THE INTEGRATION OF A SEQUENCE OF FUNCTIONS AND ITS APPLICATION TO ITERATED INTEGRALS

BY

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Introduction.

The fundamental principle involved in this memoir is the integration of a sequence of functions. As this depends essentially upon an inversion in the order of passing to a limit, applications are readily made to several important topics. In particular, the equality of a multiple integral and the corresponding iterated integrals is here treated, also the integration of a series term by term, and the differentiation of both series and definite integrals.

Hitherto in the integration of sequences the condition of uniform convergence in some form has played an important rôle. In this paper the methods are entirely independent of any such condition of convergence. The problem is then, in this respect, brought into more intimate relation with that of the inversion of the order of summation in an absolutely convergent multiple series. The field of integration is taken in § 3 to be a limited m-dimensional region A which either possesses content or is closed. Throughout A there is defined a sequence of functions $f_1(x_1, \cdots, x_m)$, $f_2(x_1, \cdots, x_m)$, $\cdots$ converging to the function $f(x_1, \cdots, x_m)$. If $f_1, f_2, \cdots$ form a monotone increasing sequence of integrable positive functions, it is shown that

$$\lim_{n=\infty} \int_A f_n = \underline{\int}_A f.$$

This is included in a more comprehensive result stated in theorem 13.

If for some points of A the functions $f_1, f_2, \cdots$ form a monotone increasing sequence and for the remaining points they form a monotone decreasing sequence, then provided $f, f_1, f_2, \cdots$ are integrable,

$$\lim_{n=\infty} \int_A f_n = \int_A f. \tag{1}$$

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The equality (1) holds also for a sequence of functions, the totality of which is limited ($|f_i| \leqq G$, $i = 1, 2, \cdots$). Corresponding theorems for differentiation are developed.

The relations

$$
(2)\quad
\begin{aligned}
\overline{\int_T} f(x, y)\,dT \geqq \overline{\int_X} dx \overline{\int_Y} f(x, y)\,dy \geqq \overline{\int_X} dx \underline{\int_Y} f(x, y)\,dy \geqq \underline{\int_T} f(x, y)\,dT \\
\overline{\int_T} f(x, y)\,dT \geqq \underline{\int_X} dx \overline{\int_Y} f(x, y)\,dy \geqq \underline{\int_X} dx \underline{\int_Y} f(x, y)\,dy \geqq \underline{\int_T} f(x, y)\,dT
\end{aligned}
$$

are proved in the standard texts for the case that f is a limited function and T a portion of the plane bounded by a simple curve. The most general treatment is that by Professor PIERPONT.* Various simple examples may be given where the existence of one or more of the integrals

$$
\int_T f(x, y)\,dT, \qquad \int_X dx \int_Y f(x, y)\,dy, \qquad \int_Y dy \int_X f(x, y)\,dx
$$

is not a sufficient condition for the existence of the others.† When the function f has infinite discontinuities, the problem is much more complex. For the case where the discontinuities are arranged on a finite number of regular curves, relations similar to (2) have been derived. Under this restriction, if the double integral and the iterated integral exist, they are equal. The first *general* treatment of the subject is that of DE LA VALLÉE-POUSSIN. The problem was later considered by SCHÖNFLIES, PIERPONT and HOBSON, the most general treatment from some standpoints being that of HOBSON. For a field comprising *all* the points of a *rectangle*, he has shown that if the double integral and the iterated integral of a *positive* function exist they are equal. In this article it will be proved (§ 4) that for *any* function f the existence (finite or infinite) of the integrals

$$
\int_T f(x, y)\,dT, \qquad \int_X dx \int_Y f(x, y)\,dy
$$

is a sufficient condition for their equality, the only conditions on the field being that T possess content and that X possess content or is closed. In the discussions of this memoir we use the broad definition of integral which allows the existence of non-absolutely convergent single integrals, while Hobson's work is limited to the narrower absolutely convergent integrals. Corresponding to (2) very general relations for an unlimited function f are developed. For example, if $f \geqq -G$, it is shown under the same limitations for T and X that

* *Lectures on the Theory of Functions of Real Variables*, vol. 1, Ginn & Co., Boston (1905). This will be referred to as *Lectures*.

† See p. 361 of this article.

$$\overline{\int_T} f(x, y)\, dT \geqq \underline{\int_X} dx \overline{\int_Y} f(x, y)\, dy \geqq \underline{\int_X} dx \underline{\int_Y} f(x, y)\, dy \geqq \underline{\int_T} f(x, y)\, dT,$$

where one or more of the integrals may be unlimited. An attempt is made in § 4 to discuss the results of DE LA VALLÉE-POUSSIN, SCHÖNFLIES and HOBSON.

In § 5 the results of § 3 are applied to the integration of a convergent series

$$U(x_1, \cdots, x_m) = u_1(x_1, \cdots, x_m) + u_2(x_1, \cdots, x_m) + \cdots.$$

If $u_1, u_2, \cdots$ are positive integrable functions, then

$$\underline{\int_A} U = \int_A u_1 + \int_A u_2 + \cdots.$$

If for some points of A, the functions $u_1, u_2, \cdots$ are all positive or zero, while for the remaining points $u_1, u_2, \cdots$ are all negative or zero, then provided U, $u_1, u_2, \cdots$ are integrable

$$\int_A U = \int_A u_1 + \int_A u_2 + \cdots. \tag{3}$$

Relation (3) is true also if for all values of n we have $|\sum_{i=1}^{n} u_i| \leqq G$. These relations are generalized and the corresponding theorems for differentiation developed.

In a later article these topics will be discussed for an infinite field, the relation to the integrals of LEBESGUE will be shown, and the conditions for inversion of order in an iterated integral will be treated.

§ 1. *Preliminary notions and definitions.*

Let $\mathfrak{A}$ denote a limited aggregate in m-dimensional space. We may effect a division of space into cells each of which has its greatest dimension equal to or less than δ. This is indicated by saying that the division is of *norm* δ. When δ approaches zero, the sum of those cells containing at least one point of $\mathfrak{A}$ approaches a limit which is called the *upper content* of $\mathfrak{A}$ and is written $\overline{\mathfrak{A}}$. In a similar manner we designate by $\underline{\mathfrak{A}}$ the *lower content* of $\mathfrak{A}$, which is the limit of the sum of those cells all of whose points belong to $\mathfrak{A}$. If $\overline{\mathfrak{A}} = \underline{\mathfrak{A}}$, the aggregate is said to possess *content* $\mathfrak{A}$; if $\overline{\mathfrak{A}} = 0$, it is called *discrete*. The *frontier* of $\mathfrak{A}$ consists of an aggregate F such that in the infinitesimal vicinity of each point of F there is at least one point of $\mathfrak{A}$ and one not of $\mathfrak{A}$. If $\mathfrak{A}$ possesses content, $\overline{F} = 0$; and conversely. $\mathfrak{A}$ is *everywhere dense* in a region R_m when no cell can be found within R_m which does not contain at least one point of $\mathfrak{A}$. An aggregate is said to be closed if it contains its first derivative. The aggregate $\mathfrak{A}$ may be enclosed in a finite or enumerably infinite set of cells. The volume of each such set of cells has a definite value. The minimum of this volume is called the *upper measure* of the aggregate. The aggregate $\mathfrak{A}_1$ is said to be a *partial aggregate* of $\mathfrak{A}$ when each point of $\mathfrak{A}_1$ is in $\mathfrak{A}$. The points $\mathfrak{A}_2$ belonging to $\mathfrak{A}$ but not to $\mathfrak{A}_1$ form the aggregate *complementary* to $\mathfrak{A}_1$. To obtain the *lower measure* of the aggregate $\mathfrak{A}$ situated in the region R_m, we subtract from R_m the upper measure of the aggregate complementary to $\mathfrak{A}$. If the upper measure and lower measure are equal, the common value is called the *measure* of $\mathfrak{A}$. The upper measure of an aggregate is at least equal to its lower content, and at most equal to its upper content. For a closed aggregate the upper measure and the upper content are identical. If we have a sequence of aggregates $\mathfrak{A}_1, \mathfrak{A}_2, \cdots$, the aggregate $\mathfrak{A}$ consisting of all points found in any one of them is called their *least common multiple*. The aggregate α containing all points common to $\mathfrak{A}_1, \mathfrak{A}_2, \cdots$ is the *greatest common divisor* of that sequence. If $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are complementary partial aggregates of $\mathfrak{A}$, they are said to be *unmixed* if the aggregate of frontier points common to them is discrete.

In this article the restriction is put on the field of integration that the frontier be discrete. The field possessing content is denoted by A to distinguish it from the general field $\mathfrak{A}$. The function $f(x_1, \cdots, x_m)$ defined over A may take on any value unless otherwise stated. If in the field A we hold p coördinates $x_1, \cdots, x_p$ fixed, the point $(x_1, \cdots, x_m)$ ranges over an aggregate of q dimensions $(p + q = m)$ which is called $\mathfrak{C}$. If the integral of f is taken over $\mathfrak{C}$, the resulting function $F(x_1, \cdots, x_p)$ is a function of p variables. The function F is defined over an aggregate $\mathfrak{B}$ in p-dimensional space and may permit of integration. The resulting integral is written

$$\int_{\mathfrak{B}} F(x_1, \cdots, x_p) = \int_{\mathfrak{B}} \int_{\mathfrak{C}} f(x_1, \cdots, x_m),$$

and is called the iterated integral. The simplest case is the double integral $(m = 2)$ which is broken up into two single integrations. Throughout the discussion of the reduction of a multiple integral, the aggregate $\mathfrak{B}$ is assumed to possess content or to be closed and is then denoted by B.

In the memoir we adopt the definition of a multiple integral of a *limited* function $f(x_1, \cdots, x_m)$ introduced by Professor PIERPONT,* which includes as special cases the integrals of JORDAN, STOLZ and DE LA VALLÉE-POUSSIN. Since in this definition the arrangement of the frontier points is in no manner restricted, the difficulties which ordinarily arise from a discussion of these points are entirely avoided. This definition is as follows.

Let $\mathfrak{A}$ be a limited field in m-dimensions, over which a limited function $f(x_1, \cdots, x_m)$ is defined. Effect a division of space of norm δ into cells $\delta_1, \delta_2, \cdots$ not necessarily rectangular. Take those cells which contain points of $\mathfrak{A}$. Let

* *Lectures*, p. 528.

M_i and m_i denote respectively the maximum and minimum of f in δ_i. When δ approaches zero, the two sums $\Sigma M_i\delta_i$, $\Sigma m_i\delta_i$ approach certain limits which are denoted by

$$\bar{\int}_{\mathfrak{A}} f(x_1, \cdots, x_m)\, d\mathfrak{A}, \qquad \underline{\int}_{\mathfrak{A}} f(x_1, \cdots x_m)\, d\mathfrak{A},$$

and are called the upper and lower integrals of f in $\mathfrak{A}$. If these are equal, the common value is called the integral of f in $\mathfrak{A}$ and is written

$$\int_{\mathfrak{A}} f(x_1, \cdots, x_m) = \int_{\mathfrak{A}} f.$$

Two methods for defining *improper* integrals in a limited field $\mathfrak{A}$ are available, being generalizations respectively of the definition ordinarily followed and of the definition due to DE LA VALLÉE-POUSSIN. The first method is that which is followed by Professor PIERPONT,* while the second is used by the writer in a previous article.† For convenience we insert here the definitions used in this previous article.

With the function f is associated the auxiliary function $f_{\lambda_1\lambda_2}$ defined thus:

$$f_{\lambda_1\lambda_2} = f \quad \text{if } -\lambda_2 \leqq f \leqq \lambda_1; \quad f_{\lambda_1\lambda_2} = \lambda_1 \quad \text{if } f > \lambda_1; \quad f_{\lambda_1\lambda_2} = -\lambda_2 \quad \text{if } f < -\lambda_2.$$

Provided the limits exist, we define the improper integrals as follows:

$$(1) \qquad \int_{\mathfrak{A}} f = \lim_{\lambda_1=\infty,\, \lambda_2=\infty} \int_{\mathfrak{A}} f_{\lambda_1\lambda_2}, \qquad (2) \quad \bar{\int}_{\mathfrak{C}} f = \lim_{\lambda_1=\infty,\, \lambda_2=\infty} \bar{\int}_{\mathfrak{C}} f_{\lambda_1\lambda_2},$$

and so forth. The integral itself is said "to exist" only if the limit is finite. In case the limit is definitely infinite (either $+\infty$ or $-\infty$), this will be expressed by saying that the integral exists infinitely. If the limit does not exist, the function is not integrable. In the iterated integral

$$\int_{\mathfrak{B}} \int_{\mathfrak{C}} f$$

the integrand

$$\int_{\mathfrak{C}} f$$

may be infinite, and we introduce the notation:

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*Transactions of the American Mathematical Society, vol. 7 (1906), p. 158.

†Transactions, vol. 7 (1906), p. 449. It may be noted that the discussion given in the present article will hold if in place of f_λ as here defined we read f_n where the latter symbol is that defined by DE LA VALLÉE-POUSSIN (*Cours d'Analyse*, vol. 2, p. 89) and used by him in the exposition of the reduction of the double integral.

$$\left[\int_{\mathfrak{C}} f\right]_{\mu_1\mu_2} = \int_{\mathfrak{C}} f \quad \text{if } -\mu_2 \leqq \int_{\mathfrak{C}} f \leqq \mu_1,$$

$$\left[\int_{\mathfrak{C}} f\right]_{\mu_1\mu_2} = \mu_1 \quad \text{if} \int_{\mathfrak{C}} f > \mu_1,$$

$$\left[\int_{\mathfrak{C}} f\right]_{\mu_1\mu_2} = -\mu_2 \quad \text{if} \int_{\mathfrak{C}} f < -\mu_2.$$

If when μ_1, μ_2 approach infinity in any manner a unique limit is obtained, we take by definition

$$\underline{\int}_{\mathfrak{B}} \int_{\mathfrak{C}} f = \lim_{\mu_1=\infty,\ \mu_2=\infty} \underline{\int}_{\mathfrak{B}} \left[\int_{\mathfrak{C}} f\right]_{\mu_1\mu_2},$$

and so forth.

Whenever

$$(1) \qquad \int_{\mathfrak{B}} \overline{\int}_{\mathfrak{C}} f = \int_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} f,$$

the integral

$$\int_{\mathfrak{B}} \int_{\mathfrak{C}} f = \int_{\mathfrak{B}} \overline{\underline{\int}}_{\mathfrak{C}} f$$

is said to exist although for some points of $\mathfrak{B}$ the integrand

$$\int_{\mathfrak{C}} f$$

is not determinate. If we denote by $\mathfrak{B}_\sigma$ the partial aggregate of $\mathfrak{B}$ where

$$\overline{\int}_{\mathfrak{C}} f - \underline{\int}_{\mathfrak{C}} f > \sigma,$$

it is evident that if $\sigma > 0$, $\bar{\mathfrak{B}}_\sigma = 0$. Throughout this article the notation

$$\int_{\mathfrak{B}} \overline{\underline{\int}}_{\mathfrak{C}} f$$

is used implying the existence and equality of the integrals of (1).

The writer has shown * that the improper integrals defined by the two methods are absolutely convergent. One advantage of the second method is that it may be shown (theorem 4) that in order that the integral may exist, the points of infinite discontinuity of f must be discrete; if the first method is used, this must be expressly assumed. Another advantage of this method is the pos-

* Transactions, loc. cit., theorems 6 and 12. For a correction of the proof of theorem 6 see the note on page 371 of the present memoir.

sibility of integrating along a section for every point of which the function is infinite, the other definition not permitting of this operation. Apart from these differences, as I have shown,* the existence of one of the improper integrals defined by these methods is a necessary and sufficient condition for the existence of the other. The *lower* integral as defined by the second method may exist for a positive function when it has points of infinite discontinuity everywhere dense; while no lower integral is possible by the first method.

The various properties of these integrals have already been developed.† We mention a few that are characteristic.

I. If f is integrable in $\mathfrak{A}$, it is integrable in any partial field $\mathfrak{A}_1$.

II. If $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are unmixed complementary partial aggregates of $\mathfrak{A}$ in which f is integrable, then

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{A}_1} f + \int_{\mathfrak{A}_2} f.$$

III. If $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are complementary partial aggregates of $\mathfrak{A}$, over which a function $f \geqq 0$ is defined, then

$$\int_{\mathfrak{A}} f \leqq \int_{\mathfrak{A}_1} f + \int_{\mathfrak{A}_2} f,$$

provided the integrals exist.

IV. If the integral over $\mathfrak{A}$ exists, then for every $\epsilon > 0$ there is a $\delta > 0$ such that for any set of unmixed partial aggregates $\mathfrak{A}_1, \mathfrak{A}_2, \cdots$ of which the total upper content is less than δ the sum of the integrals over these partial fields is less in absolute value than ϵ.

V. It is possible to change the value of the integrand at a discrete aggregate of points without altering the value of the integral. The new values may be finite or infinite.

VI. If f and g are integrable in $\mathfrak{A}$, the integral of the sum is equal to the sum of the integrals.

It may be noted that the theorems of this paper hold in case the integrals exist when the definition of HARNACK or any of its extensions is used. If $\mathfrak{C}$ is a one-dimensional aggregate, the integral of f over $\mathfrak{C}$ may exist only non-absolutely. In this case the ordinary (HARNACK) definition is more general than that of DE LA VALLÉE-POUSSIN. Although the notation used throughout this memoir is that of the narrower integral, the discussion will hold for the more general case.

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* Loc. cit., theorems 16 and 17. This result was given to the Society December 28, 1905. HOBSON has proved this for a less general case in the Proceedings of the London Mathematical Society, ser. 2, vol. 4 (June, 1906), p. 139.

† Loc. cit.

§ 2. *Theorems on aggregates and integrals.*

THEOREM 1. *Let $\mathfrak{A}$ be a limited point aggregate in m-dimensional space, and $\mathfrak{B}$ the p-dimensional ($p < m$) aggregate which is the projection of $\mathfrak{A}$ on a p-dimensional section. Denote by $\mathfrak{B}_1$ a discrete partial aggregate of $\mathfrak{B}$ and by $\mathfrak{A}_1$ the partial aggregate of $\mathfrak{A}$ which has for its projection $\mathfrak{B}_1$. Then $\mathfrak{A}_1$ is discrete.*

Enclose the points of $\mathfrak{B}_1$ in p-dimensional cubes of volume small at pleasure. Complete the m-dimensional parallelopipeds enclosing $\mathfrak{A}_1$. The volume of these can evidently be made as small as is desired.

THEOREM 2. *Let $A_1 \leqq A_2 \leqq A_3 \leqq \cdots$ be a series of partial aggregates of a limited point aggregate $\mathfrak{A}$ and let $\lim_{n=\infty} \bar{A}_n = \eta$. Denote by L the least common multiple of the series and by α the partial aggregate of $\mathfrak{A}$ complementary to L. Then if $\mathfrak{A}$ is closed or possesses content, $\bar{\mathfrak{A}} \geqq \bar{\alpha} \geqq \bar{\mathfrak{A}} - \eta$.*

Since α is a partial aggregate of $\mathfrak{A}$ it is evident that $\bar{\mathfrak{A}} \geqq \bar{\alpha}$. Denote by σ_n the upper content of A_n. Then $\sigma_1 \leqq \sigma_2 \leqq \cdots \leqq \eta$. By definition it is possible to enclose the points of A_1 in cells of volume less than $\sigma_1 + \frac{1}{2}\epsilon$ and the points of A_n, not already enclosed, in cells of volume less than $\sigma_n - \sigma_{n-1} + \epsilon/2^n$. The sequence $A_1, A_2, \cdots$ may then be enclosed in an enumerable set of cells of volume less than $\eta + \epsilon$. Since the points of α may be enclosed in cells of volume less than $\bar{\alpha} + \epsilon$, it follows that the points of $\mathfrak{A}$ may be enclosed in a finite or enumerably infinite set of cells in volume less than $\eta + \bar{\alpha} + 2\epsilon$.

If $\mathfrak{A}$ possess content, it is evident that $\mathfrak{A} = \underline{\mathfrak{A}} < \eta + \bar{\alpha} + 2\epsilon$, since in calculating the lower content only those cells are taken which contain no points except those of $\mathfrak{A}$. This being true for all values of ϵ, we have $\bar{\alpha} \geqq \bar{\mathfrak{A}} - \eta$.

If $\mathfrak{A}$ is closed, upper measure $\mathfrak{A} = \bar{\mathfrak{A}}$. Since $\bar{\mathfrak{A}}$ is the minimum volume of cells enclosing $\mathfrak{A}$, it follows that $\eta + \bar{\alpha} + 2\epsilon \geqq \bar{\mathfrak{A}}$, or $\bar{\alpha} \geqq \bar{\mathfrak{A}} - \eta - 2\epsilon$. Passing to the limit we have $\bar{\alpha} \geqq \bar{\mathfrak{A}} - \eta$.

COROLLARY. *Let $\mathfrak{A}_1 \leqq \mathfrak{A}_2 \leqq \cdots$ be a sequence of aggregates whose least common multiple is $\mathfrak{A}$. If $\mathfrak{A}$ possesses content or is closed,* $\lim_{n=\infty} \bar{\mathfrak{A}}_n = \bar{\mathfrak{A}}$.

In this case α possesses no points. Hence $0 = \bar{\alpha} \geqq \bar{\mathfrak{A}} - \eta$, or $\bar{\mathfrak{A}} \leqq \eta$. Since however $\bar{\mathfrak{A}} \geqq \eta$, it follows that $\lim_{n=\infty} \bar{\mathfrak{A}}_n = \eta = \bar{\mathfrak{A}}$.

THEOREM 3. *Let $\mathfrak{A}$ be an aggregate discrete in m dimensions, and $\mathfrak{B}$ and $\mathfrak{C}$ the corresponding* aggregates in p and q dimensions. Let $\mathfrak{B}_1$ denote the partial aggregate of $\mathfrak{B}$ for which $\bar{\mathfrak{C}} = 0$, and $\mathfrak{B}_2$ the complementary aggregate of $\mathfrak{B}$. Then measure $\mathfrak{B}_2 = 0$, and if $\mathfrak{B}$ possesses content, $\bar{\mathfrak{B}} = \bar{\mathfrak{B}}_1$.*

Choose a sequence $\sigma_1 > \sigma_2 > \sigma_3 > \cdots$ with the limit 0 and denote by $\mathfrak{B}_{\sigma_i}$ the partial aggregate of $\mathfrak{B}$ where $\bar{\mathfrak{C}} \geqq \sigma_i$. It has been shown by SCHÖNFLIES† that $\mathfrak{B}_{\sigma_i}$ is discrete. Since the measure of each of the enumerable sequence of aggregates $\mathfrak{B}_{\sigma_1}$, $\mathfrak{B}_{\sigma_2} - \mathfrak{B}_{\sigma_1}$, $\mathfrak{B}_{\sigma_3} - \mathfrak{B}_{\sigma_2}$, $\cdots$ is zero, it follows by a familiar

* See p. 342 for definition.

† *Die Entwickelung der Lehre von den Punktmannigfaltigkeiten* (1900), p. 96.

theorem that measure $\mathfrak{B}_2 = 0$. Since now $\mathfrak{B}_2$ is the least common multiple of $\mathfrak{B}_{\sigma_1}, \mathfrak{B}_{\sigma_2}, \cdots$ and since $\lim_{n=\infty} \mathfrak{B}_{\sigma_n} = 0$, it follows from theorem 2 that if $\mathfrak{B}$ possesses content, $\bar{\mathfrak{B}}_1 = \mathfrak{B}$.

COROLLARY. *Let $\mathfrak{A}$ be an aggregate which possesses content. Denote by $\mathfrak{B}_2$ the points of $\mathfrak{B}$ when the section $\mathfrak{C}$ does not possess content and by $\mathfrak{B}_1$ the complementary partial aggregate of $\mathfrak{B}$. Then measure $\mathfrak{B}_2 = 0$, and if $\mathfrak{B}$ possesses content, $\bar{\mathfrak{B}} = \bar{\mathfrak{B}}_1$.*

Since the frontier points of $\mathfrak{A}$ form a discrete aggregate, this is an immediate consequence of the theorem.

THEOREM 4. *Let the function f be defined over the limited aggregate $\mathfrak{A}$ and be integrable in $\mathfrak{A}$. If $\mathfrak{A}_\lambda$ denotes the partial aggregate where $|f| > \lambda$, it is possible to take λ so large that $\bar{\mathfrak{A}}_\lambda$ is small at pleasure.**

Since the integral converges it converges absolutely, as was shown in the former paper.† Let

$$\int_{\mathfrak{A}} |f| = M.$$

Choose ϵ small at pleasure and take $\lambda > M/\epsilon$. Then

$$M = \int_{\mathfrak{A}} |f| \geqq \int_{\mathfrak{A}_\lambda} |f| \geqq \int_{\mathfrak{A}} \lambda \geqq \lambda \bar{\mathfrak{A}}_\lambda.$$

Therefore

$$\bar{\mathfrak{A}}_\lambda \leqq \frac{M}{\lambda} < \epsilon.$$

The following theorem is obvious.

THEOREM 5. *Let $f \geqq 0$ be a limited function defined over a limited aggregate $\mathfrak{A}$. Adjoin to $\mathfrak{A}$ any aggregate of points and call the resulting aggregate R. Define the new function g as follows:*

$$g = f \text{ for points of } \mathfrak{A}, \qquad g = 0 \text{ for other points of } R.$$

Then

$$\bar{\int_R} g = \bar{\int_{\mathfrak{A}}} f.$$

THEOREM 6. *Let $\mathfrak{A}$ be a limited aggregate, not necessarily possessing content, over which a limited function $f \geqq 0$ is defined. Then*

$$\bar{\int_{\mathfrak{A}}} f \geqq \bar{\int_{\mathfrak{B}}} \bar{\int_{\mathfrak{C}}} f. \tag{1}$$

* This theorem is true for a function of more than one variable. It is true for a function of one variable that is absolutely integrable, in other words for a function which is integrable by the method of DE LA VALLÉE-POUSSIN.

† Theorem 12, Transactions, vol. 7 (1906), p. 455.

Enclose the aggregate $\mathfrak{A}$ in a cube the totality of whose points is denoted by R. Let T and S be the aggregates of R corresponding to $\mathfrak{C}$ and $\mathfrak{B}$. Define the auxiliary function g as follows:

$$g = f \text{ for points of } \mathfrak{A}, \qquad g = 0 \text{ for other points of } R.$$

Now R possesses content, and by Professor PIERPONT's *Lectures*, § 733,

$$\overline{\int_R} g \geqq \overline{\int_S} \overline{\int_T} g,$$

and this by theorem 5 gives relation (1).

THEOREM 7. *Let the functions* $f(x_1, \cdots, x_m) \geqq 0$, $g(x_1, \cdots, x_m) \geqq 0$ *be defined over a limited aggregate* $\mathfrak{A}$. *Then*

$$(1) \quad \overline{\int_{\mathfrak{A}}} f + \overline{\int_{\mathfrak{A}}} g \geqq \overline{\int_{\mathfrak{A}}} (f+g) \geqq \overline{\int_{\mathfrak{A}}} f + \underline{\int_{\mathfrak{A}}} g \geqq \underline{\int_{\mathfrak{A}}} (f+g) \geqq \underline{\int_{\mathfrak{A}}} f + \underline{\int_{\mathfrak{A}}} g,$$

and provided the upper and lower integrals of $f - g$ *exist finitely or infinitely and the inequalities have a sense,*

$$(2) \quad \overline{\int_{\mathfrak{A}}} f - \underline{\int_{\mathfrak{A}}} g \geqq \overline{\int_{\mathfrak{A}}} (f-g) \geqq \underline{\int_{\mathfrak{A}}} f - \underline{\int_{\mathfrak{A}}} g \geqq \underline{\int_{\mathfrak{A}}} (f-g) \geqq \underline{\int_{\mathfrak{A}}} f - \overline{\int_{\mathfrak{A}}} g.$$

If f and g are limited functions, for every cell of $\mathfrak{A}$ we have the following inequalities:

$$\max f + \max g \geqq \max (f+g) \geqq \max f + \min g \geqq \min (f+g) \geqq \min f + \min g,$$

$$\max f - \min g \geqq \max (f-g) \geqq \min f - \min g \geqq \min (f-g) \geqq \min f - \max g.$$

Relations (1) and (2) follow immediately from these inequalities. When f and g are unlimited, we note that

$$f_{2\lambda} + g_{2\lambda} \geqq (f+g)_{2\lambda} \geqq f_\lambda + g_\lambda.$$

Hence from the previous proof for f and g limited,

$$\overline{\int_{\mathfrak{A}}} f_{2\lambda} + \overline{\int_{\mathfrak{A}}} g_{2\lambda} \geqq \overline{\int_{\mathfrak{A}}} (f_{2\lambda} + g_{2\lambda}) \geqq \overline{\int_{\mathfrak{A}}} (f+g)_{2\lambda} \geqq \overline{\int_{\mathfrak{A}}} (f_\lambda + g_\lambda) \geqq \overline{\int_{\mathfrak{A}}} f_\lambda + \underline{\int_{\mathfrak{A}}} g_\lambda.$$

On passing to the limit for $\lambda = \infty$, we obtain

$$\overline{\int_{\mathfrak{A}}} f + \overline{\int_{\mathfrak{A}}} g \geqq \overline{\int_{\mathfrak{A}}} (f+g) \geqq \overline{\int_{\mathfrak{A}}} f + \underline{\int_{\mathfrak{A}}} g.$$

In a similar manner the other inequalities of (1) are proved.

To establish (2), let us first assume that f is unlimited and g limited. Then for $\lambda > g$ we have the inequality

$$f_{2\lambda} - g \geqq (f - g)_\lambda \geqq f_\lambda - g.$$

Hence from the previous proof for f and g limited,

$$\overline{\int_{\mathfrak{A}}} f_{2\lambda} - \underline{\int_{\mathfrak{A}}} g \geqq \overline{\int_{\mathfrak{A}}} (f_{2\lambda} - g) \geqq \overline{\int_{\mathfrak{A}}} (f - g)_\lambda \geqq \overline{\int_{\mathfrak{A}}} (f_\lambda - g) \geqq \underline{\int_{\mathfrak{A}}} f_\lambda - \underline{\int_{\mathfrak{A}}} g.$$

On passing to the limit for $\lambda = \infty$ we have

$$\overline{\int_{\mathfrak{A}}} f - \underline{\int_{\mathfrak{A}}} g \geqq \overline{\int_{\mathfrak{A}}} (f - g) \geqq \underline{\int_{\mathfrak{A}}} f - \underline{\int_{\mathfrak{A}}} g,$$

and the remaining inequalities of (2) are similarly established.

When f and g are unlimited, it may be noted that for all values of λ

$$\overline{\int_{\mathfrak{A}}} f - \underline{\int_{\mathfrak{A}}} g_\lambda \geqq \overline{\int_{\mathfrak{A}}} (f - g_\lambda) \geqq \overline{\int_{\mathfrak{A}}} (f - g).$$

A passage to the limit for $\lambda = \infty$ gives the first inequality of (2). If the second inequality of (2) does not hold, λ can be found so large that

$$\overline{\int_{\mathfrak{A}}} (f - g) < \underline{\int_{\mathfrak{A}}} f_\lambda - \underline{\int_{\mathfrak{A}}} g,$$

or, by taking negatives,

$$\underline{\int_{\mathfrak{A}}} g - \underline{\int_{\mathfrak{A}}} f_\lambda < \underline{\int_{\mathfrak{A}}} (g - f) \leqq \underline{\int_{\mathfrak{A}}} (g - f_\lambda).$$

Here g is unlimited and f_λ limited, and this relation contradicts the third inequality of (2) which has been proved for such a case. Hence the second inequality of (2) holds for all cases. The remaining inequalities may be established by taking the negative of the first two.

Corollary 1. *The relations* (1) *and* (2) *hold if* $f \geqq -G$, $g \geqq -G$, *where* G *is arbitrarily large but fixed.*

For

$$\underline{\int_{\mathfrak{A}}} (f - g) = \underline{\int_{\mathfrak{A}}} [(f + G) - (g + G)] \geqq \underline{\int_{\mathfrak{A}}} (f + G) - \overline{\int_{\mathfrak{A}}} (g + G)$$

$$= \underline{\int_{\mathfrak{A}}} f + \overline{\int_{\mathfrak{A}}} G - \overline{\int_{\mathfrak{A}}} g - \overline{\int_{\mathfrak{A}}} G = \underline{\int_{\mathfrak{A}}} f - \overline{\int_{\mathfrak{A}}} g,$$

which gives the last relation of (2). The other relations are established in a similar manner.

COROLLARY 2. *If g is integrable,*

$$\overline{\int_{\mathfrak{A}}}(f+g)=\overline{\int_{\mathfrak{A}}}f+\int_{\mathfrak{A}}g, \qquad \underline{\int_{\mathfrak{A}}}(f+g)=\underline{\int_{\mathfrak{A}}}f+\int_{\mathfrak{A}}g,$$

$$\overline{\int_{\mathfrak{A}}}(f-g)=\overline{\int_{\mathfrak{A}}}f-\int_{\mathfrak{A}}g, \qquad \underline{\int_{\mathfrak{A}}}(f-g)=\underline{\int_{\mathfrak{A}}}f-\int_{\mathfrak{A}}g.$$

THEOREM 8. *Suppose the upper or lower integral of $f(x_1, \cdots, x_m)$ over $\mathfrak{A}$ to exist. Then the upper and lower integral respectively of $f-f_{\lambda_1\lambda_2}$ exists, and correspondingly*

$$(1) \qquad \overline{\int_{\mathfrak{A}}}f-\overline{\int_{\mathfrak{A}}}f_{\lambda_1\lambda_2}=\overline{\int_{\mathfrak{A}}}(f-f_{\lambda_1\lambda_2}), \qquad \underline{\int_{\mathfrak{A}}}f-\underline{\int_{\mathfrak{A}}}f_{\lambda_1\lambda_2}=\underline{\int_{\mathfrak{A}}}(f-f_{\lambda_1\lambda_2}).$$

For a limited function f this is true, since for every cell,

$$\max f-\max f_{\lambda_1\lambda_2}=\max(f-f_{\lambda_1\lambda_2}), \qquad \min f-\min f_{\lambda_1\lambda_2}=\min(f-f_{\lambda_1\lambda_2}).$$

If f is unlimited, the following relation holds for all points of $\mathfrak{A}$:

$$f_{\lambda_1+\lambda,\,\lambda_2+\lambda}-f_{\lambda_1\lambda_2}=(f-f_{\lambda_1\lambda_2})_{\lambda,\,\lambda}.$$

By means of this equality, it is evident that

$$\overline{\int_{\mathfrak{A}}}f_{\lambda_1+\lambda,\,\lambda_2+\lambda}-\overline{\int_{\mathfrak{A}}}f_{\lambda_1\lambda_2}=\overline{\int_{\mathfrak{A}}}(f-f_{\lambda_1\lambda_2})_{\lambda,\,\lambda}; \quad \underline{\int_{\mathfrak{A}}}f_{\lambda_1+\lambda,\,\lambda_2+\lambda}-\underline{\int_{\mathfrak{A}}}f_{\lambda_1\lambda_2}=\underline{\int_{\mathfrak{A}}}(f-f_{\lambda_1\lambda_2})_{\lambda,\,\lambda}.$$

By passing to the limit for $\lambda=\infty$ the theorem is established.

It may be noted that if one of the integrals exists infinitely, the theorem is still valid.

THEOREM 9. *Let $f(x_1, \cdots, x_m)$ be defined over $\mathfrak{A}$. Define the auxiliary functions $g(x_1, \cdots, x_m)$, $h(x_1, \cdots, x_m)$ as follows:*

$$g=f \text{ for } f>0,\ g=0 \text{ for } f\leqq 0; \qquad h=-f \text{ for } f<0,\ h=0 \text{ for } f\geqq 0.$$

Then

$$(1) \qquad \overline{\int_{\mathfrak{A}}}f=\overline{\int_{\mathfrak{A}}}g-\underline{\int_{\mathfrak{A}}}h,$$

$$(2) \qquad \underline{\int_{\mathfrak{A}}}f=\underline{\int_{\mathfrak{A}}}g-\overline{\int_{\mathfrak{A}}}h,$$

provided in each equation one of the integrals on the right exists finitely.

If first f is assumed to be a limited function, it will be observed that the cells of a division may be separated into two classes; 1) those in which $g = 0$ throughout the cell, and 2) those in which $f > 0$ for at least one point of the cell. In either case $\max(g - h) = \max g - \min h$. By the usual process this gives (1).

If now f is unlimited, we have for all values of λ_1, λ_2

$$\overline{\int}_{\mathfrak{A}} f_{\lambda_1\lambda_2} = \overline{\int}_{\mathfrak{A}} (g - h)_{\lambda_1\lambda_2} = \overline{\int}_{\mathfrak{A}} g_{\lambda_1} - \underline{\int}_{\mathfrak{A}} h_{\lambda_2},$$

whence equation (1) at once follows by passing to the limit for $\lambda_1 = \infty$, $\lambda_2 = \infty$. Relation (2) is established in a similar manner.

THEOREM 10. *Let $f(x_1, \cdots, x_m)$ be defined over a limited aggregate $\mathfrak{A}$. Let $\mathfrak{B}$ and $\mathfrak{C}$ be the corresponding aggregates in p and q dimensions. If the integrals*

$$\int_{\mathfrak{B}} \overline{\underline{\int}}_{\mathfrak{C}} f, \qquad \int_{\mathfrak{B}} \overline{\underline{\int}}_{\mathfrak{C}} f_{\lambda_1\lambda_2}$$

exist, then

$$(1) \qquad \left| \int_{\mathfrak{B}} \overline{\underline{\int}}_{\mathfrak{C}} (f - f_{\lambda_1\lambda_2}) \right| \leqq \underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} | f - f_{\lambda_1\lambda_2} | \leqq \underline{\int}_{\mathfrak{B}} \overline{\int}_{\mathfrak{C}} | f - f_{\lambda_1\lambda_2} |.$$

For consider the upper and lower integrals of f over the sections $\mathfrak{C}$. For various points of $\mathfrak{B}$ the following cases arise: (i) At least one of these integrals does not exist either finitely or infinitely; (ii) one integral (or both) exists infinitely; (iii) the upper integral exceeds the lower integral by an amount equal to or greater than a given $\eta > 0$; (iv) the difference between the upper and lower integrals is less than η. The points of $\mathfrak{B}$ belonging to cases (i) and (ii), and to (iii) for a given value of η, form a discrete aggregate. Denoting by $\mathfrak{B}_\sigma$ the points of $\mathfrak{B}$ remaining when this discrete set is enclosed in cells of volume less than σ, it follows from property (iv) of integrals (p. 345) that for a sufficiently small σ

$$\left| \int_{\mathfrak{B}} \overline{\underline{\int}}_{\mathfrak{C}} f - \int_{\mathfrak{B}_\sigma} \overline{\underline{\int}}_{\mathfrak{C}} f \right| < \epsilon.$$

In a similar manner, we have for a properly chosen σ

$$(2) \quad \left| \int_{\mathfrak{B}} \overline{\underline{\int}}_{\mathfrak{C}} f_{\lambda_1\lambda_2} - \int_{\mathfrak{B}_\sigma} \overline{\underline{\int}}_{\mathfrak{C}} f_{\lambda_1\lambda_2} \right| < \epsilon, \quad \left| \int_{\mathfrak{B}} \overline{\underline{\int}}_{\mathfrak{C}} (f - f_{\lambda_1\lambda_2}) - \int_{\mathfrak{B}_\sigma} \overline{\underline{\int}}_{\mathfrak{C}} (f - f_{\lambda_1\lambda_2}) \right| < \epsilon.$$

Since in any cell of $\mathfrak{C}$ the oscillation of the function $| f - f_{\lambda_1\lambda_2} | = | f | - | f_{\lambda_1\lambda}$ is equal to or less than the oscillation of $f - f_{\lambda_1\lambda_2}$, it follows that

$$\overline{\int_{\mathfrak{C}}}|f-f_{\lambda_1\lambda_2}|-\underline{\int_{\mathfrak{C}}}|f-f_{\lambda_1\lambda_2}|\leqq\overline{\int_{\mathfrak{C}}}(f-f_{\lambda_1\lambda_2})-\underline{\int_{\mathfrak{C}}}(f-f_{\lambda_1\lambda_2}).$$

By means of theorem 8 this becomes for every point of $\mathfrak{B}_\sigma$

$$0\leqq\overline{\int_{\mathfrak{C}}}|f-f_{\lambda_1\lambda_2}|-\underline{\int_{\mathfrak{C}}}|f-f_{\lambda_1\lambda_2}|\leqq\overline{\int_{\mathfrak{C}}}f-\underline{\int_{\mathfrak{C}}}f-\left(\overline{\int_{\mathfrak{C}}}f_{\lambda_1\lambda_2}-\underline{\int_{\mathfrak{C}}}f_{\lambda_1\lambda_2}\right)<\eta,$$

whence we have with the aid of the second inequality of (2), theorem 7,

$$0\leqq\underline{\int_{\mathfrak{B}_\sigma}}\overline{\int_{\mathfrak{C}}}|f-f_{\lambda_1\lambda_2}|-\underline{\int_{\mathfrak{B}_\sigma}}\underline{\int_{\mathfrak{C}}}|f-f_{\lambda_1\lambda_2}|\leqq\eta\bar{\mathfrak{B}}_\sigma. \tag{3}$$

Since now

$$\left|\overline{\int_{\mathfrak{C}}}(f-f_{\lambda_1\lambda_2})\right|\leqq\overline{\int_{\mathfrak{C}}}|f-f_{\lambda_1\lambda_2}|,$$

and since an integrable function is absolutely integrable, we have

$$\int_{\mathfrak{B}_\sigma}\left|\overline{\int_{\mathfrak{C}}}(f-f_{\lambda_1\lambda_2})\right|\leqq\underline{\int_{\mathfrak{B}_\sigma}}\overline{\int_{\mathfrak{C}}}|f-f_{\lambda_1\lambda_2}|.$$

And *a fortiori*

$$\left|\int_{\mathfrak{B}_\sigma}\overline{\int_{\mathfrak{C}}}(f-f_{\lambda_1\lambda_2})\right|\leqq\underline{\int_{\mathfrak{B}_\sigma}}\overline{\int_{\mathfrak{C}}}|f-\quad{}_{\lambda_1\lambda_2}|. \tag{4}$$

By combining (2) and (3) with (4) we obtain

$$\left|\int_{\mathfrak{B}}\overline{\int_{\mathfrak{C}}}(f-f_{\lambda_1\lambda_2})\right|-\epsilon\leqq\underline{\int_{\mathfrak{B}_\sigma}}\underline{\int_{\mathfrak{C}}}|f-f_{\lambda_1\lambda_2}|+\eta\bar{\mathfrak{B}}_\sigma\leqq\underline{\int_{\mathfrak{B}}}\underline{\int_{\mathfrak{C}}}|f-f_{\lambda_1\lambda_2}|+\eta\bar{\mathfrak{B}}_\sigma.$$

Since ϵ and η are small at pleasure, this gives

$$\left|\int_{\mathfrak{B}}\overline{\int_{\mathfrak{C}}}(f-f_{\lambda_1\lambda_2})\right|\leqq\underline{\int_{\mathfrak{B}}}\underline{\int_{\mathfrak{C}}}|f-f_{\lambda_1\lambda_2}|\leqq\underline{\int_{\mathfrak{B}}}\overline{\int_{\mathfrak{C}}}|f-f_{\lambda_1\lambda_2}|.$$

THEOREM 11. *Let the limited function $f\geqq 0$ be defined over a limited aggregate $\mathfrak{A}$. Let $A_1\leqq A_2\leqq\cdots$ be a series of partial aggregates of $\mathfrak{A}$ such that $\lim_{n=\infty}\bar{A}_n=\bar{\mathfrak{A}}$. At every point of any A_n let $f<\sigma$. Then*

$$\underline{\int_{\mathfrak{A}}}f\leqq\sigma\bar{\mathfrak{A}}.$$

Denote by L the least common multiple of $A_1, A_2, \cdots$. Then since L is a partial aggregate of $\bar{\mathfrak{A}}$, $\bar{A}_n\leqq\bar{L}\leqq\bar{\mathfrak{A}}$. This is true for all values of n, and passing to the limit we have $\bar{L}=\bar{\mathfrak{A}}$. For ϵ small at pleasure it is possible to effect a division of space such that the volume of cells containing at least one point of $\mathfrak{A}$ is less than $\bar{\mathfrak{A}} + \epsilon$. Since the minimum volume of cells containing at least one point of L is $\bar{\mathfrak{A}}$, the volume of cells containing a point of $\mathfrak{A}$ and not containing one of L must be less than ϵ. Now f is limited, and the integral over the points in these cells is small at pleasure. In the remaining cells the minimum of f is certainly not greater than σ. Hence

$$\underline{\int}_{\mathfrak{A}} f \leqq \sigma \bar{\mathfrak{A}}.$$

COROLLARY. *Denote by* f_n *a limited function which in the points of* A_n *is equal to* f *and which at the remaining points of* $\mathfrak{A}$ *is arbitrary except that it be positive and limited. Then for* η *small at pleasure,* n *may be taken so great that*

$$\underline{\int}_{\mathfrak{A}} f_n < \sigma \bar{\mathfrak{A}} + \eta. \tag{1}$$

For n may be so chosen that $\bar{A}_n > \bar{\mathfrak{A}} - \epsilon$. In other words, the volume of the cells where the minimum of f_n is greater than σ, while the minimum of f is equal to or less than σ, is small at pleasure. By reasoning analogous to that in theorem 11 we arrive at relation (1).

THEOREM 12. *Let* $f(x_1, \cdots, x_m)$ *be limited in the field* A *which possesses content. Then*

$$\underline{\int}_A f \leqq \underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} f \leqq \overline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} f \leqq \overline{\int}_A f, \qquad \underline{\int}_A f \leqq \underline{\int}_{\mathfrak{B}} \overline{\int}_{\mathfrak{C}} f \leqq \overline{\int}_{\mathfrak{B}} \overline{\int}_{\mathfrak{C}} f \leqq \overline{\int}_A f.$$

The proof is essentially that given by Professor PIERPONT* for a slightly less general case.

COROLLARY. *If the multiple integral exists, then*

$$\int_A f = \int_{\mathfrak{B}} \overline{\int}_{\mathfrak{C}} f = \int_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} f = \int_{\mathfrak{B}} \overline{\underline{\int}}_{\mathfrak{C}} f.$$

§ 3. *Integration of a sequence of functions.*

Let us consider the integration of a sequence of functions $f_1(x_1, \cdots, x_m)$, $f_2(x_1, \cdots, x_m)$, $\cdots$ *defined over a limited aggregate* A *which either possesses content or is closed.* We assume that for each point of A the sequence f_1, f_2, $\cdots$ has a limiting value which is finite or definitely infinite. The aggregate of such limiting values defines a new function

* *Lectures*, § 733.

$$f(x_1, \cdots, x_m) = \lim_{n=\infty} f_n(x_1, \cdots, x_m).$$

Unless otherwise stated, no restriction is put on the continuity of the functions $f, f_1, f_2, \cdots$, which may take on finite or infinite values.

THEOREM 13. *Let $f_1(x_1, \cdots, x_m), f_2(x_1, \cdots, x_m), \cdots$ be for each $(x_1, \cdots, x_m)$ in A a monotone sequence of functions converging to $f(x_1, \cdots, x_m)$.*

1) *If the functions are positive and the sequence is increasing,*

$$\overline{\int_A} f \geqq \lim_{n=\infty} \overline{\int_A} f_n \geqq \underline{\int_A} f \geqq \lim_{n=\infty} \underline{\int_A} f_n. \tag{1}$$

2) *If the functions are negative* and the sequence is decreasing,*

$$\lim_{n=\infty} \overline{\int_A} f_n \geqq \overline{\int_A} f \geqq \lim_{n=\infty} \underline{\int_A} f_n \geqq \underline{\int_A} f. \tag{2}$$

To prove (1) let us assume in the first place that f is limited throughout A. Choose $\sigma > 0$ small at pleasure, and denote by A_n the partial aggregate of A where $f - f_n \leqq \sigma$. It is evident that $A_n \leqq A_{n+1}$. The least common multiple of the sequence $A_1, A_2, \cdots$ is the aggregate A. For, since $f_n(P)$ converges to $f(P)$, it follows that any point P of A belongs to the sequence $A_\nu, A_{\nu+1}, \cdots$ for a sufficiently large ν. Hence by the corollary of theorem 2, $\lim_{n=\infty} \bar{A}_n = \bar{A}$. Applying now the corollary to theorem 11, we have by proper choice of n, η, and σ,

$$\underline{\int_A} (f - f_n) < \sigma \bar{A} + \eta < \epsilon.$$

Hence from theorem 7

$$\underline{\int_A} f - \overline{\int_A} f_n < \epsilon,$$

where the expression on the left is not necessarily positive or zero. By passing to the limit the second relation of (1) is established.

If now f is unlimited, two cases arise according as the lower integral of f is finite or infinite. If finite, then for proper choice of λ

$$\lim_{n=\infty} \overline{\int_A} f_n \geqq \lim_{n=\infty} \overline{\int_A} (f_n)_\lambda \geqq \underline{\int_A} f_\lambda \geqq \underline{\int_A} f - \epsilon.$$

Since this is true for infinitesimal values of ϵ, we have the desired relation. If, on the other hand, the lower integral of f is infinite, λ may be so chosen that for M great at pleasure,

* It will hereafter be assumed without express statement that if a theorem is true for $f \geqq 0$, a corresponding theorem is true for $f \leqq 0$.

$$\underline{\int_A} f_\lambda > M.$$

Hence

$$\lim_{n=\infty} \overline{\int_A} f_n \geqq \lim_{n=\infty} \overline{\int_A} (f_n)_\lambda \geqq \underline{\int_A} f_\lambda > M,$$

and

$$\lim_{n=\infty} \overline{\int_A} f_n = \infty .$$

This establishes the second relation of (1) for all cases. The other parts are obvious since $f \geqq f_n$. To prove (2) it suffices to take the negative of the terms of (1).

Corollary. *If the positive functions $f_1, f_2, \cdots$ are integrable in A, then*

$$\lim_{n=\infty} \int_A f_n = \underline{\int_A} f.$$

It may be remarked that the relations (1) of this theorem and corollary are true if each function of the sequence is greater than $-G$, where G is arbitrarily large but fixed. (See theorem 7, corollary 1.)

The points where $f - f_n \geqq \sigma > 0$ may be everywhere dense for all values of n. The complementary aggregate, however, is such that its upper content may, by proper choice of n, be made to differ by an infinitesimal from the upper content of A. Examples may be given where

$$\lim_{n=\infty} \overline{\int_A} f_n = 0, \qquad \overline{\int_A} f = + M,$$

and also where

$$\lim_{n=\infty} \underline{\int_A} f_n = 0, \qquad \underline{\int_A} f = + M,$$

M being finite or definitely infinite.

Theorem 14. *Let $f_1(x_1, \cdots, x_m)$, $f_2(x_1, \cdots, x_m)$, $\cdots$ be for each $(x_1, \cdots, x_m)$ of A a monotone decreasing sequence of functions converging to $f(x_1, \cdots, x_m) \geqq 0$. If the upper integral of f_1 over A exists, then*

$$\lim_{n=\infty} \overline{\int_A} f_n \geqq \overline{\int_A} f \geqq \lim_{n=\infty} \underline{\int_A} f_n \geqq \underline{\int_A} f. \tag{1}$$

We need only to establish the middle inequality.

If f_1 is limited, it is readily shown as in theorem 13 that n may be chosen such that for $\epsilon > 0$,

$$\underline{\int_A} (f_n - f) < \epsilon.$$

Hence from theorem 7

$$\underline{\int}_A f_n - \overline{\int}_A f < \epsilon .$$

This establishes the desired inequality.

If f_1 is unlimited, λ_0 may be found such that for $\lambda > \lambda_0$ (theorem 8)

$$\overline{\int}_A [f_1 - (f_1)_\lambda] = \overline{\int}_A f_1 - \overline{\int}_A (f_1)_\lambda < \epsilon .$$

And since $f_n - (f_n)_\lambda \leqq f_1 - (f_1)_\lambda$ for all values of n, we have

$$\overline{\int}_A [f_n - (f_n)_\lambda] < \epsilon . \tag{2}$$

From the preceding part of the proof n may be chosen such that

$$\overline{\int}_A [(f_n)_\lambda - f_\lambda] < \epsilon . \tag{3}$$

Hence from theorem 7 and relations (2) and (3) it follows that

$$\underline{\int}_A f_n - \overline{\int}_A f \leqq \underline{\int}_A f_n - \overline{\int}_A f_\lambda \leqq \underline{\int}_A (f_n - f_\lambda) \leqq \overline{\int}_A [f_n - (f_n)_\lambda] + \underline{\int}_A [(f_n)_\lambda - f_\lambda] < 2\epsilon ,$$

which establishes (1).

Corollary. *If $f_1, f_2, \cdots$ are integrable in A, then*

$$\lim_{n=\infty} \int_A f_n = \overline{\int}_A f .$$

Theorem 15. *Let $f_1(x_1, \cdots, x_m)$, $f_2(x_1, \cdots, x_m)$, $\cdots$ be a monotone sequence of integrable functions which increases simultaneously or decreases simultaneously at every $(x_1, \cdots, x_m)$ of A and converges to $f(x_1, \cdots, x_m)$. If f is integrable,*

$$\lim_{n=\infty} \int_A f_n = \int_A f .$$

Define the auxiliary functions g_n, g, h_n, h as follows:

$$g_n = f_n \text{ for } f_n > 0 ; \quad g_n = 0 \text{ for } f_n \leqq 0 ; \quad h_n = -f_n \text{ for } f_n < 0 ; \quad h_n = 0 \text{ for } f_n \geqq 0 ;$$
$$g = f \text{ for } f > 0 ; \quad g = 0 \text{ for } f \leqq 0 ; \quad h = -f \text{ for } f < 0 ; \quad h = 0 \text{ for } f \geqq 0 .$$

Then it follows from the corollaries to theorems 13 and 14, that

$$\lim_{n=\infty}\int_A h_n = \int_A h, \qquad \lim_{n=\infty}\int_A g_n = \int_A g.$$

Hence

$$\lim_{n=\infty}\int_A f_n = \lim_{n=\infty}\int_A g_n - \lim_{n=\infty}\int_A h_n = \int_A g - \int_A h = \int_A f.$$

THEOREM 16. *In A let the positive functions $f_1(x_1, \cdots, x_m), f_2(x_1, \cdots, x_m)$, $\cdots$ converge to $f(x_1, \cdots, x_m)$. At the points of a partial aggregate A_1 let the sequence be monotone and increasing and at the points of the complementary aggregate A_2 let the sequence be monotone and decreasing. If the upper integral of f_1 exists, then*

$$(1) \qquad \lim_{n=\infty}\overline{\int_A} f_n \geqq \underline{\int_A} f, \qquad (2) \quad \lim_{n=\infty}\underline{\int_A} f_n \leqq \int_A f.$$

Let the auxiliary functions ϕ, ϕ_n, ψ, ψ_n be defined as follows:

$$\phi_n = 0 \text{ for points of } A_2, \qquad \phi_n = f_n \text{ for points of } A_1, \qquad \lim_{n=\infty}\phi_n = \phi.$$

$$\psi_n = f_n \text{ for points of } A_2, \qquad \psi_n = 0 \text{ for points of } A_1, \qquad \lim_{n=\infty}\psi_n = \psi.$$

Then $f = \phi + \psi$, $f_n = \phi_n + \psi_n$. If the lower integral of ϕ is limited, it follows from the proof of theorem 13 that for $\epsilon > 0$, n may be found such that

$$\underline{\int_A}(\phi - \phi_n) \leqq \epsilon.$$

Since $f - f_n = \phi - \phi_n + \psi - \psi_n \leqq \phi - \phi_n$, this gives

$$\underline{\int_A}(f - f_n) \leqq \underline{\int_A}(\phi - \phi_n) \leqq \epsilon.$$

Hence from theorem 7,

$$\underline{\int_A} f - \overline{\int_A} f_n \leqq \epsilon,$$

and relation (1) is proved.

When the lower integral of ϕ is infinite λ may be chosen such that for M large at pleasure

$$\underline{\int_A}\phi_\lambda \geqq M.$$

Hence

$$\lim_{n=\infty}\overline{\int_A} f_n \geqq \lim_{n=\infty}\overline{\int_A}(f_n)_\lambda \geqq \underline{\int_A} f_\lambda \geqq \underline{\int_A}\phi_\lambda \geqq M$$

and

$$\lim_{n=\infty} \overline{\int}_A f_n = \infty.$$

This completes the proof of (1).

If we use the inequality $f_n - f \leqq \psi_n - \psi$, relation (2) is readily established by means of theorem 14.

THEOREM 17. *Let the convergence* of* $f_1(x_1, \cdots, x_m), f_2(x_1, \cdots, x_m), \cdots$ *to* $f(x_1, \cdots, x_m)$ *be monotone (for some points of A monotone and increasing and for some points monotone and decreasing). Then if the upper integral of* $|f_1|$ *exists,*

$$\lim_{n=\infty} \overline{\int}_A f_n \geqq \int_A f \tag{1}$$

provided this limit and integral exist, finitely or infinitely. And further

$$\lim_{n=\infty} \underline{\int}_A f_n \leqq \overline{\int}_A f \tag{2}$$

provided the limit and integral exist, finitely or infinitely.

Define the auxiliary functions g and h as follows:

$$g = f \text{ for } f > 0, \quad g = 0 \text{ for } f \leqq 0; \quad h = -f \text{ for } f < 0, \quad h = 0 \text{ for } f \geqq 0.$$

Then by theorem 16

$$\lim_{n=\infty} \overline{\int}_A g_n \geqq \underline{\int}_A g, \qquad \lim_{n=\infty} \underline{\int}_A h_n \leqq \overline{\int}_A h.$$

It follows from theorem 9 that when $\int_A f$ exists finitely or infinitely, either the lower integral of g or the upper integral of h is limited. Subtracting, we have by means of theorem 9

$$\lim_{n=\infty} \overline{\int}_A f_n = \lim_{n=\infty} \overline{\int}_A (g_n - h_n) = \lim_{n=\infty} \left[\overline{\int}_A g_n - \underline{\int}_A h_n \right] \geqq \underline{\int}_A g - \overline{\int}_A h = \underline{\int}_A (g - h) = \underline{\int}_A f.$$

Relation (2) is established in a similar manner.

COROLLARY. *Let the convergence* of* $f_1, f_2, \cdots$ *to* f *be monotone. Then provided the integrals of* $f, f_2, f_3, \cdots$ *exist finitely or infinitely and the integral of* f_1 *exists finitely,*

$$\lim_{n=\infty} \int_A f_n = \int_A f.$$

THEOREM 18. *Let* $f_1(x_1, \cdots, x_m), f_2(x_1, \cdots, x_m), \cdots$ *be a sequence of functions integrable in A and converging to* $f(x_1, \cdots, x_m)$. *Let*

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* This theorem is true for $m \geqq 2$. If $m = 1$, it is true if the integrals (finite or infinite) exist absolutely.

$$| f_n(x_1, \cdots, x_m) | < M, \qquad n \geqq n_0$$

where M is arbitrarily large but fixed. If

$$\lim_{n=\infty} \int_A f_n, \qquad \int_A f$$

exist, they are equal.

On denoting by A_n the partial aggregate of A for which $| f - f_m | \leqq \sigma$, where m takes on the values $n, n+1, \cdots$, it may be shown as in theorem 13 that for proper choice of n

$$\int_A | f - f_n | = \int_A | f - f_n | < \epsilon.$$

Hence

$$\left| \int_A f - \int_A f_n \right| = \left| \int_A (f - f_n) \right| < \epsilon,$$

and the theorem is proved.

A corresponding theorem for the differentiation of a sequence of functions may be easily derived.

THEOREM 19. *Let $f_1(x), f_2(x), \cdots$ be a sequence of differentiable functions in an interval (a, b) which converge to a differentiable limit. Let $f_1'(x)$, $f_2'(x), \cdots$ be integrable and suppose also* $\lim_{n=\infty} f_n'(x) \equiv \phi(x)$ *to exist either finitely or infinitely in (a, b). If then either*

1) *$f_1', f_2', \cdots$ form a monotone sequence at each point of (a, b) (increasing at some or all points, decreasing at the points of the complementary aggregate, the infinite discontinuities in each case forming at most an aggregate of the first species);*

or 2) $| f_i'(x) | \leqq G (i = 1, 2, \cdots)$, *$G$ being arbitrarily large but fixed; then for each point of continuity of ϕ,*

$$D_x(f) = \phi(x).$$

For by theorems 13 (corollary), 17, and 18, and the fundamental theorem of the integral calculus *

$$\int_a^x \phi(x) = \lim_{n=\infty} \int_a^x f_n'(x) = \lim_{n=\infty} \{ f_n(x) - f_n(a) \}.$$

Differentiating this expression, we have for each point of continuity of ϕ, †

$$\phi(x) = D_x [\lim_{n=\infty} f_n(x)] = D_x f.$$

* DE LA VALLÉE-POUSSIN, Journal de Mathématiques, ser. 4, vol. 8 (1892), p. 430.

† *Lectures*, § 604.

§ 4. *Reduction of multiple integrals.*

The problem of the reduction of a multiple integral depends in large measure upon the nature of the field, and the arrangement of the discontinuities. If a function f is limited $(|f| < G)$ throughout a discrete aggregate $\mathfrak{A}_1$, the multiple integral and the iterated integral of f over $\mathfrak{A}_1$ are both zero. For in this case the points of $\mathfrak{B}_1$ where $\bar{\mathfrak{C}}_1 \geqq \epsilon > 0$, or where

$$\overline{\int_{\mathfrak{C}_1}} |f| > G\,\epsilon$$

form a discrete aggregate. This being true for every $\epsilon > 0$, we have

$$\left| \int_{\mathfrak{B}_1} \underline{\overline{\int_{\mathfrak{C}_1}}} f \right| = \overline{\int_{\mathfrak{B}_1}} \overline{\int_{\mathfrak{C}_1}} |f| = 0 \,.$$

For an unlimited function this reasoning will no longer hold, as is seen by a reference to Example I below. The aggregate of points of $\mathfrak{B}_1$ where $\bar{\mathfrak{C}} > 0$ form an aggregate of measure zero (theorem 3), but this aggregate may be everywhere dense in $\mathfrak{B}_1$. Now if $\mathfrak{A}_1$ represents the discrete partial aggregate of $\mathfrak{A}$ where the integrable function has discontinuities equal to or greater than $\sigma > 0$, we see that if f is limited, the multiple integral may be replaced by the iterated integral. If, however, these discontinuities are infinite, this is not always possible. This is the fundamental difference between the problems of reduction of multiple integrals of limited and unlimited functions.

Before entering upon a discussion of the theory, it may be well to consider some examples.

Example I. Denote by R the rectangle $(1, 0)$, $(1, 1)$ $(-1, 1)$, $(-1, 0)$, and by T those points whose coördinates are of the form $|x| \leqq 1/2^n$, $y = (2m + 1)/2^n$ where m and n are positive integers. The aggregate T is discrete in two dimensions and hence the double integral of any function $f(x, y)$ over T is zero. Giving to $f(x, y)$ the value 2^n for the points of T, we note that

$$\int_{X_T} f(x, y) = 2, \qquad \int_{Y_T} \int_{X_T} f(x, y) = 2, \qquad \int_T f(x, y) = 0,$$

where X_T, Y_T denote the sections of the aggregate T. Assigning to $f(x, y)$ the value 2^n at the points T and the value zero at the remaining points of R, we have

$$\int_R f = 0, \qquad \underline{\int_Y} \int_X f = 0, \qquad \overline{\int_Y} \int_X f = 2.$$

Example II. In the rectangle R of the preceding example assign to $f(x, y)$ the value 2^n for points of T, the value 1 for points whose ordinates are not of the form $y = (2m + 1)/2^n$ and the value 0 for the remaining points. The integral over each section X is 2, and hence the iterated integral

$$\int_Y \int_X f$$

exists. However, the integral

$$\int_Y \int_X f_\lambda$$

exists for no value of λ.

Example III. Assign to $f(x, y)$ the value $2^n(1/x \cos 1/x)$ at the points of T (Example I) and the value zero for the remaining points of R. Then since T is discrete in two dimensions,

$$\int_R f = \int_R |f| = 0.$$

We have also

$$\int_Y \int_X f = 0.$$

On the other hand

$$\int_X |f| = \infty$$

for all points of Y whose coördinates are of the form $y = (2m + 1)/2^n$, and the iterated integral of $|f|$ does not exist.

Example IV. At the points of T (Example I) whose abscissæ are rational numbers let $f(x, y) = 1$, and at the remaining points of T let $f(x, y) = 0$. Then

$$\int_T f = \int_Y \overline{\underline{\int_X}} f = 0.$$

Example V. At the points of T (Example I) at which $x > 0$ let $f(x, y) = 2^n$; where $x = 0$ let $f(x, y) = 0$; and where $x < 0$ let $f(x, y) = -2^n$. Then

$$\int_R f = \int_Y \int_X f = 0.$$

Example VI. PRINGSHEIM* has defined an aggregate everywhere dense in two dimensions and yet having only a finite number of points on any line parallel to the X or Y axis. Denote by P the aggregate of such points contained in a unit square S. Assign to $f(x, y)$ the value $M > 0$ throughout the aggregate P. Then

$$\int_P f = M, \qquad \int_{Y_P} \int_{X_P} f = \int_{X_P} \int_{Y_P} f = 0.$$

*Münchener Sitzungsberichte (1899), or PIERPONT's *Lectures*, p. 546.

If now we consider the square S and assign to f the value 0 at the points of S not belonging to P,

$$\int_X\int_Y f=\int_Y\int_X f=\underline{\int_S} f=0, \qquad \overline{\int_S} f=M.$$

In his classic memoir * DE LA VALLÉE-POUSSIN has shown that in case the double integral and the iterated integral exist, a sufficient condition for their equality is the regular convergence of the integral

$$\int_X f(x, y)\,dx.$$

In undertaking to prove this condition necessary he restricts his discussion to the case of a function of one sign only. Let T be the fundamental rectangle with sides parallel to the axes. Denote by T_σ the points of T at which the discontinuity of f exceeds $\sigma>0$, and by X_σ the points of T_σ in any section parallel to the x axis. Let Y_1 denote the partial aggregate of Y such that $\overline{X}_\sigma>0$. If Y_1 is discrete, DE LA VALLÉE-POUSSIN proved that the condition of regular convergence is necessary. It may be shown that the points of Y where $\overline{X}_\sigma \geqq \delta>0$ form a discrete aggregate and that consequently Y_1 is of measure zero. That Y_1 is not necessarily discrete is seen from example IV.

In 1899 DE LA VALLÉE-POUSSIN returned to the problem and devoted a memoir † to its discussion. He states that, although he has not been able to remove the restrictions cited, he believes them unnecessary. A minimum function mf is introduced which is defined at each point as the minimum value of f in the infinitesimal vicinity of that point. For a positive function it is shown that

$$\overline{\int_T} f \geqq \underline{\int_Y}\,\overline{\int_X} mf \geqq \underline{\int_X}\,\underline{\int_Y} mf \geqq \underline{\int_T} f. \tag{1}$$

A comparison of (1) with the results of theorem 20 show that the introduction of the minimum function is unnecessary.

This problem was later discussed by SCHÖNFLIES,‡ his methods involving the introduction of a "most nearly continuous function," corresponding in some measure to the minimum function of DE LA VALLÉE POUSSIN. Although the double integral of the auxiliary function is equal to the double integral of f, the existence of the iterated integral of f is not a necessary condition for the existence and equality of the iterated integral of the "most nearly continuous function." The statement of SCHÖNFLIES that the convergence of

* Loc. cit.

† Journal de Mathématiques, ser. 5, vol. 5 (1899).

‡ *Die Entwickelung der Lehre von den Punktmannigfaltigkeiten* (1900), p. 210.

$$\int_Y \int_X f_\lambda$$

to the iterated integral of f is a simple process (though apparently a double limit) is open to criticism. Example I furnishes an illustration of the fact that the double integral and the iterated integral may exist in a field possessing content and that at the same time the values of these integrals may be different.

The results of DE LA VALLÉE-POUSSIN have been extended in a recent article by HOBSON.* He also introduces an auxiliary function. The field G which possesses content is enclosed in a rectangle, the auxiliary function f being equal to the original function ϕ at the points where the latter is defined, and equal to zero at the remaining points of the rectangle. There is no investigation of the relations between the integrals of f and ϕ over the rectangle and G respectively. It is evident that the double integrals are equal, but that this is not necessarily true for the iterated integrals is shown by Example I. For the auxiliary function f, HOBSON shows that in case it is positive the existence of the double integral and the iterated integrals is a sufficient condition for their equality.† To obtain this result there is no need of his generalization of the notion of regular convergence introduced by DE LA VALLÉE POUSSIN. A much more general result for a function of both signs is here obtained without it and is given in theorem 21.‡

When the double integral and the iterated integral exist by the definition of HARNACK, there is no necessity for any sort of convergence (regular, or regular except for a set of points of zero measure) of the integral

$$\int_X f_{\lambda_1\lambda_2}.$$

This is readily seen from Example III.

We proceed now to build up the theory of reduction of multiple integrals for a limited field. It is assumed as already stated that A *possesses content* and that B either *possesses content* or *is closed.* § No restriction is put on the sections $\mathfrak{C}$. *The results of this memoir are true if the integrals over* $\mathfrak{C}$ *exist non-*

*Proceedings of the London Mathematical Society, ser. 2, vol. 4 (1906), p. 139.

† This result was announced to the Society at the meeting in April, 1906. HOBSON's article appeared in June, 1906.

‡ Since the definition of DE LA VALLÉE-POUSSIN is used, this integral can exist only absolutely. From this standpoint, it seems that the example given on page 157 of HOBSON's article has no meaning. Since the integral

$$\int_0^1 \frac{1}{y} \sin \frac{1}{y}\, dy$$

exists only non-absolutely, it cannot illustrate any argument of his paper.

§ This restriction on A is necessary for a limited function f (Example VI), while that on B is necessary only when f is unlimited at some points of A.

absolutely,* in accordance with the definition ordinarily used. The theory, however, is developed here only when the integrals over $\mathfrak{C}$ are defined by the method of DE LA VALLÉE-POUSSIN.

THEOREM 20. *Let $f(x_1, \cdots, x_m)$ be a function defined over A. Then if $f \geqq -G$,*

$$\overline{\int}_A f \geqq \underline{\int}_B \overline{\int}_{\mathfrak{C}} f \geqq \underline{\int}_B \underline{\int}_{\mathfrak{C}} f \geqq \underline{\int}_A f. \tag{1}$$

If $f \leqq G$,

$$\overline{\int}_A f \geqq \overline{\int}_B \overline{\int}_{\mathfrak{C}} f \geqq \overline{\int}_B \underline{\int}_{\mathfrak{C}} f \geqq \underline{\int}_A f. \tag{2}$$

To deduce (1), let $\lambda_1 < \lambda_2 < \cdots$ be a series of values increasing to infinity. Then the functions

$$\overline{\int}_{\mathfrak{C}} f_{\lambda_1},\ \overline{\int}_{\mathfrak{C}} f_{\lambda_2},\ \cdots; \qquad \underline{\int}_{\mathfrak{C}} f_{\lambda_1},\ \underline{\int}_{\mathfrak{C}} f_{\lambda_2},\ \cdots$$

form monotone increasing sequences which by definition converge to

$$\overline{\int}_{\mathfrak{C}} f,\ \underline{\int}_{\mathfrak{C}} f.$$

For any value of λ it follows from theorem 12 that

$$\overline{\int}_A f_\lambda \geqq \overline{\int}_B \overline{\int}_{\mathfrak{C}} f_\lambda; \qquad \underline{\int}_B \underline{\int}_{\mathfrak{C}} f_\lambda \geqq \underline{\int}_A f_\lambda.$$

A passage to the limit gives by means of the extension of theorem 13 for $f \geqq -G$ the following relations:

$$\overline{\int}_A f \geqq \lim_{\lambda=\infty} \overline{\int}_B \overline{\int}_{\mathfrak{C}} f_\lambda \geqq \underline{\int}_B \overline{\int}_{\mathfrak{C}} f, \tag{3}$$

$$\underline{\int}_B \underline{\int}_{\mathfrak{C}} f \geqq \lim_{\lambda=\infty} \underline{\int}_B \underline{\int}_{\mathfrak{C}} f_\lambda \geqq \underline{\int}_A f. \tag{4}$$

Since, however,

$$\underline{\int}_B \overline{\int}_{\mathfrak{C}} f \geqq \underline{\int}_B \underline{\int}_{\mathfrak{C}} f,$$

relations (3) and (4) give (1). In a similar manner (2) is established. It may be remarked that relations (1) and (2) are true if one or more of the integrals are infinite.

COROLLARY 1. *If $f \geqq -G$ is integrable in A,*

$$\int_A f = \underline{\int}_B \overline{\int}_{\mathfrak{C}} f.$$

* P. 345.

COROLLARY 2. *If $f \geqq -G$ is integrable in A and the iterated integral also exists, then*

$$\int_A f = \int_B \overline{\underline{\int}}_{\mathfrak{C}} f.$$

COROLLARY 3. *If the integral of $f \geqq -G$ in A is infinite, then both the integrals*

$$\underline{\int}_B \overline{\underline{\int}}_{\mathfrak{C}} f, \quad \overline{\int}_B \overline{\underline{\int}}_{\mathfrak{C}} f$$

are infinite.

COROLLARY 4. *If for $f \geqq -G$ one of the integrals*

$$\underline{\int}_B \underline{\int}_{\mathfrak{C}} f, \quad \underline{\int}_B \overline{\int}_{\mathfrak{C}} f, \quad \overline{\int}_B \underline{\int}_{\mathfrak{C}} f, \quad \overline{\int}_B \overline{\int}_{\mathfrak{C}} f \tag{5}$$

is limited, then

$$\underline{\int}_A f$$

is limited. If one of the integrals (5) *is unlimited, then the integral*

$$\overline{\int}_A f$$

is unlimited.

COROLLARY 5. *If for any $\epsilon > 0$ the points where the discontinuity of $f \geqq -G$ exceeds ϵ form a discrete aggregate, and if one of the integrals* (5) *is limited, then the multiple integral exists and*

$$\int_A f = \underline{\int}_B \overline{\underline{\int}}_{\mathfrak{C}} f. \tag{6}$$

For since the discontinuities of f form a discrete aggregate,

$$\overline{\int}_A f_\lambda = \underline{\int}_A f_\lambda = \int_A f_\lambda.$$

By letting λ increase indefinitely it follows from the preceding corollary that the integral of f in A exists. Relation (6) then results by corollary 1.

THEOREM 21. *Let $f(x_1, \cdots, x_m)$ take on both positive and negative values in A. If*

$$\int_A f, \quad \int_B \underline{\int}_{\mathfrak{C}} f$$

exist, they are equal.

Since f is integrable in A, it is absolutely integrable, and by the corollaries of theorems 20 and 12 we have for sufficiently great values of λ, λ_1, λ_2

$$(1)\qquad \epsilon > \int_A |f| - \int_A |f|_\lambda = \underline{\int}_B \overline{\int}_{\mathfrak{C}} |f| - \underline{\int}_B \overline{\int}_{\mathfrak{C}} |f|_\lambda ,$$

$$(2)\qquad \epsilon > \left| \int_A f - \int_A f_{\lambda_1\lambda_2} \right| = \left| \int_A f - \underline{\int}_B \overline{\int}_{\mathfrak{C}} f_{\lambda_1\lambda_2} \right| .$$

The functions f, $f_{\lambda_1\lambda_2}$ for any given point of A have the same sign and hence

$$(3)\qquad |f - f_{\lambda_1\lambda_2}| = |f| - |f_{\lambda_1\lambda_2}| .$$

Now by means of theorems 8 and 10

$$\left| \underline{\int}_B \overline{\int}_{\mathfrak{C}} f - \underline{\int}_B \overline{\int}_{\mathfrak{C}} f_{\lambda_1\lambda_2} \right| = \left| \underline{\int}_B \left[\overline{\int}_{\mathfrak{C}} f - \overline{\int}_{\mathfrak{C}} f_{\lambda_1\lambda_2} \right] \right| = \left| \underline{\int}_B \overline{\int}_{\mathfrak{C}} (f - f_{\lambda_1\lambda_2}) \right| \leqq \underline{\int}_B \overline{\int}_{\mathfrak{C}} |f - f_{\lambda_1\lambda_2}| .$$

Applying equation (3) and theorem 8, we obtain

$$(4)\qquad \left| \underline{\int}_B \overline{\int}_{\mathfrak{C}} f - \underline{\int}_B \overline{\int}_{\mathfrak{C}} f_{\lambda_1\lambda_2} \right| \leqq \underline{\int}_B \overline{\int}_{\mathfrak{C}} (|f| - |f|_\lambda) = \underline{\int}_B \left[\overline{\int}_{\mathfrak{C}} |f| - \overline{\int}_{\mathfrak{C}} |f|_\lambda \right] ,$$

where λ is the smaller of the two numbers λ_1, λ_2. Since the integral

$$\int_B \overline{\int}_{\mathfrak{C}} |f|_\lambda$$

exists, the last relation in corollary 2 of theorem 7 gives the equation

$$\underline{\int}_B \left[\overline{\int}_{\mathfrak{C}} |f| - \overline{\int}_{\mathfrak{C}} |f|_\lambda \right] = \underline{\int}_B \overline{\int}_{\mathfrak{C}} |f| - \underline{\int}_B \overline{\int}_{\mathfrak{C}} |f|_\lambda .$$

Hence from (1) and (4)

$$(5)\qquad \left| \underline{\int}_B \overline{\int}_{\mathfrak{C}} f - \underline{\int}_B \overline{\int}_{\mathfrak{C}} f_{\lambda_1\lambda_2} \right| \leqq \underline{\int}_B \overline{\int}_{\mathfrak{C}} |f| - \underline{\int}_B \overline{\int}_{\mathfrak{C}} |f|_\lambda < \epsilon .$$

Combining (2) and (5), we obtain

$$\left| \int_A f - \underline{\int}_B \overline{\int}_{\mathfrak{C}} f \right| < 2\epsilon$$

and the theorem is proved.

COROLLARY. *Let the points where the discontinuity of f exceeds $\epsilon > 0$ form a discrete aggregate. Let*

$$\underline{\int}_B \underline{\int}_{\mathfrak{C}} |f|$$

exist. Then f is integrable in A.

Since the points of discontinuity of $f_{\lambda_1\lambda_2}$ form a discrete aggregate,

$$\overline{\int}_A f_{\lambda_1\lambda_2} = \underline{\int}_A f_{\lambda_1\lambda_2} = \int_A f_{\lambda_1\lambda_2}.$$

Now

$$\left|\int_A f_{\lambda_1\lambda_2}\right| = \left|\int_A g_{\lambda_1} - \int_A h_{\lambda_2}\right| \leqq \int_A (g_{\lambda_1} + h_{\lambda_2}) = \int_A |f_{\lambda_1\lambda_2}| \leqq \underline{\int}_A |f|.$$

The lower integral of $|f|$ is limited (corollary 4, theorem 20) and on passing to the limit for $\lambda_1 = \infty$, $\lambda_2 = \infty$, the corollary is established.

Upon assuming that the upper integral of f exists finitely or infinitely, various relations may be derived. By means of theorems 9, 12 and 7 we have the relation

$$\overline{\int}_A f_{\lambda_1\lambda_2} = \overline{\int}_A g_{\lambda_1} - \underline{\int}_A h_{\lambda_2} \geqq \overline{\int}_B \overline{\int}_{\mathfrak{C}} g_{\lambda_1} - \underline{\int}_B \underline{\int}_{\mathfrak{C}} h_{\lambda_2} \geqq \overline{\int}_B \left[\overline{\int}_{\mathfrak{C}} g_{\lambda_1} - \underline{\int}_{\mathfrak{C}} h_{\lambda_2}\right].$$

On passing to the limit for $\lambda_1 = \infty$ and placing $f_{\lambda_2} = \lim_{\lambda_1=\infty} f_{\lambda_1\lambda_2}$, it follows from theorem 13 that

$$(6)\quad \overline{\int}_A f_{\lambda_2} \geqq \lim_{\lambda_1=\infty} \overline{\int}_B \left[\overline{\int}_{\mathfrak{C}} g_{\lambda_1} - \underline{\int}_{\mathfrak{C}} h_{\lambda_2}\right] \geqq \underline{\int}_B \lim_{\lambda_1=\infty} \left[\overline{\int}_{\mathfrak{C}} g_{\lambda_1} - \underline{\int}_{\mathfrak{C}} h_{\lambda_2}\right] = \underline{\int}_B \left[\overline{\int}_{\mathfrak{C}} g - \underline{\int}_{\mathfrak{C}} h_{\lambda_2}\right].$$

And by theorem 9

$$(7)\qquad \underline{\int}_B \left[\overline{\int}_{\mathfrak{C}} g - \underline{\int}_{\mathfrak{C}} h_{\lambda_2}\right] = \underline{\int}_B \overline{\int}_{\mathfrak{C}} (g - h_{\lambda_2}) = \underline{\int}_B \overline{\int}_{\mathfrak{C}} f_{\lambda_2}.$$

Hence from (6) and (7)

$$\overline{\int}_A f_{\lambda_2} \geqq \underline{\int}_B \overline{\int}_{\mathfrak{C}} f_{\lambda_2}.$$

Finally by passing to the limit for $\lambda_2 = \infty$ we obtain

$$(8) \qquad \overline{\int}_A f \geqq \lim_{\lambda_2=\infty} \underline{\int}_B \overline{\int}_{\mathfrak{C}} f_{\lambda_2} \geqq \underline{\int}_B \lim_{\lambda_2=\infty} \overline{\int}_{\mathfrak{C}} f_{\lambda_2}.$$

If the integral

$$\underline{\int}_B \overline{\int}_{\mathfrak{C}} f$$

exists finitely or infinitely, relation (8) gives

$$\overline{\int}_A f \geqq \underline{\int}_B \lim_{\lambda_2=\infty} \overline{\int}_{\mathfrak{C}} f_{\lambda_2} = \underline{\int}_B \overline{\int}_{\mathfrak{C}} f.$$

If $\mathfrak{C}$ is a one-dimensional aggregate, it is evident that the upper integral of f over $\mathfrak{C}$ may exist non-absolutely for an everywhere dense set of points of B. For these sections the expressions

$$\lim_{\lambda_2=\infty} \overline{\int}_{\mathfrak{C}} f_{\lambda_2}, \qquad \overline{\int}_{\mathfrak{C}} f$$

have different values. If we denote by

$$\overline{\overline{\int}}_{\mathfrak{C}} f$$

the maximum value obtained from

$$\overline{\int}_{\mathfrak{C}} f_{\lambda_1 \lambda_2}$$

by letting λ_1, λ_2 increase indefinitely in any manner whatsoever, we have

$$\overline{\int}_A f \geqq \underline{\int}_B \overline{\overline{\int}}_{\mathfrak{C}} f.$$

We have then the exceedingly general result:

THEOREM 22. *Let the upper integral of f exist finitely or infinitely in A. Then*

$$\overline{\int}_A f \geqq \underline{\int}_B \overline{\overline{\int}}_{\mathfrak{C}} f.$$

And further,

$$\overline{\int}_A f \geqq \underline{\int}_B \overline{\int}_{\mathfrak{C}} f \geqq \underline{\int}_B \underline{\int}_{\mathfrak{C}} f,$$

provided the last two integrals exist finitely or infinitely.

Corresponding relations exist when it is assumed that the lower integral of f exists finitely or infinitely. It may be noted that theorem 21 follows as a corollary to this theorem.

§ 5. *Integration and differentiation of series.*

Let $u_1(x_1, \cdots, x_m)$, $u_2(u_1, \cdots, x_m) \cdots$ be a sequence of functions defined over a limited field A which either possesses content or is closed. Unless otherwise stated no restriction is put on the discontinuities of the functions $u_1, u_2, \cdots$, which may take on either finite or infinite values. Form the sum

$$U_n(x_1, \cdots, x_m) = u_1(x_1, \cdots, x_m) + u_2(x_1, \cdots, x_m) + \cdots + u_n(x_1, \cdots, x_m).$$

For a point P of A chosen at pleasure and then fixed we shall assume that when n increases indefinitely, $U_n(P)$ converges toward a limit, finite or definitely infinite. Denote the aggregate of limiting values by U and write

$$U(x_1, \cdots, x_m) = \lim_{n=\infty} U_n(x_1, \cdots, x_m) = u_1 + u_2 + \cdots. \tag{1}$$

The object of the following discussion is the determination of some simple conditions under which the series (1) can be integrated term by term; in other words, under which

$$\lim_{n=\infty} \int_A U_n = \int_A \lim_{n=\infty} U_n = \int_A U.$$

The conditions under which this inversion of the order is permissible have been investigated by OSGOOD, ARZELÀ and others. Uniform convergence in a more or less modified form is shown to be a sufficient condition. In this section it is shown that for certain classes of functions the notion of uniformity is superfluous.

THEOREM 23. *Let all terms of the series $U(x_1, \cdots, x_m)$ be of like sign and integrable in A. If $u_i(x_1, \cdots, x_m) \geqq 0$*

$$\underline{\int_A} U = \int_A u_1 + \int_A u_2 + \cdots,$$

and if $u_i(x_1, \cdots, x_m) \leqq 0$

$$\overline{\int_A} U = \int_A u_1 + \int_A u_2 + \cdots.$$

Since the sequence of functions $U_1, U_2, \cdots$ is monotone, this theorem is the same as the corollary of theorem 13. The integral on the left may be finite or infinite.

THEOREM 24. *Let*

$$\int_A |u_1| + \int_A |u_2| + \cdots$$

be finite. Then

$$\int_A U = \int_A u_1 + \int_A u_2 + \cdots,$$

provided these integrals exist.

Denoting by V the sum

$$\sum_{i=1}^{\infty} |u_i(x_1, \cdots, x_m)|,$$

we have from the preceding theorem and the second corollary of theorem 7

$$\lim_{n=\infty} \underline{\int_A} \left[V - \sum_{i=1}^{n} |u_i| \right] = \lim_{n=\infty} \left[\underline{\int_A} V - \sum_{i=1}^{n} \int |u_i| \right] = 0.$$

Now

$$\left| \int_A U - \sum_{i=1}^{n} \int_A u_i \right| = \left| \int_A \left[U - \sum_{i=1}^{n} (u_i) \right] \right| \leqq \underline{\int_A} \left[V - \sum_{i=1}^{n} |u_i| \right].$$

By passing to the limit for $n = \infty$, we have the theorem :

THEOREM 25. *Let $U(x_1, \cdots, x_m)$ be a series for which there exists a fixed number G and a positive integer ν such that $|U_n| < G$ for $n > \nu$. If $U, u_1, u_2, \cdots$ are integrable in A,*

$$\int_A U = \int_A u_1 + \int_A u_2 + \cdots.$$

This is the same as theorem 18.

THEOREM 26. *At each point of A let all the functions $u_1, u_2, \cdots$ have the same sign (at some points of A positive and at others negative). Then provided the integrals (finite or infinite) of $U, u_1, u_2, \cdots$ exist absolutely,*

$$\int_A U = \int_A u_1 + \int_A u_2 + \cdots.$$

This follows immediately from the corollary to theorem 17.

THEOREM 27. *Let $U(x) = u_1(x) + u_2(x) + \cdots$, be a series of functions defined in the interval (a, b). Let each of the terms $u_i(x)$ be differentiable and denote by $V(x)$ the series of derived terms $u_1'(x) + u_2'(x) + \cdots$. Let $V(x)$ and $u_i'(x)$ be integrable in (a, b). Let either 1) all the functions $u_i'(x)$ have the same sign at each particular point of A (at all points positive or at all points negative or at some points positive and at some negative), the* *points of infinite discontinuity of each term forming at most an aggregate of the first species; or* 2)

$$\sum_{i=1}^{\infty}\int_a^b |u_i'(x)|$$

be limited; or 3) $|\sum_{i=1}^{n} u_i'(x)| < G$ *for all values of* n, G *being arbitrarily large but fixed. Then for each point of continuity of* V

$$D_x U(x) = V(x).$$

By theorems 23, 24, 25, 26 and the fundamental theorem of the integral calculus *

$$\int_a^x V(x) = \sum_{i=1}^{\infty}\int_a^x u_i'(x)\,dx = \sum_{i=1}^{\infty}[u_i(x) - u_i(a)] = U(x) - U(a).$$

Differentiating this expression we have for every point of continuity † of V

$$V(x) = D_x U(x).$$

BROWN UNIVERSITY, PROVIDENCE,
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CORRECTION. The author wishes to call attention to an error which occurs in a former paper [Transactions, vol. 7 (1906), p. 451]. In the proof there of theorem 3 (3°), it is erroneously assumed that the limit of the upper content $\bar{B}_i$ of a sequence of aggregates $B_1 \geqq B_2 \geqq \cdots$ is zero provided that these aggregates have no point in common. This error causes an hiatus in the proof of theorem 6 of that paper. To correct this proof let the part after relation (2) on page 452 be neglected and replaced by the following discussion.

Denote by $\mathfrak{C}$ the points of $\mathfrak{B}$ where $f \geqq 0$. Let $\Delta_1, \Delta_2, \cdots$ be a set of superimposed divisions whose norms converge to zero. Denote by $\mathfrak{B}_k$ all points of $\mathfrak{B}$ in the cells of Δ_k which contain at least one point of $\mathfrak{C}$. The norm of the division may be so taken that the difference of the upper contents of $\mathfrak{C}$ and

* DE LA VALLÉE-POUSSIN, Journal de Mathématiques, ser. 4, vol. 8 (1892), p. 430.
† *Lectures*, § 604.

$\mathfrak{B}_k$ is small at pleasure. Since f is limited throughout $\mathfrak{B}_k$ it follows as a corollary to theorem 4 that for proper choice of k

$$\left| \overline{\int_{\mathfrak{B}_k}} f - \overline{\int_{\mathfrak{C}}} f \right| < \frac{\epsilon}{2}.$$

Combined with (2) of page 452 this gives

$$\overline{\int_{\mathfrak{C}}} f < \epsilon.$$

Hence by theorem 5,

$$\overline{\int_{\mathfrak{B}}} g < \epsilon.$$

Likewise it may be shown that

$$\overline{\int_{\mathfrak{B}}} h < \epsilon.$$

Then

$$\underline{\int_{\mathfrak{B}}} |f| \leqq \overline{\int_{\mathfrak{B}}} |f| = \overline{\int_{\mathfrak{B}}} (g + h) \leqq \overline{\int_{\mathfrak{B}}} g + \overline{\int_{\mathfrak{B}}} h < 2\epsilon.$$