Sigma\beta_1 y, \qquad y_{uuvv} = \Sigma\gamma_1 y, \\ &y_{uvvv} = \Sigma\delta_1 y, \qquad y_{vvvv} = \Sigma\epsilon_1 y, \end{aligned}$$

where the symbols on the right members are abbreviations for expressions of the form

$$\alpha_1 y + \alpha_2 z + \alpha_3 \rho + \alpha_4 \sigma, \text{ etc.},$$

and where

$$(7) \quad \begin{array}{llll} \alpha_1 = f^2 - 4b^2 g + 2bf_v - f_{uu}, & \alpha_2 = -2f_u, & \alpha_3 = 2(2bf + 2bb_v - b_{uu}), & \alpha_4 = -4b_u, \\ \beta_1 = 2b_u g - f_{uv}, & \beta_2 = 2bg - f_v, & \beta_3 = -(f_u + 2b_{uv}), & \beta_4 = -(f + 2b_v), \\ \gamma_1 = fg, & \gamma_2 = 0, & \gamma_3 = 2bg, & \gamma_4 = 0, \\ \delta_1 = 0, & \delta_2 = -g_v, & \delta_3 = 0, & \delta_4 = -g, \\ \epsilon_1 = -g_{vv} + g^2, & \epsilon_2 = 0, & \epsilon_3 = -2g_v, & \epsilon_4 = 0. \end{array}$$

Similarly we may write

$$(8) \quad \begin{aligned} &y_{uuuuu} = \Sigma\alpha_1' y, \qquad y_{uuuuv} = \Sigma\beta_1' y, \qquad y_{uuuvv} = \Sigma\gamma_1' y, \\ &y_{uuvvv} = \Sigma\delta_1' y, \qquad y_{uvvvv} = \Sigma\epsilon_1' y, \qquad y_{vvvvv} = \Sigma\zeta_1' y, \end{aligned}$$

where

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* These Transactions, vol. 9 (1908), pp. 98–99, eq. (78) to (80).

$$(9)\qquad \begin{aligned} \alpha_4' &= \frac{\partial \alpha_4}{\partial u} + \alpha_3, & \beta_4' &= \frac{\partial \alpha_4}{\partial v} + \alpha_2, & \gamma_4' &= \frac{\partial \beta_4}{\partial v} + \beta_2, \\ \delta_4' &= \frac{\partial \gamma_4}{\partial v} + \gamma_2, & \epsilon_4' &= \frac{\partial \delta_4}{\partial v} + \delta_2, & \zeta_4' &= \frac{\partial \epsilon_4}{\partial v} + \epsilon_2, \end{aligned}$$

or explicitly,

$$(10)\qquad \begin{aligned} \alpha_4' &= -6b_{uu} + 4bf + 4bb_v, & \beta_4' &= -4b_{uv} - 2f_u, & \gamma_4' &= 2bg, \\ \delta_4' &= 0, & \epsilon_4' &= -2g_v, & \zeta_4' &= 0, \end{aligned}$$

while the expressions of the other coefficients in (8) are not needed for our present purpose.

Let us consider the point $u = u_0$, $v = v_0$ of an integral surface of system (1). The values of y, y_u, y_v, etc., for this point can be computed by the formulæ just developed. Let Y or (u, v) represent a point in the vicinity of (u_0, v_0), and let us assume that the point (u_0, v_0) is an ordinary point of the surface, i. e., let us assume that the coefficients of (1) are regular about this point. Then we shall have by TAYLOR's development

$$Y = y + y_u(u - u_0) + y_v(v - v_0) + \cdots,$$

But, making use of (4) to (8), this may be put into the form

$$Y = x_1 y + x_2 z + x_3 \rho + x_4 \sigma,$$

where $x_1, \cdots, x_4$ are series proceeding according to positive integral powers of $u - u_0$ and $v - v_0$. As in previous papers we shall introduce $P_y P_z P_\rho P_\sigma$ as tetrahedron of reference in such a way that $(x_1, \cdots, x_4)$ become the homogeneous coördinates of the point Y with respect to it. Moreover, in place of $u - u_0$ and $v - v_0$ we shall write u and v respectively.

We obtain in this way the following series for the homogeneous coördinates of a point of the surface in the vicinity of P_y, the terms up to and including those of the fifth order being actually computed:

$$(11)\qquad \begin{aligned} x_1 &= 1 - \tfrac{1}{2} f u^2 - \tfrac{1}{2} g v^2 - \tfrac{1}{6} f_u u^3 + \tfrac{1}{2}(2bg - f_v) u^2 v - \tfrac{1}{6} g_v v^3 \\ &\quad + \tfrac{1}{24}(\alpha_1 u^4 + 4\beta_1 u^3 v + 6\gamma_1 u^2 v^2 + 4\delta_1 u v^3 + \epsilon_1 v^4) \\ &\quad + \tfrac{1}{120}(\alpha_1' u^5 + 5\beta_1' u^4 v + 10\gamma_1' u^3 v^2 + 10\delta_1' u^2 v^3 + 5\epsilon_1' u v^4 + \zeta_1' v^5) + \cdots, \\ x_2 &= u - \tfrac{1}{6} f u^3 - \tfrac{1}{2} g u v^2 + \tfrac{1}{24}(\alpha_2 u^4 + 4\beta_2 u^3 v + 6\gamma_2 u^2 v^2 + 4\delta_2 u v^3 + \epsilon_2 v^4) \\ &\quad + \tfrac{1}{120}(\alpha_2' u^5 + 5\beta_2' u^4 v + 10\gamma_2' u^3 v^2 + 10\delta_2' u^2 v^3 + 5\epsilon_2' u v^4 + \zeta_2' v^5) + \cdots \\ x_3 &= v - b u^2 - \tfrac{1}{3} b_u u^3 - \tfrac{1}{2}(f + 2b_v) u^2 v - \tfrac{1}{6} g v^3 \\ &\quad + \tfrac{1}{24}(\alpha_3 u^4 + 4\beta_3 u^3 v + 6\gamma_3 u^2 v^2 + 4\delta_3 u v^3 + \epsilon_3 v^4) \end{aligned}$$

$$+\tfrac{1}{120}(\alpha_3' u^5 + 5\beta_3' u^4 v + 10\gamma_3' u^3 v^2 + 10\delta_3' u^2 v^3 + 5\epsilon_3' u v^4 + \zeta_3' v^5) + \cdots,$$

$$x_4 = uv - \tfrac{1}{3}bu^3 + \tfrac{1}{24}(\alpha_4 u^4 + 4\beta_4 u^3 v + 6\gamma_4 u^2 v^2 + 4\delta_4 u v^3 + \epsilon_4 v^4)$$
$$+\tfrac{1}{120}(\alpha_4' u^5 + 5\beta_4' u^4 v + 10\gamma_4' u^3 v^2 + 10\delta_4' u^2 v^2 + 5\epsilon_4' u v^4 + \zeta_4' v^5) + \cdots.$$

We find therefore

$$(12)\quad \frac{1}{x_1} = 1 + \tfrac{1}{2}fu^2 + \tfrac{1}{2}gv^2 + \tfrac{1}{6}f_u u^3 - (bg - \tfrac{1}{2}f_v)u^2 v + \tfrac{1}{6}g_v v^3 + \tfrac{1}{24}(6f^2 - \alpha_1)u^4$$
$$-\tfrac{1}{6}\beta_1 u^3 v + \tfrac{1}{4}(2fg - \gamma_1)u^2 v^2 - \tfrac{1}{6}\delta_1 u v^3 + \tfrac{1}{24}(6g^2 - \epsilon_1)v^4 + \cdots;$$

whence, if we introduce non-homogeneous coördinates

$$\xi = \frac{x_2}{x_1} = u + \tfrac{1}{3}fu^3 + \tfrac{1}{24}(4f_u + \alpha_2)u^4 + \tfrac{1}{6}(\beta_2 - 6bg + 3f_v)u^3 v + \tfrac{1}{4}\gamma_2 u^2 v^2$$
$$+\tfrac{1}{6}(g_v + \delta_2)uv^3 + \tfrac{1}{24}\epsilon_2 v^4 + \tfrac{1}{120}(20f^2 - 5\alpha_1 + \alpha_2')u^5$$
$$+\tfrac{1}{24}(\beta_2' - 4\beta_1)u^4 v + \tfrac{1}{12}(2fg + \gamma_2' - 3\gamma_1)u^3 v^2 + \tfrac{1}{12}(\delta_2' - 2\delta_1)u^2 v^3$$
$$+\tfrac{1}{24}(\epsilon_2' - \epsilon_1)uv^4 + \tfrac{1}{120}\zeta_2' v^5 + \cdots,$$

$$\eta = \frac{x_3}{x_1} = v - bu^2 - \tfrac{1}{3}b_u u^3 - b_v u^2 v + \tfrac{1}{3}gv^3 + \tfrac{1}{24}(\alpha_3 - 12bf)u^4 + \tfrac{1}{6}(f_u + \beta_3)u^3 v$$
$$+\tfrac{1}{4}(2f_v - 6bg + \gamma_3)u^2 v^2 + \tfrac{1}{6}\delta_3 uv^3 + \tfrac{1}{24}(\epsilon_3 + 4g_v)v^4 + \tfrac{1}{120}(\alpha_3' - 40bf_u)u^5$$
$$(13)\quad +\tfrac{1}{24}(\beta_3' - \alpha_1 + 24b^2 g - 12bf_v - 12b_v f)u^4 v + \tfrac{1}{12}(\gamma_3' - 2\beta_1 - 2b_u g)u^3 v^2$$
$$+\tfrac{1}{12}(\delta_3' - 3\gamma_1 + 2fg - 2g_v b - 6gb_v)u^2 v^3 + \tfrac{1}{24}(\epsilon_3' - 4\delta_1)uv^4$$
$$+\tfrac{1}{120}(\zeta_3' - 5\epsilon_1 + 20g^2)v^5 + \cdots,$$

$$\zeta = \frac{x_4}{x_1} = uv - \tfrac{1}{3}bu^3 + \tfrac{1}{24}\alpha_4 u^4 + \tfrac{1}{6}(\beta_4 + 3f)u^3 v + \tfrac{1}{4}\gamma_4 u^2 v^2 + \tfrac{1}{6}(\delta_4 + 3g)uv^3$$
$$+\tfrac{1}{24}\epsilon_4 v^4 + \tfrac{1}{120}(\alpha_4' - 20bf)u^5 + \tfrac{1}{24}(\beta_4' + 4f_u)u^4 v$$
$$+\tfrac{1}{12}(\gamma_4' - 14bg + 6f_v)u^3 v^2 + \tfrac{1}{12}\delta_4' u^2 v^3 + \tfrac{1}{24}(\epsilon_4' + 4g_v)uv^4$$
$$+\tfrac{1}{120}\zeta_4' v^5 + \cdots.$$

This gives

$$\zeta - \xi\eta = \tfrac{2}{3}bu^3 + \tfrac{1}{6}b_u u^4 + \tfrac{2}{3}b_v u^3 v + \tfrac{1}{30}(b_{uu} + 16bf - 4bb_v)u^5 + \tfrac{1}{6}b_{uv}u^4 v$$
$$+\tfrac{1}{6}(4bg - 3f_v - b_{vv})u^3 v^2 + \cdots,$$

$$(14)\quad \xi^3 = u^3 + fu^5 + \cdots, \qquad \xi^3\eta = u^3 v - bu^5 + \cdots,$$
$$\xi^4 = u^4 + \cdots, \qquad \xi^3\eta^2 = u^3 v^2 + \cdots,$$
$$\xi^5 = u^5 + \cdots, \qquad \xi^4\eta = u^4 v + \cdots,$$

so that we obtain the following development

$$(15)\quad \begin{aligned}\zeta = \xi\eta + \tfrac{2}{3}b\xi^3 + \tfrac{1}{6}b_u\xi^4 + \tfrac{2}{3}b_v\xi^3\eta + \tfrac{1}{30}(b_{uu} - 4bf + 16bb_v)\xi^5 + \tfrac{1}{6}b_{uv}\xi^4\eta \\ + \tfrac{1}{6}(4bg - 3f_v - b_{vv})\xi^3\eta^2 + \cdots.\end{aligned}$$

Notice that, in this series, the terms which involve ξ^0, ξ, η, ξ^2, η^2, $\xi^2\eta$, $\xi\eta^2$, η^3, $\xi^2\eta^2$, $\xi\eta^3$, η^4, $\xi^2\eta^3$, $\xi\eta^4$, η^4 are absent, while the coefficient of $\xi\eta$ is unity. Just as in the case of a non-ruled surface, it may be shown that the most general transformation of coördinates which preserves this form of the development is the following:

$$(16)\quad x = \frac{\lambda(\xi + \beta\zeta)}{1+\alpha\xi+\beta\eta+\alpha\beta\zeta}, \qquad y = \frac{\mu(\eta + \alpha\zeta)}{1+\alpha\xi+\beta\eta+\alpha\beta\zeta}, \qquad z = \frac{\lambda\mu\zeta}{1+\alpha\xi+\beta\eta+\alpha\beta\zeta},$$

where α, β, λ, μ are constants at our disposal, which we shall choose in such a way as to simplify further the form of the development.

We find, to terms of the third order inclusive,

$$(17)\quad \begin{aligned}\frac{1}{1 + \alpha\xi + \beta\eta + \alpha\beta\zeta} = 1 - \alpha\xi - \beta\eta + \alpha^2\xi^2 + \alpha\beta\xi\eta + \beta^2\eta^2 \\ - (\alpha^3 + \tfrac{2}{3}b\alpha\beta)\xi^3 - \alpha^2\beta\xi^2\eta - \alpha\beta^2\xi\eta^2 - \beta^3\eta^3 + \cdots,\end{aligned}$$

so that we shall have, to terms of the fourth order,

$$(18)\quad \begin{aligned}\frac{x}{\lambda} &= \xi - \alpha\xi^2 + (\alpha^2 + \tfrac{2}{3}b\beta)\xi^3 + \tfrac{1}{6}(-8b\alpha\beta - 6\alpha^3 + b_u\beta)\xi^4 + \tfrac{2}{3}(-b\beta^2 + b_v\beta)\xi^3\eta + \cdots, \\ \frac{y}{\mu} &= \eta - \beta\eta^2 + \tfrac{2}{3}\alpha b\xi^3 + \beta^2\eta^3 + \tfrac{1}{6}(-4\alpha^2 b + b_u\alpha)\xi^4 \\ &\qquad + \tfrac{2}{3}(-2b\alpha\beta + b_v\alpha)\xi^3\eta - \beta^3\eta^4 + \cdots,\end{aligned}$$

whence, to terms of the fifth order inclusive,

$$\begin{aligned}\frac{xy}{\lambda\mu} &= \xi\eta - \alpha\xi^2\eta - \beta\xi\eta^2 + \tfrac{2}{3}\alpha b\xi^4 + (\alpha^2 + \tfrac{2}{3}b\beta)\xi^3\eta + \alpha\beta\xi^2\eta^2 + \beta^2\xi\eta^3 \\ &\quad + \tfrac{1}{6}\alpha(b_u - 8b\alpha)\xi^5 + \tfrac{1}{6}(b_u\beta + 4b_v\alpha - 16b\alpha\beta - 6\alpha^3)\xi^4\eta \\ &\quad + \tfrac{1}{3}(-4b\beta^2 + 2b_v\beta - 3\alpha^2\beta)\xi^3\eta^2 - \alpha\beta^2\xi^2\eta^3 - \beta^3\xi\eta^4 + \cdots,\end{aligned}$$

$$(19)\quad \begin{aligned}\frac{z}{\lambda\mu} &= \xi\eta + \tfrac{2}{3}b\xi^3 - \alpha\xi^2\eta - \beta\xi\eta^2 + \tfrac{1}{6}(b_u - 4\alpha b)\xi^4 + \tfrac{1}{3}(3\alpha^2 - 2b\beta + 2b_v)\xi^3\eta \\ &\quad + \alpha\beta\xi^2\eta^2 + \beta^2\xi\eta^3 + \tfrac{1}{30}(b_{uu} - 4bf + 16bb_v - 5b_u\alpha + 20b\alpha^2)\xi^5 \\ &\quad + \tfrac{1}{6}(-6\alpha^3 - b_u\beta - 4b_v\alpha + b_{uv})\xi^4\eta + \tfrac{1}{6}(4bg - 3f_v - b_{vv} - 6\alpha^2\beta \\ &\quad + 4b\beta^2 - 4b_v\beta)\xi^3\eta^2 - \alpha\beta^2\xi^2\eta^3 - \beta^3\xi\eta^4 + \cdots.\end{aligned}$$

Consequently we obtain:

$$
(20)\qquad \begin{aligned}\frac{z-xy}{\lambda\mu} = \tfrac{4}{3}b\xi^3 + \tfrac{1}{6}(b_u - 8ab)\xi^4 + \tfrac{4}{3}(b_v - 2b\beta)\xi^3\eta + \tfrac{1}{30}(b_{uu} - 4bf\\ + 16bb_v - 10b_u\alpha + 60b\alpha^2)\xi^5 + \tfrac{1}{6}(b_{uv} - 8b_v\alpha - 2b_u\beta + 16b\alpha\beta)\xi^4\eta\\ + \tfrac{1}{6}(4bg - 3f_v - b_{vv} - 8b_v\beta + 12b\beta^2)\xi^3\eta^2 + \cdots.\end{aligned}
$$

Again we find, to terms of the fifth order inclusive,

$$
\left(\frac{x}{\lambda}\right)^3 = \xi^3 - 3\alpha\xi^4 + (6\alpha^2 + 2b\beta)\xi^5 + \cdots,
$$

$$
\left(\frac{x}{\lambda}\right)^4 = \xi^4 - 4\alpha\xi^5 + \cdots, \quad \left(\frac{x}{\lambda}\right)^5 = \xi^5 + \cdots, \quad \left(\frac{x}{\lambda}\right)^4\frac{y}{\mu} = \xi^4\eta + \cdots,
$$

$$
\left(\frac{x}{\lambda}\right)^3\left(\frac{y}{u}\right)^2 = \xi^3\eta^2 \cdots, \quad \left(\frac{x}{\lambda}\right)^2\left(\frac{y}{\mu}\right)^3 = \xi^2\eta^3 + \cdots, \quad \frac{x}{\lambda}\left(\frac{y}{\mu}\right)^4 = \xi\eta^4 + \cdots
$$

whence

$$
\xi^4 = \left(\frac{x}{\lambda}\right)^4 + 4\alpha\left(\frac{x}{\lambda}\right)^5 + \cdots,
$$

$$
\xi^3 = \left(\frac{x}{\lambda}\right)^3 + 3\alpha\left(\frac{x}{\lambda}\right)^4 + (6\alpha^2 - b_v)\left(\frac{x}{\lambda}\right)^5 + \cdots.
$$

Let us substitute these values in (20), and put

$$
(21)\qquad \beta = \frac{b_v}{2b}
$$

as in the case of a non-ruled surface. Then (20) becomes

$$
(22)\qquad \begin{aligned}\frac{z-xy}{\lambda\mu} = \tfrac{4}{3}b\left(\frac{x}{\lambda}\right)^3 + \tfrac{1}{6}(b_u + 4b\alpha)\left(\frac{x}{\lambda}\right)^4 + \tfrac{1}{30}(20b\alpha^2 + 10b_u\alpha + b_{uu} - 4bf - 4bb_v)\left(\frac{x}{\lambda}\right)^5\\ + \tfrac{1}{6}\left(b_{uv} - \frac{b_ub_v}{b}\right)\left(\frac{x}{\lambda}\right)^4\frac{y}{\mu} + \tfrac{1}{6}\left(4bg - 3f_v - b_{vv} - \frac{b_v^2}{b}\right)\left(\frac{x}{\lambda}\right)^3\left(\frac{y}{\mu}\right)^2 + \cdots.\end{aligned}
$$

We may choose α so as to make the coefficient of $(x/\lambda)^4$ equal to zero, i. e., we may put

$$
(23)\qquad \alpha = -\frac{b_u}{4b}.
$$

Then the series becomes

$$
(24)\qquad \frac{z-xy}{\lambda\mu} = \tfrac{4}{3}b\left(\frac{x}{\lambda}\right)^3 - \tfrac{2}{15}\frac{h}{b}\left(\frac{x}{\lambda}\right)^5 + \frac{C'}{48}\left(\frac{x}{\lambda}\right)^4\frac{y}{\mu} - \frac{\theta'}{384b}\left(\frac{x}{\lambda}\right)^3\left(\frac{y}{\mu}\right)^2 + \cdots,
$$

where h, C', and θ' are the same quantities (invariants) which have been denoted by these symbols in the first and second memoirs, viz.;

$$
\begin{aligned}
h &= b^2(f + b_v) - \tfrac{1}{4}bb_{uu} + \tfrac{5}{16}b_u^2, \\
\theta' &= 64\,[b_v^2 + 2bf_v - 4b^2 g], \qquad (25)\\
C' &= 8\left(b_{uv} - \frac{b_u b_v}{b}\right).
\end{aligned}
$$

Of course, in making the substitutions (21) and (23) it is assumed that b is not equal to zero; this means that the point of the ruled surface, in the vicinity of which the development (24) is valid, is not a flecnode.* Any ruled surface which is not a quadric will therefore admit of a development of the form (24) in the vicinity of any one of its ordinary points which is not a flecnode, since the quadric is the only ruled surface all of whose points are flecnodes.

The constants λ and μ are still at our disposal. We shall choose them in such a way as to make the coefficients of the development absolute invariants. If h is different from zero, assume

$$\lambda^2 = \tfrac{4}{3}b\mu, \qquad \lambda^4 = -\frac{2h}{15b}\mu,$$

whence

$$\lambda = \frac{1}{b}\sqrt{-\tfrac{1}{5}h}, \qquad \mu = -\frac{3h}{10b^3}, \qquad (26)$$

where either of its two values may be given to the square root. This gives us the final form of the canonical development for $h \neq 0$,

$$z = xy + x^3 + x^5 - \frac{5b^3C'}{48h\sqrt{-\frac{1}{5}h}}x^4y - \frac{5^2b^4\theta'}{2^6\cdot 3^2 h^2}x^3y^2 + \cdots. \qquad (27)$$

The equation of any ruled surface may be developed in two ways into a power-series of form (27) *in the vicinity of any ordinary point of the surface which is not a flecnode and for which the invariant h does not vanish.*

If $h = 0$ while $b \neq 0$, (24) is still valid. Assume $C' \neq 0$, and determine λ and μ so that

$$\lambda^2 = \tfrac{4}{3}b\mu, \qquad \lambda^3 = \frac{C'}{48},$$

whence

$$\lambda = \tfrac{1}{2}\sqrt[3]{\tfrac{1}{6}C'}, \qquad \mu = \frac{3\sqrt[3]{\frac{1}{36}C'^2}}{8b},$$

which gives *the canonical development for this case* ($h = 0$, $C' \neq 0$),

*Cf. second memoir, eq. (15). This gives $u'_{12} = -8b$. But $u'_{12} = 0$ is the condition that P_y be a flecnode. (*Proj. Diff. Geom.*, p. 150.)

(28) $$z = xy + x^3 + x^4 y - \frac{\theta'}{6\,C'\sqrt[3]{\frac{1}{6}C'}}x^3y^2 + \cdots.$$

If $h = C' = 0$, $\theta' \neq 0$ we put

$$\lambda^2 = \tfrac{2}{3}b\mu, \qquad \lambda^2\mu = -\frac{\theta'}{384b},$$

with the resulting development

(29) $$z = xy + x^3 + x^3y^2 + \cdots.$$

If $h = C' = \theta' = 0$ the establishment of a definite canonical form would require the computation of terms of the sixth order. We shall see, however, in a later paragraph that if the conditions

$$b \neq 0, \qquad h = C' = \theta' = 0,$$

are fulfilled identically, not only will the development

(30) $$z = xy + x^3$$

be exact up to terms of the fifth order, inclusive, but the series will break off at this point, so that the surface is of the third degree. Finally if the condition $b = 0$ is fulfilled identically, the surface is a quadric

(31) $$z = xy.$$

Every ruled surface may be represented by at least one of the developments (27), (28), (29), (30), *or* (31) *in the vicinity of one of its ordinary points. The conditions under which one or the other of these developments breaks down are given by the vanishing of one or several of the invariants* b, h, c', θ'. *The geometrical significance of these conditions will become apparent subsequently.*

§ 2. *Geometrical determination of the canonical tetrahedron. The osculating Cayley's cubic scroll.*

The canonical development having been found, we proceed to investigate the geometrical significance of the tetrahedron of reference which is connected with it. As in the case of the non-ruled surfaces it becomes necessary, for this purpose, to consider a unodal cubic surface which has contact of the fourth order with the given surface S at the given point P.

The equation

(32) $$Z(\omega+pX+qY+rZ)^2+\phi(X+\lambda Z,\ Y+\mu Z,\ \omega+pX+qY+rZ)=0,$$

in which p, q, r, λ, μ are arbitrary constants and where ϕ denotes a general ternary cubic involving ten further arbitrary constants, represents the most general cubic surface with a unode. The intersection of the three planes

(33) $$x + \lambda Z = 0, \qquad Y + \mu Z = 0, \qquad \omega + pX + qY + rZ = 0$$

will be its unode, and the plane

$$\omega + pX + qY + rZ = 0 \tag{34}$$

its uniplane.*

We may introduce homogeneous coördinates into (27) by putting

$$x = \frac{X}{\omega}, \qquad y = \frac{Y}{\omega}, \qquad z = \frac{Z}{\omega}. \tag{35}$$

We intend to determine the fifteen constants of (32) in such a way that the unodal cubic shall have contact of the fourth order with S at P. Then, to begin with, the plane $Z = 0$ (the tangent plane) must intersect the cubic surface in a curve which has a double point at $X = Y = 0$ (the point of contact), the tangents at the double point being $X = 0$ and $Y = 0$ respectively (the two asymptotic tangents of the surface S at the point P). Now the curve, in which $Z = 0$ intersects the cubic surface (32), is

$$Z = 0, \qquad \phi(X, Y, \omega + pX + qY) = 0.$$

The conditions just mentioned require that ϕ shall be of the form

$$aX^3 + bY^3 + dX^2Y + eXY^2 + lXY(\omega + pX + qY),$$

so that our unodal cubic will have an equation of the form

$$\begin{aligned} Z(\omega+pX+qY+rZ)^2&+a(X+\lambda Z)^3+b(Y+\mu Z)^3+d(X+\lambda Z)^2(Y+\mu Z) \\ &+e(X+\lambda Z)(Y+\mu Z)^2+l(X+\lambda Z)(Y+\mu Z)(\omega+pX+qY+rZ) = 0, \end{aligned}$$

or in non-homogeneous coördinates

$$\begin{aligned} z(1+px+qy+rz)^2&+ a(x+\lambda z)^3+ b(y+\mu z)^3+ d(x+\lambda z)^2(y+\mu z) \\ &+ e(x+\lambda z)(y+\mu z)^2 + l(x+\lambda z)(y+\mu z)(1+px+qy+rz) = 0. \end{aligned} \tag{36}$$

If this cubic surface has fourth order contact with S at P the development (27), when substituted for z in (36), must satisfy the equation identically up to terms of the fourth order inclusive in x and y. The conditions that this be so are found to be as follows:

$$\begin{gathered} l+1=0, \qquad a+1=0, \qquad b=0, \\ 2p+d+l\mu+pl=0, \qquad 2q+e+l(\lambda+q)=0, \\ p^2+2q+3a\lambda+d\mu+l(\lambda+p\mu)=0, \\ 2(pq+r)+2\lambda d+2\mu e+l(\lambda\mu+p\lambda+q\mu+r)=0, \\ q^2+3b\mu+e\lambda+lq\lambda=0, \qquad 2p+l\mu=0, \end{gathered} \tag{37}$$

* Second Memoir, eq. (104).

the solution of which offers no difficulty. We find:

(38) $a=-1,\ b=0,\ l=-1,\ \lambda=\frac{1}{2}p^2,\ \mu=2p,\ d=p,\ e=0,\ q=\frac{1}{2}p^2,\ r=\frac{1}{2}p^3,$

so that (in homogeneous form),

$$
(39)\quad \begin{aligned} F\equiv Z(\omega+pX+\tfrac{1}{2}p^2Y+\tfrac{1}{2}p^3Z)^2-(X+\tfrac{1}{2}p^2Z)^3+p(X+\tfrac{1}{2}p^2Z)^2(Y+2pZ)& \\ -(X+\tfrac{1}{2}p^2Z)(Y+2pZ)(\omega+pX+\tfrac{1}{2}p^2Y+\tfrac{1}{2}p^3Z)&=0 \end{aligned}
$$

is the equation of the most general unodal cubic surface which has contact of the fourth order with the ruled surface S at one of its points P. Since p remains arbitrary, we see that there exists a singly infinite family of such surfaces.

In order to investigate these surfaces a little more in detail, let us put

$$(40)\quad \xi=X+\tfrac{1}{2}p^2Z,\quad \eta=Y+2pZ,\quad \zeta=Z,\quad \Omega=\omega+pX+\tfrac{1}{2}p^2Y+\tfrac{1}{2}p^3Z,$$

so that (39) becomes

$$(41)\qquad F\equiv\zeta\Omega^2-\xi^3+p\xi^2\eta-\xi\eta\Omega=0.$$

Indicate the partial derivatives of F by subscripts. Then

$$
\begin{aligned}
F_\xi&=-3\xi^2+2p\xi\eta-\eta\Omega, & F_\eta&=p\xi^2-\xi\Omega,\\
F_\zeta&=\Omega^2, & F_\Omega&=2\zeta\Omega-\xi\eta.
\end{aligned}
$$

These equations show that (41) has a nodal line $\xi=\Omega=0$ for every value of p, so that all of these cubic surfaces are scrolls. The locus of these nodal lines

$$\xi=X+\tfrac{1}{2}p^2Z=0,\qquad \Omega=\omega+\tfrac{1}{2}p^2Y+p(X+\tfrac{1}{2}p^2Z)=0,$$

obtained by eliminating p, is the hyperboloid

$$\omega Z-XY=0$$

which osculates the ruled surface S along the generator g that passes through P. We find further

$$
\begin{aligned}
F_{\xi\xi}&=-6\xi+2p\eta, & F_{\eta\eta}&=0, & F_{\zeta\zeta}&=0, & F_{\Omega\Omega}&=2\zeta.\\
F_{\xi\eta}&=2p\xi-\Omega, & F_{\eta\zeta}&=0, & F_{\zeta\Omega}&=2\Omega,\\
F_{\xi\zeta}&=0, & F_{\eta\Omega}&=-\xi,\\
F_{\xi\Omega}&=-\eta.
\end{aligned}
$$

Let $(0,\eta',\zeta',0)$ be a point of the nodal line. The equation of the pair of planes tangent to (41) at this point will be

$$\eta'(p\xi^2 - \xi\Omega) + \zeta'\Omega^2 = 0. \tag{42}$$

which will coincide if and only if

$$\eta'(\eta' - 4p\zeta') = 0, \tag{43}$$

so that the ruled cubic will have two distinct unodes

$$(0, 0, 1, 0) \qquad \text{and} \qquad (0, 4p, 1, 0),$$

the two *pinch points* of the cubic scroll, if p is not equal to zero. If p is equal to zero, the two pinch-points coincide, so that we obtain a CAYLEY's *cubic scroll.* Equation (42) then becomes

$$\Omega(-\eta'\xi + \zeta'\Omega) = 0$$

or

$$\omega(-Y'X + Z'\omega) = 0,$$

so that the plane $\omega = 0$ is tangent to the ruled cubic along the whole nodal line, while the other plane touches it at just one point. We shall speak of $\omega = 0$ as the *singular tangent plane* of the CAYLEY cubic. The equation of the CAYLEY cubic itself becomes

$$Z\omega^2 - X^3 - XY\omega = 0, \tag{44}$$

or in non-homogeneous coördinates

$$z = xy + x^3. \tag{45}$$

If we compare this with the canonical development (27), it becomes apparent that one of the faces of the canonical tetrahedron is the singular tangent plane, and one of its vertices the pinch-point of the osculating CAYLEY cubic. This gives rise to the following theorem.

1. *The unodal cubic surfaces which have fourth order contact with a given ruled surface S at a given point P, which is not a flecnode, are themselves ruled surfaces. They form a one-parameter family. Just one surface of this family is a Cayley cubic scroll, and may be called the osculating Cayley cubic scroll of the point P of the ruled surface S.*

2. *The canonical tetrahedron has the given point P of the ruled surface S as one vertex, and the two asymptotic tangents of that point (one of which is the generator g of S), as two of its edges. The pinch-point of the osculating Cayley cubic scroll, its nodal line, and its singular tangent plane constitute three further elements of the canonical tetrahedron, which is thus completely determined geometrically. The nodal line of the osculating Cayley cubic scroll, moreover, meets the generator g of S.*

§ 3. *The family of cubic scrolls having fourth order contact with the ruled surface S at the point P.*

We have noted already that the locus of the nodal lines of the cubic scrolls (41) is the osculating hyperboloid. What is the locus of their pinch-points? We have seen that the pinch-points are determined by the conditions

$$\xi = \Omega = 0, \qquad \eta(\eta - 4p\zeta) = 0,$$

where ξ, η, ζ, Ω are defined by (40). We find, therefore, that the coördinates of these two pinch-points are given by

(46) $$X_1 = -\tfrac{1}{2}p^2, \qquad Y_1 = -2p, \qquad Z_1 = 1, \qquad \omega_1 = p^3,$$

and

(47) $$X_2 = -\tfrac{1}{2}p^2, \qquad Y_2 = +2p, \qquad Z_2 = 1, \qquad \omega_2 = -p^3,$$

respectively. For $p = 0$, of course, the two points coincide. It is easy to verify that both $X_1, \cdots, \omega_1$ and $X_2, \cdots, \omega_2$ satisfy the three quadratic equations

(48) $$\omega Z - XY = 0, \qquad Y^2 + 8XZ = 0, \qquad 8X^2 + Y\omega = 0,$$

so that the locus of the two pinch-points is a single twisted cubic curve upon the osculating hyperboloid. Let us speak of that set of generators of the osculating hyperboloid to which the generator g of the given ruled surface belongs as being of the *first kind*. It will then be apparent that the following theorem is true.

The nodal lines of the cubic scrolls, which have fourth order contact with a given ruled surface S at a given point P, are the generators of the second kind upon the hyperboloid H, which osculates S along the generator g which passes through P. The pinch-points of these cubic scrolls are the points of a twisted cubic curve on H which intersects every generator of the second kind in two points. This twisted cubic passes through P, is tangent at P to the curved asymptotic line of S which passes through that point, and has the plane tangent to S at P as its osculating plane. It also passes through the pinch-point of the osculating Cayley cubic scroll and there has the nodal line and singular tangent plane of that surface as its tangent and osculating plane respectively.

We may notice further that the two cubic surfaces of the one-parameter family, for which the parameter p has numerically equal but opposite values, have the same nodal lines and the same pair of pinch-points, as follows from (46) and (47). This may be expressed as follows:

There exists a two to two correspondence between the cubic surfaces of our one-parameter family and the points of the twisted cubic.

Finally it may be verified that the *locus of the two planes, which are tangent to one of the cubic scrolls of the one-parameter family at its pinch-points, is a developable whose cuspidal edge is again a twisted cubic curve.*

§ 4. *Osculating pseudo-paraboloids of various orders.*

If x and y are cartesian coördinates, a curve whose equation is of the form

$$y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

is sometimes known as a parabola of the nth order. Similarly we might speak of a surface whose cartesian equation is of the form

$$z = a_0 + a_1 x + a_2 y + \cdots + u_n,$$

where u_n is homogeneous and of the nth degree in x and y, as a paraboloid of the nth order. This latter equation may be written in the homogeneous form

$$Z\omega^{n-1} + u_0\omega^n + u_1\omega^{n-1} + u_2\omega^{n-2} + \cdots + u_{n-1}\omega + u_n = 0,$$

where u_k is a binary form of degree k in X and Y, and where $\omega = 0$ represents the plane at infinity. We shall have to consider surfaces of the nth order whose equations are of this form but in which the plane $\omega = 0$ is not the plane at infinity and in which the three planes $X = 0$, $Y = 0$, $Z = 0$ are not mutually perpendicular. These surfaces, the projective generalizations of the paraboloids of the nth order, shall be known as pseudo-paraboloids. We may write the equation of the surface in the form

$$Z\omega^{n-1} + \phi(X, Y, \omega) = 0,$$

where $\phi(X, Y, \omega)$ is the most general ternary form of the nth order in the variables X, Y, ω, and in which Z does not occur. *The point* $X = Y = \omega = 0$, $Z = 1$ *is an* $n - 1$ *fold point of the surface whose tangent cone degenerates into the plane* $\omega = 0$ *counted* $n - 1$ *times. This property is characteristic of pseudo-paraboloids.* We may speak of the singular point as a *unode of the* $n - 1$*th order*, and of the plane as its *uniplane.*

If the canonical development (27) be broken off with the terms of any finite degree n, a pseudo-paraboloid of the nth order will be obtained, whose unode and uniplane coincide with those of the osculating Cayley cubic, which has contact of the nth order with the given ruled surface at the point considered, and which is uniquely determined by these conditions.

We shall make use of one of these pseudo-paraboloids in the following consideration. We have already ascertained the geometrical significance of the tetrahedron which gives rise to the canonical development. But this does not suffice to determine the coördinates themselves. In fact any transformation of the form

$$X = \lambda_1\bar{X},\ Y = \lambda_2\bar{Y},\ Z = \lambda_3\bar{Z},\ \omega = \lambda_4\bar{\omega},$$

or, in non-homogeneous coördinates,

$$x = \lambda\bar{x},\ y = \mu\bar{y},\ z = \nu\bar{z}, \tag{49}$$

where λ, μ, ν are constants will leave unaltered the tetrahedron of reference. However, in the system of coördinates as employed by us, the equations of the osculating hyperboloid and of the osculating Cayley cubic were

$$z - xy = 0 \qquad \text{and} \qquad z - xy - x^3 = 0$$

respectively. These equations are left invariant by (49) only if

$$\nu = \lambda\mu, \qquad \lambda^2 = \mu,$$

i. e., if

$$\mu = \lambda^2, \qquad \nu = \lambda^3. \tag{50}$$

But the osculating pseudo-paraboloid of the fifth order was also endowed with an absolutely fixed equation, and we find that λ must be equated to unity in (49) and (50) so that this equation also may be left invariant. Therefore, *the constants, which are still available after the tetrahedron of the canonical system of coördinates has been selected, are chosen in such a way that the osculating hyperboloid, the osculating Cayley cubic scroll, and the osculating pseudo-paraboloid of the fifth order may have the equations*

$$\begin{gathered} z - xy = 0, \qquad z - xy - x^3 = 0, \\ z - xy - x^3 - x^5 + \frac{5b^3C'}{48h\sqrt{-\frac{1}{5}h}}x^4y + \frac{5^2b^4\theta'}{2^6\cdot 3^2h^2}x^3y^2 = 0, \end{gathered} \tag{51}$$

respectively.

§ 5. *Tangents and curves of fifth-order contact. Discussion of the exceptional cases.*

The osculating Cayley cubic scroll has contact of the fourth order with S at the point P, i. e., all sections of the two surfaces made by a plane through P will have contact of at least the fourth order. It is apparent, however, from (27) that there will be certain tangents of S at P such that, in their directions, the contact of the two surfaces will be of the fifth order at least. We shall speak of these as the tangents of fifth-order contact. The equation of these tangents, obtained by equating to zero the terms of the fifth order in (27), shows that there are five such tangents, of which, however, the generator g of S counts as three. The other two are given by the equation

$$x^2 - \frac{5b^3C'}{48h\sqrt{-\frac{1}{5}h}}xy - \frac{5^2b^4\theta'}{2^6\cdot 3^2\cdot h^2}y^2 = 0, \tag{52}$$

and may be spoken of more specifically as the *proper tangents of fifth order contact.* They obviously determine two families of ∞^1 curves on the surface, two of which (in general) pass through every point of the surface. They may be called the *curves of fifth-order contact.*

A few special cases are of interest. *If* $h = 0$, *while* b, C', θ' *do not vanish, one of the tangents of fifth-order contact coincides with the asymptotic tangent of the point* P [cf., eq. (24)]. *If* $C' = 0$, *while* b, h, *and* θ' *are non-vanishing quantities, the two proper tangents of fifth order contact separate the generator and the asymptotic tangent harmonically. If* $\theta' = 0$ *while* b, h, C' *are different from zero, one of the proper tangents of fifth-order contact coincides with the generator, which in that case counts for four. This latter case is to be well distinguished, of course, from that in which the two proper tangents of fifth-order contact coincide, i. e., in which the discriminant of* (52) *is equal to zero.* Let us remember that the condition $\theta' = 0$ characterizes those generators of the ruled surface whose flecnodes coincide. Therefore: *if the two flecnodes of a generator coincide, four of the five tangents of fifth-order contact at every point of that generator coincide with the generator itself, and conversely.*

If $b \neq 0$, $\theta' \neq 0$, $h = C' = 0$, *two tangents of fifth-order contact coincide with the asymptotic tangent, the other three, of course, with the generator. Finally if*

$$h = C' = \theta' = 0, \qquad b \neq 0, \tag{53}$$

the fifth-order contact tangents become indeterminate, i. e., in this case the Cayley cubic scroll has contact of at least the fifth order with the given ruled surface. If the conditions (53) *are satisfied, not merely at a particular point* u, v *of the surface but identically, the surface must itself be a Cayley cubic scroll.*

It is of interest to prove this directly. If the conditions (53) are satisfied identically, we shall have, making use of the expressions (25) for h, C' and θ',

$$\begin{gathered} b^2(f + b_v) - \tfrac{1}{4} b b_{uu} + \tfrac{5}{16} b_u^2 = 0, \qquad b_v^2 + 2bf_v - 4b^2 g = 0, \\ b_{uv} - \frac{b_u b_v}{b} = 0. \end{gathered} \tag{54}$$

If $b_u \neq 0$, the last equation may be written

$$\frac{b_{uv}}{b_u} = \frac{b_v}{b},$$

whence

$$b = \phi(u)\psi(v), \tag{55}$$

where $\phi(u)$ and $\psi(v)$ are functions of u and v alone respectively. If $b_u = 0$, b is a function of v alone, which case is also included in (55) if there we put $\phi(u) = 1$.

The substitution of (55) into the first equation of (54) gives

$$f = \frac{1}{4}\frac{\phi_{uu}}{\phi} - \frac{5}{16}\frac{\phi_u^2}{\phi^2} - \phi\psi_v. \tag{56}$$

On account of the integrability conditions (2), g is a function of v alone, say

$$g = \chi(v),$$

so that the second equation (54) gives

$$\phi^2\psi_v^2 - 2\phi^2\psi\psi_{vv} - 4\phi^2\psi^2\chi = 0.$$

Since $b \neq 0$, neither ϕ nor ψ can be identically zero, so that

$$\psi_v^2 - 2\psi\psi_{vv} - 4\psi^2\chi = 0. \tag{57}$$

The two remaining integrability conditions (2), or

$$b_{vv} + f_v = 0, \qquad -f_{vv} + 4gb_v + 2bg_v = 0,$$

are satisfied. This is obvious for the first. The second gives, after division by ϕ,

$$\psi_{vvv} + 4\chi\psi_v + 2\psi\chi_v = 0,$$

which is satisfied because this same equation can be obtained from (57) by differentiation.

The differential equations of the surface become, therefore,

$$\begin{aligned} &y_{uu} + 2\phi(u)\psi(v)y_v + \left[\frac{1}{4}\frac{\phi_{uu}}{\phi} - \frac{5}{16}\frac{\phi_u^2}{\phi^2} - \phi\psi_v\right]y = 0, \\ &y_{vv} - \frac{1}{2}\left(\frac{\psi_{vv}}{\psi} - \frac{1}{2}\frac{\psi_v^2}{\psi^2}\right)y = 0; \end{aligned} \tag{58}$$

this system we shall now proceed to integrate.

Let us substitute

$$y = \sqrt{\psi(v)}\,\bar{y}$$

in the second equation of (58). We shall find

$$\bar{y}_{vv} + \frac{\psi_v}{\psi}\bar{y}_v = 0,$$

whence

$$\bar{y} = U + U'\bar{v}, \tag{59}$$

where

$$\bar{v} = \int \frac{dv}{\psi(v)}, \tag{60}$$

and where U and U' are functions of u only. Let us substitute the resulting value of y, i. e.,

$$y = \sqrt{\psi}(U + U'\bar{v}) \tag{61}$$

in the first equation (58). We shall find

$$\sqrt{\psi}(U_{uu} + U'_{uu}\bar{v}) + 2\phi\sqrt{\psi}\,[U' + \tfrac{1}{2}(U + U'\bar{v})\psi_v]$$
$$+\left(\frac{1}{4}\frac{\phi_{uu}}{\phi} - \frac{5}{16}\frac{\phi_u^2}{\phi^2} - \phi\psi_v\right)\sqrt{\psi}(U + U'\bar{v}) = 0,$$

or

$$U_{uu} + U'_{uu}\bar{v} + 2\phi U' + \left(\frac{1}{4}\frac{\phi_{uu}}{\phi} - \frac{5}{16}\frac{\phi_u^2}{\phi^2}\right)(U + U'\bar{v}) = 0.$$

Since u and v are independent variables, that part of the left member of this equation which is independent of v and that part which contains $\bar{v}$ as a factor must vanish independently, i. e., we must have

(62)
$$U_{uu} + 2\phi(u)U' + \left(\frac{1}{4}\frac{\phi_{uu}}{\phi} - \frac{5}{16}\frac{\phi_u^2}{\phi^2}\right)U = 0,$$
$$U'_{uu} + \left(\frac{1}{4}\frac{\phi_{uu}}{\phi} - \frac{5}{16}\frac{\phi_u^2}{\phi^2}\right)U' = 0.$$

Put

$$U = \frac{\overline{U}}{\sqrt[4]{\phi(u)}}, \qquad U' = \frac{\overline{U}'}{\sqrt[4]{\phi(u)}}.$$

The resulting differential equations for $\overline{U}$ and $\overline{U}'$ are

$$\overline{U}_{uu} - \frac{1}{2}\frac{\phi_u}{\phi}\overline{U}_u + 2\phi\overline{U}' = 0,$$
$$\overline{U}'_{uu} - \frac{1}{2}\frac{\phi_u}{\phi}\overline{U}'_u = 0,$$

whence, if we put

(63)
$$\bar{u} = \int\sqrt{\phi(u)}\,du,$$

we obtain

$$\overline{U}' = a\bar{u} + b,$$
$$\overline{U} = -\tfrac{1}{3}a\bar{u}^3 - b\bar{u}^2 + c\bar{u} + d,$$

where a, b, c, d are arbitrary constants.

The general integral of the system (58) becomes, therefore,

(64)
$$y = \frac{\sqrt{\psi(v)}}{\sqrt[4]{\phi(u)}}\left[-\tfrac{1}{3}a\bar{u}^3 - b\bar{u}^2 + c\bar{u} + d + (a\bar{u} + b)\bar{v}\right]$$

where

(65)
$$\bar{u} = \int\sqrt{\phi(u)}\,du, \qquad \bar{v} = \int\frac{dv}{\psi(v)}.$$

Let us denote the four linearly independent solutions which are multiplied by a, b, c, d respectively by y_1, y_2, y_3, y_4, and let ω be the factor $\phi^{\frac{1}{3}}\psi^{-\frac{1}{3}}$; then

$$(66)\quad \omega y_1 = -\tfrac{1}{3}\bar{u}^3 + \bar{u}\bar{v}, \qquad \omega y_2 = -\bar{u}^2 + \bar{v}, \qquad \omega y_3 = \bar{u}, \qquad \omega y_4 = 1.$$

The elimination of $\bar{u}$ and $\bar{v}$ gives

$$(67)\qquad 3(y_2y_3 - y_1y_4)y_4 - 2y_3^3 = 0,$$

so that the integral surface of (58) is indeed a Cayley cubic scroll as we had intended to show. Incidentally equations (66) show that the asymptotic curves of the Cayley cubic scroll are twisted cubic curves, and that all of them pass through the pinch-point, a known result.

Analytically the following remark is of interest. If $\phi(u)$ and $\psi(v)$ are rational functions of their arguments, the general solution of (58) can always be expressed by hyperelliptic integrals. Let us look upon the variables u and v as being capable of all complex values and construct an appropriate Riemann's surface for each of the integrals (65). The totality of values which a system of solutions of (58) may assume for given values of u and v can be obtained from one of them by such quaternary linear transformations as leave the surface (67) unchanged.

Similar remarks may be made about the general case. Such systems of multiform functions of two independent variables connected with ternary instead of quaternary linear substitutions have been considered to some extent by HORN, PICARD and APPELL.*

§ 5. *The ∞^1 osculating Cayley cubic scrolls which belong to the ∞^1 points of a generator.*

There is associated with every point P of a ruled surface S its osculating Cayley cubic scroll. Let P move along a generator g of S; then the pinch-point of the osculating Cayley cubic will describe a curve, its nodal line will describe a ruled surface, and its singular tangent plane will envelop a developable. We wish to investigate these loci.

In the canonical system of coördinates (X, Y, Z, ω), the coördinates of the pinch-point are $(0, 0, 1, 0)$. Consequently the coördinates of this point referred to our original tetrahedron $P_yP_zP_\rho P_\sigma$ may be easily found. This gives the expression

$$(68)\qquad \pi = -b_u b_v y - 4bb_v z + 2bb_u\rho + 8b^2\sigma$$

for the pinch-point where, be it remembered, y is a solution of the system of differential equations (1) and where z, ρ, σ are defined by (3).

* HORN, Mathematische Annalen, vols. 33 and 34 (1889); Acta Mathematica, vol. 12 (1888–89). APPELL, Journal de Mathematiques, series 3, vol. 8 (1882). PICARD, Annales de l'Ecole Normale, series 2, vol. 10 (1881).

The generator g of S which passes through P contains also the point P_ρ, so that

(69) $$Y = y + l\rho,$$

where l is an arbitrary constant, will represent an arbitrary point Q of g. We shall set up a system of differential equations for Y of form (1), compute its semi-covariants and set up the expression Π for the pinch-point of the Cayley cubic which osculates S at Q, making use of the expression (68).

We begin by deducing a system of differential equations for ρ. We have

(70) $$\begin{gathered}\rho = y_v, \qquad \rho_u = y_{uv} = \sigma, \qquad \rho_v = y_{vv} = -gy,\\ \rho_{uu} = y_{uuv} = (2bg - f_v)y - (f + 2b_v)\rho,\\ \rho_{vv} = y_{vvv} = -g_v y - g\rho, \qquad \rho_{uv} = y_{uvv} = -gz,\end{gathered}$$

whence the system of differential equations for ρ;

(71) $$\rho_{uu} + \frac{2bg - f_v}{g}\rho_v + (f + 2b_v)\rho = 0, \qquad \rho_{vv} - \frac{g_v}{g}\rho_v + g\rho = 0.$$

We have to consider this system together with system (1) for y.

From (69) we find

(72) $$\begin{gathered}Y_u = z + l\sigma, \qquad Y_v = \rho - lgy,\\ Y_{uu} = [-f + l(2bg - f_v)]y - [2b + l(f + 2b_v)]\rho,\\ Y_{vv} = -(g + lg_v)y - lg\rho,\end{gathered}$$

whence

(73) $$\begin{gathered}Y_{uu} + \frac{l^2(2bg - f_v) + 2b + 2lb_v}{1 + l^2 g}Y_v + \frac{f + lf_v + l^2 g(f + 2b_v)}{1 + l^2 g}Y = 0,\\ Y_{vv} - \frac{l^2 g_v}{1 + l^2 g}Y_v + \frac{g + l^2 g^2 + lg_v}{1 + l^2 g}Y = 0.\end{gathered}$$

This system is not yet quite of the desired form, in so far as the coefficient of Y_v in the second equation is not equal to zero. In order to transform it into another system which shall have that form, we put

(74) $$Y = \lambda\eta, \qquad \lambda = \sqrt{1 + l^2 g},$$

which gives

(75) $$\eta_{uu} + 2B\eta_v + F\eta = 0, \qquad \eta_{vv} + G\eta = 0,$$

where

$$B = b + \frac{lb_v - \frac{1}{2}l^2 f_v}{1 + l^2 g},$$

$$(76) \qquad F = f + \frac{l f_v + 2l^2 g b_v + l^2 b g_v}{1 + l^2 g} + \frac{l^3 g_v (b_v - \frac{1}{2} l f_v)}{(1 + l^2 g)^2},$$

$$G = g + \frac{l g_v + \frac{1}{2} l^2 g_{vv}}{1 + l^2 g} - \frac{l^4 g_v^2}{4(1 + l^2 g)^2}.$$

The fundamental semi-covariants of (75) will be

$$\mathfrak{y} = \frac{1}{\lambda}(y + l\rho), \qquad \mathrm{Z} = \mathfrak{y}_u = \frac{1}{\lambda}(z + l\sigma),$$

$$(77) \qquad \mathrm{P} = \mathfrak{y}_v = \frac{1}{\lambda}\left[- \left(lg + \frac{\frac{1}{2} l^2 g_v}{1 + l^2 g} \right) y + \left(1 - \frac{\frac{1}{2} l^3 g_v}{1 + l^2 g} \right) \rho \right],$$

$$\Sigma = \mathfrak{y}_{uv} = \frac{1}{\lambda}\left[- \left(lg + \frac{\frac{1}{2} l^2 g_v}{1 + l^2 g} \right) z + \left(1 - \frac{\frac{1}{2} l^3 g_v}{1 + l^2 g} \right) \sigma \right].$$

The expression for the pinch-point of the Cayley cubic scroll which osculates S at the point $P_{\mathfrak{y}}$ of the generator g is, according to (68),

$$\Pi = - B_u B_v \mathfrak{y} - 4 B B_v \mathrm{Z} + 2 B B_u \mathrm{P} + 8 B^2 \Sigma.$$

The substitution of the above expressions gives

$$(78) \qquad \lambda \Pi = x_1 y + x_2 z + x_3 \rho + x_4 \sigma,$$

where

$$(79) \qquad \begin{aligned} x_1 &= - B_u (B_v + 2BC), & x_2 &= - 4B(B_v + 2BC), \\ x_3 &= B_u(- l B_v + 2BD), & x_4 &= 4B(- l B_v + 2BD), \end{aligned}$$

if we write for abbreviation

$$(80) \qquad \begin{gathered} C = lg + \frac{\frac{1}{2} l^2 g_v}{1 + l^2 g}, \qquad D = 1 - \frac{\frac{1}{2} l^3 g_v}{1 + l^2 g}, \\ Cl + D = 1 + l^2 g. \end{gathered}$$

Making use of the integrability conditions, we find

$$(81) \qquad \begin{aligned} B_v + 2BC &= b_v + (b_{vv} + 2bg) l, \\ - l B_v + 2BD &= 2b + b_v l, \end{aligned}$$

so that *we obtain*

$$(82) \qquad \begin{aligned} x_1 &= - \left[b_u + b_{uv} l + \tfrac{1}{2}(b_{uvv} + 2g b_u) l^2 \right] \left[b_v + (b_{vv} + 2bg) l \right], \\ x_2 &= - \left[4b + 4b_v l + 2(b_{vv} + 2bg) l^2 \right] \left[b_v + (b_{vv} + 2bg) l \right], \\ x_3 &= + \left[b_u + b_{uv} l + \tfrac{1}{2}(b_{uvv} + 2g b_u) l^2 \right] \left[2b + b_v l \right], \\ x_4 &= + \left[4b + 4b_v l + 2(b_{vv} + 2bg) l^2 \right] \left[2b + b_v l \right], \end{aligned}$$

as the parametric equations of the locus of the pinch-points of the ∞^1 Cayley cubic scrolls which osculate the surface S at the ∞^1 points of one of its generators. We see that the locus is a twisted cubic situated (of course) upon the osculating hyperboloid.

The locus of the nodal lines of the ∞^1 Cayley cubic scrolls must be the osculating hyperboloid H, i. e., its second set of generators. For we have seen that the locus of the nodal lines of the ∞^1 cubic scrolls, which have fourth order contact with S at a fixed point P, is the second set of generators on H. But the osculating Cayley cubic on P is one of these cubic scrolls of fourth order contact, so that its nodal line is one of these generators of the second set on H. As P moves along g, this generator moves on H.

The cubic (82) intersects the generator g of S, whose equations are $x_2 = x_4 = 0$, in two points

$$y + l_k \rho \qquad (k = 1, 2),$$

where l_1 and l_2 are the roots of the quadratic

$$2b + 2b_v l + (b_{vv} + 2bg) l^2 = 0,$$

so that

$$l_1 + l_2 = -\frac{2b_v}{b_{vv} + 2bg}, \qquad l_1 l_2 = \frac{2b}{b_{vv} + 2bg}.$$

Consequently

$$(y + l_1 \rho)(y + l_2 \rho) = y^2 - \frac{2b_v}{b_{vv} + 2bg} y\rho + \frac{2b}{b_{vv} + 2bg}\rho^2. \tag{83}$$

But this quadratic is precisely the covariant which determines the flecnodes of g. In fact the flecnodes are given by the factors of

$$u'_{12}\rho^2 - u'_{21}y^2 + (u'_{11} - u'_{22}) y\rho,$$

where *

$$\begin{aligned} u'_{11} &= -4f, \qquad u'_{12} = -8b, \\ u'_{21} &= -4f_v + 8bg = 4b_{vv} + 8bg, \\ u'_{22} &= -4f - 8b_v. \end{aligned} \tag{84}$$

It also becomes apparent now why the canonical development breaks down if the point P of the surface S is a flecnode. For, in that case, the pinch-point of the osculating Cayley cubic scroll coincides with P itself, so that the canonical tetrahedron degenerates.

Let us multiply out the terms in (82). If the determinant Δ of the coefficients of l^0, l^1, l^2, l^3 in (82) is different from zero, the twisted cubic will be non-degenerate. We proceed to evaluate that determinant. We have

*Second Memoir, eq. (15).

$$\tfrac{1}{4}\Delta =$$
$$\begin{vmatrix} b_u b_v, & b_u(b_{vv}+2bg)+b_{uv}b_v, & b_{uv}(b_{vv}+2bg)+\tfrac{1}{2}b_v(b_{uvv}+2gb_u), & \tfrac{1}{2}(b_{uvv}+2gb_u)(b_{vv}+2bg) \\ 2bb_v, & 2b(b_{vv}+2bg)+2b_v^2 & 3b_v(b_{vv}+2bg) & (b_{vv}+2bg)^2 \\ 2bb_u & b_u b_v+2bb_{uv} & b_{uv}b_v+b(b_{uvv}+2gb_u) & \tfrac{1}{2}b_v(b_{uvv}+2gb_u) \\ 4b^2 & 6bb_v & 2b_v^2+2b(b_{vv}+2bg) & b_v(b_{vv}+2bg) \end{vmatrix}.$$

Multiply the elements of the second and fourth rows by $-b_u/2b$ and add to those of the first and third respectively; in the resulting determinant multiply the elements of the third and fourth rows by $-b_v/2b$ and add them to the elements of the first and second rows respectively. Making use of the equations which define C' and θ' we find

$$\frac{-\Delta}{4b^2} = \begin{vmatrix} 0 & -\dfrac{1}{2^{10}}\dfrac{C'\theta'}{b} & -\dfrac{1}{2^8}\dfrac{\theta'}{b}\left(b_{uvv}-\dfrac{b_u b_{vv}}{b}\right) \\ -\dfrac{1}{2^6}\theta' & -\dfrac{1}{2^6}\dfrac{b_v}{b}\theta' & -\dfrac{1}{2^7}(b_{vv}+2bg)\dfrac{\theta'}{b} \\ \dfrac{1}{2^2}bC' & \dfrac{1}{2^3}C'b_v+bb_{uvv}-b_u b_{vv} & \dfrac{1}{2}b_v\left(b_{uvv}-\dfrac{b_u b_{vv}}{b}\right) \end{vmatrix},$$

whence

$$(85)\qquad \Delta = -\frac{\theta'^2}{2^{10}}\left[b^2\left(b_{uvv}-\frac{b_u b_{vv}}{b}\right)^2+\frac{1}{2^5}C'^2 b(b_{vv}+2bg) - \frac{1}{2^2}bb_v C'\left(b_{uvv}-\frac{b_u b_{vv}}{b}\right)\right].$$

But we have

$$\theta' = 2^6(b_v^2-2bb_{vv}-4b^2g),$$

$$\theta'_u = 2^6(2b_v b_{uv}-2b_u b_{vv}-2bb_{uvv}-8bb_u g),$$

whence

$$b\theta'_u - 2b_u\theta' = 2^7\left[-b^2\left(b_{uvv}-\frac{b_u b_{vv}}{b}\right)+\frac{1}{2^3}bb_v C'\right],$$

so that

$$(86)\qquad \Delta = \frac{\theta'^2\mathfrak{B}}{2^{20}b^2},$$

where

$$(87)\qquad \mathfrak{B} = -2^6\left[(b\theta'_u-2b_u\theta')^2-4b^2C'^2\theta'\right]$$

is the invariant of the linear complex which osculates the ruled surface S along the generator g.* The cubic curve (82) can therefore be degenerate, only if either the two flecnodes of g coincide, or if the osculating linear complex is special.

* Second Memoir, eq. (47).

Of the problems to be considered in this paragraph there still remains the third one; what is the developable generated by the singular tangent plane of the osculating Cayley cubic scroll as P moves along g? But the answer to this question is very simple. Since the singular tangent plane is tangent to the osculating hyperboloid at the pinch-point of the osculating Cayley cubic scroll, its coördinates (u_1, u_2, u_3, u_4) will be proportional to $(x_4, -x_3, -x_2, x_1)$ where the quantities x_k are defined by equations (82). This developable is therefore circumscribed about the osculating hyperboloid, and its edge of regression is again a space cubic. We may recapitulate the main results of this paragraph in the following theorem.

At every point P of a generator g of a ruled surface S, whose osculating hyperboloid H does not hyperosculate it, there exists a unique osculating Cayley cubic scroll. As P moves along the generator g, the locus of the nodal line of the osculating Cayley cubic scroll consists of those generators of H which intersect g. The locus of the pinch-points is a twisted cubic curve C_1 on H, which passes through the two flecnodes of g and which does not degenerate, except when either the two flecnodes of g coincide, or when the linear complex which osculates S along g becomes special. The locus of the singular tangent planes of the osculating Cayley cubic scrolls is a developable which is circumscribed about the osculating hyperboloid H along the cubic curve C_1, and whose cuspidal edge is another twisted cubic curve C_2 which does not degenerate unless C_1 does also.

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