

AN APPLICATION OF THE FORM-PROBLEMS ASSOCIATED WITH
CERTAIN CREMONA GROUPS TO THE SOLUTION OF
EQUATIONS OF HIGHER DEGREE*

BY

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Introduction.

Certain groups of Cremona transformations isomorphic with the symmetric group of order $n!$ are discussed in this paper. They are seen to define form-problems in much the same way as groups of collineations, the known quantities or parameters in the problem being the invariants of the n -ic under linear transformation. The well-known theory of binary forms is thus utilized to the utmost in reducing the number of parameters that occur in the general equation of degree n .

The earlier paragraphs are devoted to a detailed study of the quintic. These have close contact at many points with KLEIN's *Ikosaeder*. In § 1 and § 2 Cremona groups of order 120 and their invariants are given. In § 3 the form-problems determined by these groups are formulated. They contain as parameters two absolute invariants of the quintic. For want of a more direct method, † in § 4 the Cremona form-problem is made to depend on KLEIN's "problem of the A 's." It is probable that this form of the reduction of the quintic to the A -problem is as simple as can be devised. ‡ It is indeterminate only so far as the choice of the two absolute invariants is concerned. Since the solution of the A -problem proposed in the *Ikosaeder* is not carried out completely, a new solution is given in § 5 and all the necessary calculations are made. The summary in § 6 shows that the solution of the quintic, after the adjunction of the square root of its discriminant, is obtained by an extraction of a square root, a determination of the ikosahedral irrationality and furthermore by substitutions in explicitly given rational expressions.

The discussion of the general case in § 8 emphasizes the importance of this

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† See the opening remarks of § 7.

‡ See other methods suggested in the *Ikosaeder*, p. 248, § 6.

novel Cremona form-problem, both as effecting an immediate reduction in the number of parameters found in the original problem, and as being the first in a series of successively simpler form-problems by means of which the given equation can be solved.

The distinction preserved by KLEIN between the "form-problem" and its accompanying "equation-problem" is here not maintained. In fact, nearly all the problems referred to hereafter as form-problems are, in reality, equation-problems. The two are very closely connected, however, and the former term is much more characteristic. Strictly speaking, a Cremona group cannot define a form-problem.

§ 1. *The Cremona group of order 120 determined by five points in a plane.**

The group is suggested by this problem: *Given two sets of n points, p_0, p_1, \dots, p_{n-1} and q_0, q_1, \dots, q_{n-1} , in the same or different planes, the two sets being ordered with regard to each other, under what conditions can points x and y be found such that the pencil of lines from x to the points p_i is projective to the pencil of lines from y to the points q_i ?*

When n is 3, there is no limitation on x and y except that if x is on a line $p_i p_k$, y is on the line $q_i q_k$.

When n is 4, if x is any point on a conic through p_0, \dots, p_3 , y is any point on a definite conic through q_0, \dots, q_3 . A one-to-one relation is thus established between the pencils of conics through the two sets of four points, which is fixed by the requirement that the three line pairs of the one pencil correspond to the three line pairs of the other.

When n is 5, there is a one-to-one correspondence between the points x and y . They are corresponding points in a Cremona transformation, T . For if x is given, to the conics $x p_0 p_1 p_2 p_3^2$ and $x p_0 p_1 p_2 p_4^2$ correspond conics $y q_0 q_1 q_2 q_3^2$ and $y q_0 q_1 q_2 q_4^2$ which meet at q_0, q_1, q_2 and the required fourth point y . To determine the order of T we examine its principal points. If x is at p_0 , it determines only a four-line pencil $p_0 - p_1, p_2, p_3, p_4$. Hence y is any point on a conic through q_1, q_2, q_3, q_4 . The five points p_i are, then, double principal points of T and the five points q_i are double principal points of T^{-1} . If x is any point on the conic $p_0 p_1 p_2 p_3 p_4^2$, y is a definite point q . Therefore q is a sixth double principal point of T^{-1} . If y is any point on the conic $q_0 \dots q_4^2$, x is a definite point p , the sixth double principal point of T . For all other points the transformation is determinate, i. e., T is a Cremona transformation of the fifth order with six double principal points.

*The Cremona G_{120} of § 2 has been studied exhaustively by H. E. SLAUGHT, American Journal of Mathematics, vol. 22 (1900), p. 343 and vol. 23 (1901), p. 99. In the later paper the connection between the invariants of the group and the invariants of the quintic is obtained by a method different from that of § 2. The generalized group, G_n in S_{n-3} , considered in § 8 is due to E. H. MOORE, American Journal of Mathematics, vol. 22 (1900), p. 279.

Moreover T is the most general Cremona transformation of that type. If r_0, r_1, \dots, r_5 and s_0, s_1, \dots, s_5 are the two sets of double principal points of such a transformation, r_i and s_i being "opposite" points, i. e., points such that the principal conic of the one does not pass through the other, the pencil of conics through four points r_i corresponds to the pencil of conics through the four opposite points s_i in such a way that the line pairs in the two pencils correspond. Hence if x and y are a pair of corresponding points, the six lines from x to r_i are projective to the six lines from y to s_i . If x is at r_i , y is any point on the conic through the five points $q_k, k \neq i$, and the five lines from r_i to r_k are projective to the five points q_k on their conic. Two sets of six points with this property will be called a "double-six-point." *

The necessary and sufficient condition that two sets of six points be the double principal points of a Cremona transformation, T , of the fifth order is that the two sets form a double-six-point. If x and y are a pair of corresponding points the sextic line pencils from x and y , to the principal points of T and T^{-1} respectively, are projective.

T is fully defined when five pairs of opposite points are given. If the five points q coincide with the five points p, q_i falling on p_i , the transformation is the identity. And if the points q , in some permutation, fall on the points p , the transformation is of the fifth order and periodic, with the same period as the permutation. By employing the 120 permutations of the points q_i with reference to a fixed order of the points p_i , a group of 120 transformations is obtained isomorphic with the symmetric group on five things. Any transformation can be represented by a permutation. Thus $\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ a_0 & a_1 & a_2 & a_3 & a_4 \end{pmatrix}$ is the transformation in which the principal conic of p_0 passes through $p_{a_1}, p_{a_2}, p_{a_3}, p_{a_4}$, and through the sixth principal point of T^{-1} , which will be denoted by p with a subscript giving the inverse permutation in cycle form. A set of 120 conjugate points determine 120 five-line pencils to the points p_i , which are all projective in various orders. Thus a binary quintic determines a set of 120 conjugate points in the plane and a set of 120 conjugate points determines a system of ∞^3 projective binary quintics. This representation is determinate except for points on the conic $p_0 \cdots p_4^2$. All these points together with the 119 sixth principal points represent the same system of projective quintics. Some very degenerate quintics also are not represented by any points, e. g., quintics with a triple root.

* The double-six-point occurs in a number of connections: If a cubic surface be represented on a plane by means of six skew lines of the surface, the two sets of six points, arising from the two representations by means of the two opposite sets of lines in a double six, form a double-six-point; the six pairs of points, which satisfy four linearly independent bilinear relations, form a double-six-point; the six pairs of corresponding points common to two quadratic transformations form a double-six-point; in a Cremona transformation of the fourth order with one triple and six simple principal points, the two sets of six points form a double-six-point and, with reference to them, the two triple points are corresponding; the six elements common to four connexes (1, 1), as well as the six polar points and lines common to two triangles, form a double-six-point-line, the half-dual of the double-six-point.

It is clear that the locus of points x , for which an invariant of the five-line-pencil vanishes, is an invariant curve of the group. That the converse is true will be shown in the next paragraph. Hence,

There is a one-to-one correspondence between the invariant curves of the Cremona G_{120} and the invariants of the binary quintic.

These invariant loci have been derived by CLEBSCH [Mathematische Annalen, vol. 4 (1871)] by means of his principle of transference, in the dual case of five lines. However, the existence of the underlying Cremona group is not pointed out. The properties of the various loci are treated in some detail. By the use of the group some of these properties are rendered very evident. In what follows, the notation of CLEBSCH (*Binäre Formen*) for the comitants of the quintic is used.

A transformation T of the fifth order has seven fixed points or a locus of fixed points. If T is involutory it has necessarily a locus of fixed points of odd order. Considering in particular $T_{(01)}$ we see that every point on the line $\overline{p_0 p_1}$ is a fixed point. Each conic in the pencil of conics through $p_2, p_3, p_4, p_{(01)}$ is unaltered. Each conic cuts the line $\overline{p_0 p_1}$ in two points, the fixed points of the involution on that conic. The two conics through p_2, p_3, p_4 , touching $\overline{p_0 p_1}$ at p_0 and at p_1 respectively are the principal conics of p_0 and p_1 and meet at $p_{(01)}$. The only fixed points other than points on $\overline{p_0 p_1}$ are the vertices of the diagonal triangle of $p_2, p_3, p_4, p_{(01)}$.

By the transformation $T_{(01)(23)}$, a cubic curve with a double point at p_4 and simple points at p_0, p_1, p_2, p_3 is transformed into another of the same kind. Of the linear system containing the ∞^2 such curves, a pencil is unaltered, member by member, also one other cubic, $C_{(01)(23)}$. Of the points adjacent to p_4 , those two on the line pair apolar to the pairs $p_4 p_0, p_4 p_1$ and $p_4 p_2, p_4 p_3$ are fixed. They determine the tangents to $C_{(01)(23)}$ at p_4 . The pairs of tangents apolar to this pair belong to the members of the invariant pencil. Each member cuts $C_{(01)(23)}$ in a fixed point. The ninth base point of the pencil is also a fixed point. The two degenerate members of the pencil $\overline{p_4 p_0 p_4 p_1 p_2 p_3}$ and $\overline{p_4 p_2 p_4 p_3 p_0 p_1}$ show this point to be the meet of $\overline{p_2 p_3}$ and $\overline{p_0 p_1}$, say $P_{01, 23}$. The other two diagonal points $P_{02, 31}$ and $P_{03, 12}$ lie on $C_{(01)(23)}$. In the invariant pencil of cubics there is only one which breaks up into a line and a conic, $\overline{p_0 p_1 p_2 p_3 p_4^2 p_4 P_{01, 23}}$. The conic, being a principal conic, has no transform, and the transform of $\overline{p_4 P_{01, 23}}$ must be the line itself and the conic $\overline{p_0 \cdots p_4^2}$. Hence the sixth principal point, $p_{(10)(23)}$, lies on the line $\overline{p_4 P_{01, 23}}$. On this line is an involution whose double points are $P_{01, 23}$ and the third meet of the line with $C_{(01)(23)}$. Hence $p_{(01)(23)}$ must coincide with p_4 along the line $\overline{p_4 P_{01, 23}}$. Otherwise it would have two correspondents, the point p_4 and the point where the line meets the conic $\overline{p_0 \cdots p_4^2}$. Each conic of the pencil through $p_0 \cdots p_3$ is unaltered, the double points of the involution on any one being the two remaining meets with $C_{(01)(23)}$.

The two types considered above include all involutory Cremona transformations of the fifth order with six double principal points.

The transformation $T_{(012)}$ has seven fixed points, but three are accounted for by the fact that three principal conics pass through their principal points. The line $\overline{p_3 p_4}$ is unaltered and the collineation upon it has two fixed points, the Hessian pair of the three meets with the sides of the triangle p_0, p_1, p_2 . The other two fixed points are the two meets of the three cubics, $C_{(12)(34)}$, $C_{(20)(34)}$ and $C_{(01)(34)}$. These two sets of two points occur in two different sets of 20 conjugate points. The points of the first kind are invariant under the cyclic $G_6, (012)(34)$, and are, then, the fixed points of the transformations of period six. Those of the second kind are invariant under a dihedral G_6 generated by (012) and $(12)(34)$. To points of the first kind correspond quintics made up of a cubic and the square of one of its Hessian points; to those of the second kind correspond quintics made up of a cubic and its Hessian.

The transformation $T_{(0123)}$ has three fixed points. It interchanges the lines $\overline{p_0 p_2}$ and $\overline{p_1 p_3}$, and therefore $P_{02,13}$ is a fixed point. The conics of the pencil on p_0, p_1, p_2, p_3 are interchanged involutorily. The line-pair $\overline{p_0 p_2}, \overline{p_1 p_3}$ is a fixed conic and the other two line pairs are interchanged. Hence the other fixed conic is the polar of the first line pair as to the other two (in the sense of polarity in a linear system) and must be one of the three harmonic conics on the four points. The cubic $C_{(02)(31)}$ is also unchanged. The two remaining fixed points are the two meets of this cubic and the harmonic conic. The corresponding quintic is a cyclo-projective set of four points and a double point of the cycle.

Finally, the transformation $T_{(01234)}$ has two fixed points which are the points common to the cubics $C_{(14)(23)}, C_{(20)(34)}, C_{(31)(40)}, C_{(42)(01)}, C_{(03)(12)}$. Each point is unaltered by a dihedral G_{10} and represents a cyclo-projective quintic.

Summing up the above, we find eight sets of conjugate points which number less than 120.

(1) Sets of 60 conjugate points arising from a general point on the locus, $\sqrt{D} = \Pi(\overline{p_i p_k})$. The corresponding quintics are of the type $z_1^2 z_2 (z_1 + z_2)(z_1 + az_2)$, and the corresponding subgroups of the type (01).

(2) Sets of 60 conjugate points arising from a general point on the locus, $R = \Pi(C_{(ik)(lm)})$. The quintics are of the type $z_1(z_1^2 + z_2^2)(z_1^2 + az_2^2)$, and the subgroups of the type (01)(23).

(3) A set of 30 conjugate points Q . These are the diagonal points of the ten 4-points, $p_{(ik)}, p_l, p_m, p_n$. On a line $\overline{p_i p_k}$, they are the cubic covariant points of the cubic cut out by the sides of the triangle p_l, p_m, p_n . The quintics are of the type $z_1^2 z_2 (z_1^2 - z_2^2)$ and the subgroups of the type (01), (23), (01)(23).

(4) A set of 30 conjugate points Q' . They arise from the two meets of a cubic $C_{(01)(23)}$ with a definite harmonic conic on p_0, \dots, p_3 . The quintics are of the type $z_1(z_1^4 - z_2^4)$ and the subgroups of the type (0123).

(5) A set of 20 conjugate points H . They are the Hessian points on the lines $p_i p_k$. The quintics are of the type $z_1^2(z_1^3 - z_2^3)$ and the subgroups of the type (012)(34).

(6) A set of 20 conjugate points I . They are the meets of the cubics $C_{(ik)(lm)}$ by threes. The quintics are of the type $z_1 z_2(z_1^3 - z_2^3)$ and the subgroups of the type (012), (12)(34).

(7) A set of 15 conjugate points $P_{01, 23}$. They are the fifteen diagonal points of the five points taken by fours. The quintics are of the type $z_1^2 z_2^2(z_1 + z_2)$, and the subgroups of the type (0231), (03).

(8) A set of 12 conjugate points K . They are the meets of the cubics $C_{(ik)(lm)}$ by fives. The quintics are of the type $z_1^5 - z_2^5$, and the subgroups of the type (01234), (14)(23).

That this enumeration is complete follows from the fact that there are no other types of binary quintics which admit a linear transformation. The points I separate into two sets of 10, each set conjugate under the alternating G_{60} . Under the same group the 12 points K separate into two sets of 6 conjugate points. All the sets of points found above are referred to in the quoted article of CLEBSCH except the set (3). He finds that the invariant curves A, B, C, D are of orders 10, 20, 30, 45 respectively with 4-, 8-, 12-, 18-fold points respectively at p_0, \dots, p_4 . The curves B and $D = A^2 - 64B$ have already been noted. Since the pencil $A^2 + \lambda B$ contains two perfect squares each curve of order 20 breaks up into two curves of order 10, each invariant under G_{60} .

It is well known that the invariants B and C vanish if the quintic is $z_1^5 - z_2^5$. Hence the curves B and C pass through the 12 points K . CLEBSCH shows that, at each K -point, B has a double point and C a double cusp whose cusp tangents are the tangents of the double point of B . This fact, important in what follows, is easy to prove by the group properties. Let K be a definite K -point and B_1 the factor of B which passes through it. K is a fixed point of a dihedral G_{10} . Hence the directions about K divide into a set of two, two sets of five, and ∞^1 sets of 10 conjugate directions. An invariant curve through K must have there two, five or ten tangents. B_1 , a curve of order 10 with five 4-fold points at p_i , cannot have at K a point of order 4 or more, else it would have six such points and vanish identically. Therefore it must pass through K with the definite set of two tangents. B_1 and C meet in $30 \cdot 10 - 5 \cdot 4 \cdot 12 = 60$ points outside the points p_i . This set of 60 points is conjugate under G_{60} . C has at K a double point or a double cusp with the same tangents as B_1 . If it has only a double point, $6 \cdot 6$ of the 60 points are accounted for. But the remaining 24 would also be at the 6 K -points, for they only can form a set of 24 conjugate points. Also C can not have five-fold points at the 6 K -points. Else it would meet one of the 15 cubics $C_{(ik)(lm)}$ in $12 \cdot 6 + 4 \cdot 5 = 92$ points and contain it as a factor, but this is impossible. C must therefore have at K a double cusp with the same tangents as B_1 .

CLEBSCH also shows that, outside the points p_i , the curves A and B meet in the points H and the curves A and C touch at the 30 points Q' .

There are two objections to the present form of the group. One is the very high order of the invariant curves. The second — more weighty with regard to the application of the group in the solution of the quintic — is that one system of projective binary quintics is represented by all the points on a conic. If the group be transformed by a Cremona transformation, with this conic as a principal conic, introducing thus no new principal points, this objection will be removed. Such a transformation, S , is the cubic involution with one of the five points, say p_0 , as a double point and the other four as simple principal points. S is the projection of each conic of the pencil through p_1, \dots, p_4 into itself from p_0 . The transformed group will be discussed in the next paragraph.

§ 2. *The Cremona group of order 120 determined by four points in a plane.*

The transformed group is obtained most simply by a construction for the set of 120 conjugate points determined by a binary quintic. If x is one of the points in question, the five lines $\overline{xp_i}$ must be projective to the given quintic. These five lines are projective also to the five tangents at x to the five conics through x and four each of the points p_i . For x and p_i determine a pencil of cubics with a double point at x . The tangents to these cubics are pairs of an involution which includes, for the degenerate members, the five pairs mentioned. If S transforms x into y , the five conics become the conic $\overline{yp_1 p_2 p_3 p_4}$ and the four lines $\overline{yp_1}, \dots, \overline{yp_4}$. Naming the points by the subscripts only, we find therefore

(1) *In the new form of the group a point y represents the system of quintics projective to the four lines $\overline{y1}, \dots, \overline{y4}$ and the tangent at y to the conic $\overline{y1234}^2$, referred to hereafter as yT .*

The group is determined by four points only. Its properties can be deduced from those of the first group or discussed analytically with ease. Let the points 1, 2, 3, 4 have respectively the coördinates 1, -1, -1; -1, 1, -1; -1, -1, 1; 1, 1, 1 and let

$$\begin{aligned} s_1 &= y_1 - y_2 - y_3, & s_2 &= -y_1 + y_2 - y_3, \\ (2) \quad s_3 &= -y_1 - y_2 + y_3, & s_4 &= y_1 + y_2 + y_3, \\ & & s_1 + s_2 + s_3 + s_4 &= 0. \end{aligned}$$

(3) *The binary value system $\infty, s_1, s_2, s_3, s_4$ is projective to the five lines $yT, \overline{y1}, \overline{y2}, \overline{y3}, \overline{y4}$.*

This statement is verified at once by calculating two independent anharmonic ratios. If one root of a quintic is infinite and the sum of the other four zero, the four finite roots are determined projectively to within a factor of propor-

tionality. We find that, if the roots are r_0, \dots, r_4 , and r_0 is transformed to infinity and the sum of the remaining transformed roots made zero, to within a factor these are

$$(4) \quad \begin{aligned} s_1 &= (12)(30)(40) + (13)(40)(20) + (14)(20)(30), \\ s_2 &= (21)(30)(40) + (23)(40)(10) + (24)(10)(30), \\ s_3 &= (31)(40)(20) + (32)(40)(10) + (34)(10)(20), \\ s_4 &= (41)(20)(30) + (42)(30)(10) + (43)(10)(20) \quad [(ik) = r_i - r_k]. \end{aligned}$$

Therefore

$$(5) \quad \begin{aligned} s_4 - s_1 &= 2(y_2 + y_3) = 4 \cdot (41)(20)(30), & s_2 - s_3 &= 2(y_2 - y_3) = 4 \cdot (23)(10)(40), \\ s_4 - s_2 &= 2(y_3 + y_1) = 4 \cdot (42)(30)(10), & s_3 - s_1 &= 2(y_3 - y_1) = 4 \cdot (31)(20)(40), \\ s_4 - s_3 &= 2(y_1 + y_2) = 4 \cdot (43)(10)(20), & s_1 - s_2 &= 2(y_1 - y_2) = 4 \cdot (12)(30)(40) \end{aligned}$$

$$(6) \quad \begin{aligned} y_1 &= (43)(10)(20) + (12)(30)(40) = (42)(30)(10) - (31)(20)(40), \\ y_2 &= (41)(20)(30) + (23)(10)(40) = (43)(10)(20) - (12)(30)(40), \\ y_3 &= (42)(30)(10) + (31)(20)(40) = (41)(20)(30) - (23)(10)(40). \end{aligned}$$

Formulas (2) and (3) give one binary quintic determined by the point y and formulas (6) the point y determined by a given binary quintic.

To obtain the transformations of the group, we have merely to permute the differences of the roots in the required way and express the coördinates of the new point y' in terms of those of the old by eliminating the differences. First of all it is clear that any permutation of the roots r_1, \dots, r_4 alone determines the same permutation of s_1, \dots, s_4 ; i. e., the group contains the collineation group of the 4-line $s_i = 0$ or, also, of the 4-point 1, 2, 3, 4. To get the transformation $I_{(04)}$ we have

$$(7) \quad \begin{aligned} y'_2 + y'_3 &= \rho(01)(24)(34) = \sigma(01)(02)(03) \cdot (01)(24)(34) = \sigma(y_3 + y_1)(y_1 + y_2), \\ y'_2 - y'_3 &= \rho(23)(14)(04) = \sigma(01)(02)(03) \cdot (23)(14)(04) = \sigma(y_3 - y_2)(y_2 + y_3), \\ y'_1 + y'_2 &= \rho(03)(14)(24) = \sigma(01)(02)(03) \cdot (03)(14)(24) = \sigma(y_3 + y_1)(y_2 + y_3), \end{aligned}$$

a quadratic involution with the fixed point 4 and principal triangle 123. The line 31 corresponds to the point 2 and the line 23 to the point 1, i. e., the involution is non-inversive.

(8) *The group is generated by the collineation group G_{24} of the four points 1, 2, 3, 4 and a quadratic non-inversive involution with one of the points fixed and the other three as principal points.*

It is made up of the G_{24} and 96 quadratic transformations with principal points at the four points. The five pencils of conics through 4 of the 5 points

p_i are in this case the pencil of conics through 1, 2, 3, 4, and the four pencils of lines on the points i . The invariant curve \sqrt{D} is the six lines through the four points. The fifteen cubics $C_{(ik)(lm)}$ become three lines and twelve conics. Since they are the loci of fixed points of the even involutions we find the three lines from G_{24} . They are $y_i = 0$, the axes of the harmonic perspectivities with centers $u_i = 0$ respectively, and correspond to the permutations $(4i)(kl)$ $i, k, l = 1, 2, 3$. These determine simply the change of sign of y_k and y_l and with the identity form the invariant subgroup G_4 of G_{24} . If these perspectivities are transformed by $T_{(04)}$, (7), we find the three inversions in a conic which correspond to the permutations $(0i)(kl)$. Thus for the permutations $(01)(23)$, $(02)(31)$ and $(03)(12)$ respectively the conics are

$$(9) \quad \begin{aligned} y_1^2 - y_2^2 - y_3^2 - (y_2y_3 + y_3y_1 + y_1y_2) &= 0, \\ -y_1^2 + y_2^2 - y_3^2 - (y_2y_3 + y_3y_1 + y_1y_2) &= 0, \\ -y_1^2 - y_2^2 + y_3^2 - (y_2y_3 + y_3y_1 + y_1y_2) &= 0. \end{aligned}$$

The center of inversion for each is the point 4, but the axes of inversion are $y_2 + y_3$, $y_3 + y_1$, $y_1 + y_2$, respectively. The other 9 conics of inversion are found from these three by transformation with G_4 .

In giving the sets of conjugate points less than 120 in number, those obtained from one by the operations of G_{24} are not written. It is clear also that points on a line p_0p_i in § 1 are in this case directions about i . Taking up the points in the same order as before we find:

(3) The 30 points Q include the directions at 4 on the three cubic covariant lines of $4i$. These directions are on the lines $-2y_1 + y_2 + y_3$, $y_1 - 2y_2 + y_3$, $y_1 + y_2 - 2y_3$. Their transforms give twelve points like 2, 1, 1, and six like 0, 1, 2.

(4) The 30 points Q' are obtained by taking the intersections of the harmonic conic $-2y_1^2 + y_2^2 + y_3^2 = 0$ and the axis $y_1 = 0$, each determined by the cyclic $G_4(1243)$. They include the six points like 0, 1, i and their transforms like 1, $1 + 2i$, $1 - 2i$.

(5) The 20 points H include eight directions, the two at 4 being on the lines $y_1 + \omega y_2 + \omega^2 y_3 = 0$, $y_1 + \omega^2 y_2 + \omega y_3 = 0$ ($\omega = e^{2\pi i/3}$); their transforms are the 12 points like $\omega - \omega^2$, 1, -1 .

(6) The 20 points I are determined by taking the intersections of properly chosen factors of R , e. g., $y_1 = 0$ and the first conic (9). They are the transforms of the pairs $\{0, \omega, -\omega^2\}$, $\{1, \omega^2, \omega\}$, by the even G_{12} . The points in the upper, and in the lower, row, form the two sets of conjugate points under G_{60} .

(7) The 15 points $P_{ik, lm}$ are the three diagonal points of reference and the twelve directions at the four points on the six lines joining them.

(8) The 12 points K are the meets of properly chosen factors of R , e. g.,

y_1 and the second conic in (9). They are the transforms of the pair $\{0, 1; \delta_1, \delta_2\}$ by G_{12} , the points in the upper and lower rows forming two sets of 6 conjugate points under G_{60} . Here $\delta_1 = \epsilon + \epsilon^4$, $\delta_2 = \epsilon^2 + \epsilon^3$, $\epsilon = e^{2\pi i/5}$.

The invariant curves A, B, C, R , have now the orders 6, 12, 18, 27, respectively and 2-, 4-, 6-, 9-fold points respectively at the points i . Their equations are easily deduced from a typical representation of the binary quintic given in CLEBSCH's *Binäre Formen* (p. 353). The representation is as follows:

If $F(z) = a_4 z^4 + b_3 z^3 + \dots$ be a binary quartic, x any point not a root of the quartic and $a_2^3 a_0$ its linear polar as to F , x and its polar are taken as new reference points. The transformation is

$$\eta = (xz) = \eta_1 z_1 + \eta_2 z_2, \quad \xi = a_2^3 a_0 = \xi_1 z_1 + \xi_2 z_2.$$

Since $\dot{a}_2(\xi\eta) \equiv (a\eta)\xi - (a\xi)\eta$ we have, on raising to the fourth power,

$$(\xi\eta)^4 \cdot F(z) = (a\eta)^4 \cdot \xi^4 - 4(a\eta)^3(a\xi) \cdot \xi^3\eta + 6(a\eta)^2(a\xi)^2 \cdot \xi^2\eta^2 - 4(a\eta)(a\xi)^3 \cdot \xi\eta^3 + (a\xi)^4 \cdot \eta^4.$$

Here the coefficients $(\xi\eta)^4$ and $(a\eta)^i(a\xi)^{4-i}$ are invariants of F and η , i. e., covariants of F in the variable x . Evaluating the covariants in terms of the complete system and dividing out by $F(x)$, by hypothesis not zero, we have the typical representation:

$$(10) \quad F^3 \cdot F(z) = \xi^4 + 3H\xi^2\eta^2 + 4T\xi\eta^3 + \frac{1}{2}iF^2 - \frac{3}{4}H^2\eta^4.$$

Since η or x was any point, $\eta \cdot F(z)$ is a general quintic with a system of coefficients

$$(11) \quad 0, 1, 0, \frac{3H}{2}, 4T, 5\left(\frac{1}{2}iF^2 - \frac{3}{4}H^2\right).$$

CLEBSCH finds the invariants of this quintic to be

$$\begin{aligned} A &= -48F^2(4iH - jF), \\ B &= 2F^4\{18(4iH - jF^2) - 125F^2(i^3 - 6j^2)\}, \\ C &= 8F^6\{H^3(8 \cdot 48i^3 + 24 \cdot 25 \cdot 48j^2) - H^2F \cdot 48 \cdot 81i^2j \\ (12) \quad &- HF^2(7 \cdot 48 \cdot 48ij^2 + 25 \cdot 15i^4/2) + F^3(25 \cdot 80i^3j + 8 \cdot 24 \cdot 32j^3)\}, \\ R &= 12 \cdot 250TF^9\{-H^3 \cdot 48 \cdot 48(48j^3 + 14i^3j) + 6 \cdot 48H^2F(24i^2j^2 + 7i^5) \\ &- 48 \cdot 243HF^2 \cdot i^4j + F^3(625i^6 + 32 \cdot 48 \cdot 8i^3j^2)\}, \\ D &= 250 \cdot 64F^6(i^3 - 6j^2). \end{aligned}$$

These invariants are all functions of H, iF^2 , and jF^3 , except R which involves also T . Since

$$T^2 = -\frac{1}{2}(H^3 - \frac{1}{2}iHF^2 + \frac{1}{3}jF^3),$$

R^2 is a function of the same three quantities.

The quintic (11) having one root infinite and the sum of the other four zero is in precisely the form we considered earlier. Its roots can be identified with s_1, s_2, s_3, s_4 , in (2) and expressed as functions of y_1, y_2, y_3 . Equating the symmetric functions of s_i with the coefficients in (10) we find

$$(13) \quad \begin{aligned} \Sigma s_1 s_2 &= -2\Sigma y_1^2 = 3H, & \Sigma s_1 s_2 s_3 &= -8y_1 y_2 y_3 = -4T, \\ s_1 s_2 s_3 s_4 &= \Sigma y_1^4 - 2\Sigma y_1^2 y_2^2 = \frac{1}{2}iF^2 - \frac{3}{4}H^2. \end{aligned}$$

Since
$$2jF^3 = 3iF^2 H - 6H^3 - 12T^2,$$

$$(14) \quad \begin{aligned} H &= -\frac{2}{3}\Sigma y_1^2, & T &= -2y_1 y_2 y_3, & iF^2 &= \frac{8}{3}(\Sigma y_1^4 - \Sigma y_1^2 y_2^2), \\ jF^3 &= -\frac{8}{9}[2\Sigma y_1^6 - 3\Sigma y_1^4(y_2^2 + y_3^2) + 12y_1^2 y_2^2 y_3^2]. \end{aligned}$$

These values of iF^2, jF^3, H, T , in terms of y_1, y_2, y_3 , substituted in (12) give the equations of the invariant curves A, B, C, D, R .

We can now show that any invariant curve of the group corresponds to an invariant of the quintic. Since the curve admits G_{24} it must be a function of $\Sigma y_1^2, \Sigma y_2^2 y_3^2$, and $y_1 y_2 y_3$, or also of H, T , and iF^2 . Hence at all points y on the curve the line $y\overline{T}$ is the root of a definite covariant of the four lines from y to the points i . But if the curve admits also a quadratic transformation like $T_{(04)}$, the same thing is true of the line $y\overline{4}$ with regard to the four others, i. e., the covariant relation of the one line to the other four is an invariant relation of all five which is the same at all points of the curve.

In handling the invariant curves in later paragraphs we employ none of odd order and none of order as high as 54. Neither R nor R^2 can occur in the expressions used, and there is no loss of generality in making the calculation with $R = 0$ or with $y_1 = 0$. That is, the leading coefficient, or terms free of y_1 , will be sufficient for our purposes. These are somewhat simpler if we take out of the invariant of degree $4k$ the factor 2^{5k} . A direct calculation gives the results:

$$A = 4(2, 1, 1, 2\checkmark y_2^2, y_3^2),$$

$$B = (1, 1, -30, 65, -30, 1, 1\checkmark y_2^2, y_3^2),$$

$$B_1 = (-1, 2 + 5\delta_1, 2 + 5\delta_2, -1\checkmark y_2^2, y_3^2),$$

$$B_2 = (-1, 2 + 5\delta_2, 2 + 5\delta_1, -1\checkmark y_2^2, y_3^2),$$

$$C = -3(1, -11, 46, -86, 51, 51, -86, 46, -11, 1\checkmark y_2^2, y_3^2).$$

Here $B = B_1 B_2$. We have seen that B and C meet 10 times at each of the K -points, four of which are on the line $y_1 = 0$. The coördinates of these points are known and give the common factors of the leading coefficients of the curves B_i and C . In factored form the above coefficients are

$$\begin{aligned}
 A &= 4(2y_2^4 - y_2^2 y_3^2 + 2y_3^4)(y_2^2 + y_3^2), \\
 B &= (y_2^2 - \delta_1^2 y_3^2)^2 (y_2^2 - \delta_2^2 y_3^2)^2 (y_2^2 + \delta_1^4 y_3^2)(y_2^2 + \delta_2^4 y_3^2), \\
 (16) \quad B_1 &= -(y_2^2 - \delta_2^2 y_3^2)^2 (y_2^2 + \delta_1^4 y_3^2), \\
 B_2 &= -(y_2^2 - \delta_1 y_3^2)^2 (y_2^2 + \delta_2^4 y_3^2), \\
 C &= -3(y_2^2 - \delta_1^2 y_3^2)^4 (y_2^2 - \delta_2^2 y_3^2)^4 (y_2^2 + y_3^2).
 \end{aligned}$$

Certain quintic resolvents of the general equation find an immediate geometric interpretation. The absolute invariants B/A^2 and C/A^3 are functions of $iH/jF = k_1$ and $i^3/j^2 = k_2$. Since k_1 and k_2 are invariant under G_{24} , they are five valued under G_{120} and the elimination of either leads to a quintic resolvent whose coefficients are invariants. Or if \sqrt{D} be adjoined we can replace k_2 by $\frac{1}{9}\sqrt{\frac{5}{2}(k_2 - 6)} = \nu$. To obtain a simple result we shall introduce new absolute invariants J and L . Let

$$\frac{B_1}{A} = \frac{5L - 1}{8}, \quad \text{whence} \quad k_2 = \frac{3\nu + L}{4L},$$

and

$$75J = \left(-4 \cdot 24^3 \frac{C}{A^3} - 24 + 27 \cdot 25 L^2 \right);$$

then the elimination of k_1 and substitution of ν for k_2 gives the resolvent

$$(17) \quad J\nu^5 + L(4 - 125L^2)\nu^4 + \frac{5}{2}{}^5_7(J + 153L^2 - 4)\nu^3 - 50L^3\nu^2 + 5L^2\nu - 5L^3 = 0.$$

Since $j = 0$ if $\nu = \infty$, the numerator of the absolute invariant J of degree twelve is the condition that four of the roots be harmonic. L is proportional to ν and the \sqrt{D} formed for the four roots of the quintic. If (17) is made integral in the invariants by multiplication with A^3 , the resulting curve, for a given value of ν , is of order 18. It is the product of the twelve lines and three conics determined by requiring that four of the roots of a quintic or any even permutation of them have a given anharmonic ratio.

§ 3. The form problems associated with the Cremona group.

Under the group of § 1 there is a net of invariant curves of order thirty, $\rho A^3 + \sigma AB + \tau C = 0$. If numerical values are assigned to A, B, C , then ρ, σ, τ , are subject to a linear condition and a pencil of the net is determined. The various pencils in the net have 900 base points, but 780 of these are the same for all pencils. They are the $5 \cdot 12 \cdot 12 = 720$ meets at the points p_i , and the $10 \cdot 30 - 5 \cdot 4 \cdot 12$ meets of A and C outside the points p_i . Hence only 120 of the base points of pencils formed from the net are movable. They are, of course, a set of 120 conjugate points. Instead of numerical values of A, B, C , it is sufficient to give the values of $B/A^2 = k$ and $C/A^3 = \lambda$. The pencil is

then fixed by the two curves $A(B - kA^2) = 0$ and $C - \lambda A^3 = 0$. The form problem of the group can be stated as follows:

Given the numerical values of A, B, C , or of $B/A^2, C/A^3$, to find the ratios $x_1 : x_2 : x_3$ for the 120 points x for which the functions A, B, C , or the ratios $B/A^2, C/A^3$, take the assigned values.

One point x is sufficient, since the others are determined from it by a set of substitutions known in advance. The problem will be referred to as the *I*-form-problem. Its solution is equivalent to the solution of a given quintic to within rational operations. For a given quintic fixes the values of A, B, C . The solution of the form problem fixes a point x such that the five lines xp_i are projective to the given quintic. The known quintic can be transformed into the given one by a known linear substitution determined by their linear covariants. The problem has exactly the same statement for the group in § 2.

If \sqrt{D} be adjoined as a known quantity, i. e., if $\Pi(\overline{p_i p_k})$ is given a definite sign, the group reduces to the alternating G_{60} and the invariant curves are expressible by means of B_1, B_2, C , and R . In this case we consider the net of curves of order 30, $B_1^2(\rho B_1 + \sigma B_2) + \tau C = 0$. The pencils in the net have 840 fixed base points, 720 at the points p_i and 120 meets of B_1^2 and C outside the points p_i . The remaining 60 base points are movable. The form problem now reads:

Given the numerical values of B_1, B_2, C or of $B_2/B_1, C/B_1^3$, to find the ratios $x_1 : x_2 : x_3$ for the 60 points x for which B_1, B_2, C , or $B_2/B_1, C/B_1^3$, take the assigned values.

In place of B_1 and B_2, A and \sqrt{D} can be given. They are connected by the equations $4(B_1 + B_2) = -A$ and $4(B_1 - B_2) = \sqrt{D}$. As before the solution of this form problem—referred to as the *I'*-form problem—carries with it the solution of a binary quintic for which \sqrt{D} is given.

§ 4. *The reduction of the I'-form problem to Klein's "problem of the A's."*

The expressions in the *Ikosaeder* denoted by $A_0, A_1, A_2; A'_0, A'_1, A'_2; A, B, C, D$, will be referred to by the corresponding small letters to avoid confusion with the notation already introduced. In this paragraph the group of § 2 is employed.

Through five points in space q_0, q_1, \dots, q_4 there pass ∞^2 cubic curves. On each curve the five points determine a system of projective quintics and a given quintic whose roots are ordered with regard to the five points determines a definite curve. There is thus a one-to-one correspondence between the cubic curves and the systems of projective quintics or also a one-to-one correspondence between the cubic curves and the point y_1, y_2, y_3 of the plane (§ 2). To obtain the cubic curve determined by the point y , we take the five points in space with the coördinates suggested in the *Ikosaeder*, p. 187; the coördinates of q_i are

$$(1) \quad p_1 : p_2 : p_3 : p_4 = \epsilon^i : \epsilon^{2i} : \epsilon^{3i} : \epsilon^{4i}.$$

Let the parameters of the five points in order on the cubic curve be $t = \infty, s_1, s_2, s_3, s_4$, and let $c_i = (t - s_k)(t - s_l)(t - s_m)$, $i, k, l, m = 1, 2, 3, 4$. The parametral equation of the cubic is then

$$(2) \quad p_i = \epsilon^i c_1 + \epsilon^{2i} c_2 + \epsilon^{3i} c_3 + \epsilon^{4i} c_4.$$

The c_i as functions of s_i and therefore of y_i give the connection sought.

If the point y be subjected to the Cremona G_{120} , the corresponding cubic curve takes 120 positions. The 120 curves are conjugate under the collineation group, g_{120} , of the points q_i . Linear complexes are also linearly transformed by g_{120} and there arises in this way an isomorphic collineation group on the six coördinates A_{ik} . KLEIN shows (*Ikosaeder*, pp. 176-181 and pp. 245-247) that if the following combinations

$$\begin{aligned} 2a_0 &= A_{23} + A_{41}, & a_1 &= A_{42}, & a_2 &= A_{31}, \\ 2a'_0 &= -A_{23} + A_{41}, & a'_1 &= -A_{43}, & a'_2 &= A_{12}, \end{aligned}$$

of the complex coördinates be employed, the new coördinates are transformed by the even subgroup g_{60} of g_{120} in such a way that a_0, a_1, a_2 , alone experience the linear transformations of the ternary ikosahedral group and a'_0, a'_1, a'_2 , alone experience the transformations of the "contragredient" form of the same group (that form in which ϵ is replaced by ϵ^2).

The cubic curve (2) defines a linear complex and we shall find, first of all, the coördinates $a_0, a_1, a_2, a'_0, a'_1, a'_2$ of this complex. Two planes, $\sum_{i=1}^4 \rho_i p_i$ and $\sum_{i=1}^4 \sigma_i p_i$, cut the cubics in 3 points each, whose parameters are given by the binary cubics in t :

$$\sum_{i,k=1}^4 \epsilon^{ik} \rho_i c_k = 0 \quad \text{and} \quad \sum_{i,m=1}^4 \epsilon^{im} \sigma_i c_m = 0.$$

These cubics are apolar, and the line of the planes is a line of the complex, if

$$\sum_{k,m} \left\{ \sum_i \epsilon^{ik} \rho_i \sum_j \epsilon^{jm} \sigma_j - \sum_i \epsilon^{im} \rho_i \sum_j \epsilon^{jk} \sigma_j \right\} (c_k, c_m) = 0$$

where $(c_k, c_m) = -(c_m, c_k) = (s_k - s_m)(s_i - s_j)^2$ is the apolarity condition of the cubics c_k and c_m . The summation can be put in the form

$$\sum_{k,m} (c_k, c_m) \left\{ \sum_{i,l} (\rho_i \sigma_l - \rho_l \sigma_i) (\epsilon^{ik+lm} - \epsilon^{im+lk}) \right\} = 0,$$

or expressed in planar linear coördinates Π_{ii} , the equation of the complex is

$$(3) \quad \sum_{k,l} \Pi_{ii} \left[\sum_{k,m} (\epsilon^{ik+lm} - \epsilon^{im+lk}) (c_k, c_m) \right] = 0.$$

The coefficient of Π_{ii} is A_{ii} , and (c_k, c_m) is the product of the line joining the points k and m and the square of the line joining the other two points. The

six cubics (c_k, c_m) are therefore linearly independent plane cubics through the points 1, 2, 3, 4. It is convenient to replace these six by the following two sets of three:

$$(4) \quad \begin{aligned} C_1 &= y_1(y_1^2 + \delta_2 y_2^2 + \delta_1 y_3^2), & K_1 &= y_1(y_1^2 + \delta_1 y_2^2 + \delta_2 y_3^2), \\ C_2 &= y_2(\delta_1 y_1^2 + y_2^2 + \delta_2 y_3^2), & K_2 &= y_2(\delta_2 y_1^2 + y_2^2 + \delta_1 y_3^2), \\ C_3 &= y_3(\delta_2 y_1^2 + \delta_1 y_2^2 + y_3^2), & K_3 &= y_3(\delta_1 y_1^2 + \delta_2 y_2^2 + y_3^2), \end{aligned}$$

In terms of these we have

$$(5) \quad \begin{aligned} a_0 &= (\epsilon^2 - \epsilon^3)C_2 + (\epsilon^4 - \epsilon)C_3, \\ a_1 &= -(\epsilon^4 - \epsilon)(\epsilon^2 - \epsilon^3)C_1 + (\epsilon^4 - \epsilon)C_2 - (\epsilon^2 - \epsilon^3)C_3, \\ a_2 &= (\epsilon^4 - \epsilon)(\epsilon^2 - \epsilon^3)C_1 + (\epsilon^4 - \epsilon)C_2 - (\epsilon^2 - \epsilon^3)C_3, \\ a'_0 &= (\epsilon^4 - \epsilon)K_2 - (\epsilon^2 - \epsilon^3)K_3, \\ a'_1 &= (\epsilon^4 - \epsilon)(\epsilon^2 - \epsilon^3)K_1 - (\epsilon^2 - \epsilon^3)K_2 - (\epsilon^4 - \epsilon)K_3, \\ a'_2 &= -(\epsilon^4 - \epsilon)(\epsilon^2 - \epsilon^3)K_1 - (\epsilon^2 - \epsilon^3)K_2 - (\epsilon^4 - \epsilon)K_3. \end{aligned}$$

Formulas (4) and (5) give the coordinates of the complex of the cubic space curve determined by the point y . Also for every complex arising from a cubic space curve we find one point y , since a_i and a'_i determine C_i and K_i , and from the equations

$$\frac{C_1}{y_1} + \frac{C_2}{y_2} + \frac{C_3}{y_3} = 0, \quad \frac{K_1}{y_1} + \frac{K_2}{y_2} + \frac{K_3}{y_3} = 0,$$

we have

$$(6) \quad y_1 : y_2 : y_3 = \frac{1}{C_2 K_3 - C_3 K_2} : \frac{1}{C_3 K_1 - C_1 K_3} : \frac{1}{C_1 K_2 - C_2 K_1}.$$

As the variables y are subjected to the operations * of the Cremona G_{60} , the a_i and a'_i are subjected to the collineations of the two forms of the ternary ikosahedral group. Hence

(7) Any invariant of the group of the a_i , or any simultaneous invariant of the group of the a_i and the group of the a'_i , when expressed in terms of y is an invariant of the Cremona G_{60} , and therefore a rational function of B_1, B_2, C .

It is necessary, for our purpose, to find in terms of B_1, B_2, C , the invariants a, b, c (*Ikosaeder*, pp. 214-8) of orders 2, 6, 10 in a_i and the simultaneous invariants F_1, F_2, F_3 (*Ikosaeder*, pp. 231-2) linear in a'_i and of orders 3, 5, 7 respectively in a_i . The respective orders in y are 6, 18, 30, and 12, 18, 24.

* The cubics C_i and K_i are transformed into cubics multiplied by a factor which is the product of the three lines of the principal triangle. But when only the ratios are under consideration, this factor, which is the same for all six cubics, can be dropped.

Since these are all even and of order less than 54, it will be sufficient to use the leading coefficient in identifying them.

When $y_1 = 0$,

$$(8) \quad \begin{aligned} a_0 &= (\epsilon^2 - \epsilon^3)(1, -1, \delta_2, \delta_2 \bar{\eta} y_2, y_3), \\ a_1 = a_2 &= (\epsilon^2 - \epsilon^3)(\delta_2, -\delta_1, \delta_2^2, -1 \bar{\eta} y_2, y_3) = (\epsilon^4 - \epsilon)(y_2 - \delta_2 y_3)(y_2 - \delta_1 y_3)^2, \end{aligned}$$

$$(9) \quad \begin{aligned} a'_0 &= (\epsilon^4 - \epsilon)(1, -1, \delta_1, \delta_1 \bar{\eta} y_2, y_3), \\ a'_1 = a'_2 &= (\epsilon^4 - \epsilon)(\delta_1, -\delta_2, \delta_1^2, -1 \bar{\eta} y_2, y_3) = -(\epsilon^2 - \epsilon^3)(y_2 - \delta_1 y_3)(y_2 - \delta_2 y_3)^2, \\ a &= a_0^2 + a_1^2, \end{aligned}$$

$$(10) \quad \begin{aligned} b &= (8a_0^4 - 2a_0^2 a_1^2 - 2a_0 a_1^3 + a_1^4) a_1^2, \\ c &= 4(80a_0^6 - 63a_0^5 a_1 - 40a_0^4 a_1^2 + 40a_0^3 a_1^3 + 5a_0^2 a_1^4 - 10a_0 a_1^5 + 2a_1^6) a_1^4, \end{aligned}$$

$$F_1 = 2a'_0 a_0 (2a_0^2 - 3a_1^2) - 2a'_1 a_1^2 (3a_0 + a_1),$$

$$F_2 = 4a'_0 (-4a_0^3 + 3a_0 a_1^2 - a_1^3) a_1^2 + 4a'_1 (8a_0^3 - 4a_0^2 a_1 - 2a_0 a_1^2 + a_1^3) a_1^2,$$

$$\begin{aligned} F_3 &= 4a'_0 (16a_0^3 - 4a_0^2 a_1 - 8a_0 a_1^2 + 3a_1^3) a_1^4 \\ &\quad + 4a'_1 (-16a_0^5 + 24a_0^4 a_1 - 16a_0^3 a_1^2 - 2a_0^2 a_1^3 + 7a_0 a_1^4 - 2a_1^5) a_1^2. \end{aligned}$$

The values of a and F_1 are not hard to find by direct calculation. The following method reduces considerably the difficulty of finding the values of b, c, F_2, F_3 . Call the number of times that a term contains the factor $(y_2 - \delta_1 y_3)$ its "first rank"; the number of times it contains the factor $(y_2 - \delta_2 y_3)$ its "second rank"; and let $\{i, k\}$ be the "rank" of the term if its first and second ranks are i and k respectively. Then $a_0, a_1, a'_0, a'_1; C, B_2, B_1$, have the ranks $\{0, 0\}, \{2, 1\}, \{0, 0\}, \{1, 2\}; \{4, 4\}, \{2, 0\}, \{0, 2\}$ respectively. Let

$$F_3 = \mu_1 C B_1 + \mu_2 C B_2 + \mu_3 B_1^2 B_2^2 + \mu_4 B_1^3 B_2 + \mu_5 B_1 B_2^3 + \mu_6 B_1^4 + \mu_7 B_2^4.$$

Now B_2^4 and $B_1^3 B_2$ are the only terms (including of course the terms in the expression for F_3 in (11)) which have the second rank 0 and 2 respectively, hence $\mu_6 = \mu_7 = 0$. For a similar reason with regard to the first rank, $\mu_4 = \mu_5 = 0$. μ_1/μ_3 is determined by making the first rank of $C + k B_1 B_2^2$ five. We find that $k = -\frac{3}{4}$. Then μ_1 and μ_3 are obtained by making the first rank of $16a'_0 a_0^5 a_1^2 + \mu_1 (C B_1 + k B_1^2 B_2^2)$ equal to 6. The result is $\mu_1 = \frac{3}{8} 2^5 5^3, \mu_3 = -8 \cdot 5^3$. Finally μ_2 is obtained by requiring the term

$$-16a'_0 a_0^3 a_1^4 + 16a'_1 a_0^5 a_1^2 + \mu_2 C B_2 - 8 \cdot 5^3 B_1^2 B_2^2$$

to have a second rank 5, which determines μ_2 to be $4 \cdot 5^3$. We have determined thus

$$F_3 = 2^2 \cdot 5^3 \left[\frac{8}{3} CB_1 + CB_2 - 2B_1^2 B_2^2 \right].$$

The quantity c is found much more easily though of higher degree. Let

$$c = \nu_1 CB_1^2 + \nu_2 CB_1 B_2 + \nu_3 CB_2^2 + \nu_4 B_2^3 B_1^2 + \nu_5 B_2^2 B_1^3 + \nu_6 B_2^4 B_1 + \nu_7 B_2 B_1^4 + \nu_8 B_1^5 + \nu_9 B_2^5.$$

The lowest rank occurring in the expression for c in (10) is $\{8, 4\}$. Noting that there are only single terms of first rank 0 and 2 and of second rank 0 and 2 we have $\nu_6 = \nu_7 = \nu_8 = \nu_9 = 0$. ν_1 and ν_5 must be chosen so that the first rank of $B_1^2(\nu_1 C + \nu_5 B_1 B_2^2)$ is 6. But this is impossible for we have just seen that the first rank of $\nu_1 C + \nu_5 B_1 B_2^2$ can be reduced to five only by taking $\nu_5/\nu_1 = -\frac{3}{4}$. Hence $\nu_5 = \nu_1 = 0$. Then ν_2 and ν_4 must be chosen so that the first rank of $B_1 B_2(\nu_2 C + \nu_4 B_1 B_2^2)$ shall reduce from 6 to 8; this again is impossible. Hence $\nu_2 = \nu_4 = 0$. Finally we determine ν_3 so that the rank $(8, 4)$ of $-4 \cdot 80 a_0^6 a_1^4 + \nu_3 CB_2^2$ shall increase to $\{10, 5\}$. We find $\nu_3 = 5^5 \cdot 2^2/3$ and $c = 5^5 \cdot 2^2/3 CB_2^2$.

Proceeding similarly with the simpler forms b and F_2 , we obtain all the desired expressions as below:

$$\begin{aligned} a &= 4B_1 + B_2, \\ (12) \quad b &= 25 \left(\frac{4}{3} C + B_1 B_2^2 \right), \\ c &= 2^2 \cdot 5^5 \cdot \frac{1}{3} CB_2^2; \\ F_1 &= 2(16B_1^2 + 4B_2^2 + 13B_1 B_2), \\ (13) \quad F_2 &= 2^2 \cdot 5(-6C + 5B_1 B_2^2), \\ F_3 &= 2^2 \cdot 5^3 \left(\frac{8}{3} CB_1 + CB_2 - 2B_1^2 B_2^2 \right). \end{aligned}$$

To solve the I' -form problem we find, from the given quantities B_1, B_2, C , the values of a, b, c , and F_1, F_2, F_3 in (12) and (13). Taking the "problem of the a 's" in this form:

(14) *Given the numerical values of a, b, c or of the ratios $b/a^3, c/a^5$, to find the ratios of the coördinates of the 60 points $a_0 : a_1 : a_2$ for which a, b, c or $b/a^3, c/a^5$ take the assigned values;*

we have from the values of a, b, c , the values of $a_0 : a_1 : a_2$. As F_1, F_2, F_3 , are linear in a'_0, a'_1, a'_2 , their numerical values and those of a_0, a_1, a_2 determine a'_0, a'_1, a'_2 . The values of a_i and a'_i fix the values of C_i and K_i in (5). These in turn, in (6), determine the required solution, $y_1 : y_2 : y_3$.

There is, however, one case in which the method just given does not apply. The three linear equations used to obtain a'_i are of degrees 3, 5, 7 in a_i . The determinant of the system is an invariant of degree 15 and must be the curve

d (*Ikosaeder*, p. 218). The curve d is the product of the fifteen axes of the harmonic perspectivities in the group of the a_i , one of which is $a_1 - a_2 = 0$. But if $a_1 - a_2 = 0$, $C_1 = 0 = y_1(y_1^2 + \delta_2 y_2^2 + \delta_1 y_3^2)$. If $y_1 = 0$, $R = 0$. But if y is a point on the conic $y_1^2 + \delta_2 y_2^2 + \delta_1 y_3^2 = 0$, a certain alternating invariant of degree 12 vanishes. The corresponding curve is the product of the three conics and twelve lines into which the given conic is transformed by G_{60} . Calling this curve S_1 and identifying it by means of its leading coefficient as before we find that

$$(15) \quad S_1 = 32C - 24B_1B_2^2 + 3B_2^3.$$

Its transform by an odd substitution of G_{120} is

$$(16) \quad S_2 = 32C - 24B_1^2B_2 + 3B_1^3.$$

If $R = 0$, due to $y_1 = 0$, both $a_1 = a_2$ and $a'_1 = a'_2$, while if $S_1 = 0$, due to $y_1^2 + \delta_2 y_2^2 + \delta_1 y_3^2 = 0$, then $a_1 = a_2$; but $a'_1 \neq a'_2$ except for a finite number of points. We suppose (see § 6) that $R \neq 0$. If $S_1 = 0$, the given quantities B_1 and B_2 can be interchanged, and, provided also $S_2 \neq 0$, the solution can be carried through as before. The resulting $y_1 : y_2 : y_3$ must then be transformed by an odd substitution of G_{120} to afford a solution for the constants originally given. If $S_1 = 0$ and $S_2 = 0$, the form problem can be solved directly by finding the intersections of these two curves whose factors are linear or quadratic.

The invariant

$$S = S_1 S_2 = \frac{1}{2} [2^{11} C^2 + 2^4 \cdot 33 CBA - 3 CA^3 + 2 \cdot 3^6 B^3 - 3^2 B^2 A^2]$$

has for corresponding curve the locus of points whose quintics are of the type $(z_1 + az_2)(z_1^5 - z_2^5)/(z_1 - z_2)$, i. e., four of the roots belong to a cyclo-projective set of five.

§ 5. A new solution of the "Problem of the A 's."

We have considered the I -form problem and the a -form problem as depending essentially on two parameters and have avoided the use of the invariant curves R and d . The solution of the a -problem suggested by KLEIN (*Ikosaeder*, pp. 234-8) involves, at least explicitly, the use of d . The solution here given is, in some respects, quite similar, but the point of view is more nearly that of the third chapter of the second part of the *Ikosaeder* where the "principal equation" is solved by the use of a singly infinite linear system of curves on the principal surface (pp. 190-4). The a -problem can be solved by the extraction of a square root and a root of the icosahedral equation. We might infer from this the existence of singly infinite quadratic systems of invariant rational curves and, in fact, the simplest system of that sort is found below and employed to effect the solution. All the necessary rational formulæ will be given explicitly.

It is desirable, first of all, to obtain expressions that will include all rational invariant curves. If such a curve is of order n we can take it to be

$$(1) \quad U: [a_0 = g_0^{(n)}(\nu), \quad a_1 = g_1^{(n)}(\nu), \quad a_2 = g_2^{(n)}(\nu)],$$

$g_i^{(n)}(\nu)$ being a binary form of degree n in the variables ν_1, ν_2 . If U is invariant under g_{60} , the group of the a 's, to every member of g_{60} must correspond a transformation of the binary variables, and the totality of such transformations will be the binary ikosahedral group, γ_{60} . This group being put in the canonical form, there are, as KLEIN points out (*Ikosaeder*, p. 232), two possible cases; γ_{60} is either "cogredient" or "contragredient" to g_{60} . In the first case the parameter is called λ_1, λ_2 , in the second μ_1, μ_2 . The point $a_0: a_1: a_2$ is given by the coefficients of the quadratic form $a_1\lambda_1'^2 + 2a_0\lambda_1'\lambda_2' - a_2\lambda_2'^2$, an invariant if the a_i are transformed by g_{60} and λ_1', λ_2' are simultaneously transformed by the corresponding substitutions of the cogredient form of γ_{60} . Putting in this quadratic the values of a_i from (1) we have

$$(2) \quad g_1^{(n)}(\nu)\lambda_1'^2 + 2g_0^{(n)}(\nu)\lambda_1'\lambda_2' - g_2^{(n)}(\nu)\lambda_2'^2$$

and *this must be an invariant under γ_{60} in the cogredient variables λ and λ' of degrees n and 2 respectively or in the contragredient variables μ and μ' of degrees n and 2 respectively. Conversely any such form determines a rational invariant curve.*

In the first case the form (2) is a rational and integral function of the ikosahedral invariants, $f(\lambda), H(\lambda), T(\lambda)$; their first and second polars as to λ' , and $\lambda_1\lambda_2' - \lambda_2\lambda_1'$, the function being homogeneous and of the proper degree in both variables. In the second case, the form (2) is a rational and integral function of $f(\mu), H(\mu), T(\mu)$, four invariants* linear in λ' and of degrees 7, 13, 17, 23 in μ , and three invariants, quadratic in λ' and of degrees 6, 10, 16 in μ , the function being homogeneous and of the proper degrees. If f' and H' are the first polars of f and H as to λ' , the form (2) which defines the system used hereafter is

$$(3) \quad (\lambda_1\lambda_2' - \lambda_2\lambda_1')(9\rho fH' - 25Hf'),$$

ρ being the parameter of the system.

The system itself is suggested by the following considerations. The coordinates of a point, as the system of coefficients of a quadratic, determine values of λ and λ' , the parameters of the two points of contact of tangents from the point to the conic, $a = a_0^2 + a_1a_2$, i. e.,

$$(4) \quad 2a_0 = -(\lambda_1\lambda_2' + \lambda_2\lambda_1'), \quad a_1 = \lambda_2\lambda_2', \quad a_2 = -\lambda_1\lambda_1'.$$

* See the complete system, established by GORDAN, *Mathematische Annalen*, vol. 13 (1878), p. 386, for the invariants containing contragredient variables λ' and μ .

If λ and λ' are simultaneously transformed by γ_{60} , $a_0 : a_1 : a_2$ is transformed by g_{60} . The point a_i depends on two quantities, both variable under γ_{60} . If one of the quantities can be expressed in terms of the other and a third quantity ρ which is invariant under γ_{60} , we should have in (4) the equation of a system of invariant rational curves. This is possible, for example, by means of the equations

$$(5) \quad \begin{aligned} 12f'' &= \frac{\partial f}{\partial \lambda_1} \lambda'_1 + \frac{\partial f}{\partial \lambda_2} \lambda'_2 = 3f\rho_1 = r, \\ 20H' &= \frac{\partial H}{\partial \lambda_1} \lambda'_1 + \frac{\partial H}{\partial \lambda_2} \lambda'_2 = 5H\rho_2 = s. \end{aligned}$$

So defined ρ_1 and ρ_2 are invariant under γ_{60} . Solve equations (5) for λ'_1 and λ'_2 and substitute their values in (4), and we have the desired system of rational curves U :

$$(6) \quad \begin{aligned} 2a_0 &= \rho_1 3f \left(\lambda_2 \frac{\partial H}{\partial \lambda_2} - \lambda_1 \frac{\partial H}{\partial \lambda_1} \right) - \rho_2 5H \left(\lambda_2 \frac{\partial f}{\partial \lambda_2} - \lambda_1 \frac{\partial f}{\partial \lambda_1} \right), \\ a_1 &= \rho_1 3f \left(\lambda_2 \frac{\partial H}{\partial \lambda_1} \right) - \rho_2 5H \left(\lambda_2 \frac{\partial f}{\partial \lambda_1} \right), \\ a_2 &= \rho_1 3f \left(\lambda_1 \frac{\partial H}{\partial \lambda_2} \right) - \rho_2 5H \left(\lambda_1 \frac{\partial f}{\partial \lambda_2} \right); \end{aligned}$$

or again

$$(7) \quad \begin{aligned} a_0 &= \rho_1 f \cdot 6(\lambda_1^{20} - 114\lambda_1^{15}\lambda_2^5 - 114\lambda_1^5\lambda_2^{15} - \lambda_2^{20}) - \rho_2 H \cdot 5\lambda_1\lambda_2(-\lambda_1^{10} - \lambda_2^{10}), \\ a_1 &= \rho_1 f \cdot 12\lambda_1^4\lambda_2(-\lambda_1^{15} + 171\lambda_1^{10}\lambda_2^5 - 247\lambda_1^5\lambda_2^{10} - 57\lambda_2^{15}) \\ &\quad - \rho_2 H \cdot \lambda_2^2(11\lambda_1^{10} + 66\lambda_1^5\lambda_2^5 - \lambda_2^{10}), \\ a_2 &= \rho_1 f \cdot 12\lambda_1\lambda_2^4(57\lambda_1^{15} - 247\lambda_1^{10}\lambda_2^5 - 171\lambda_1^5\lambda_2^{10} - \lambda_2^{15}) \\ &\quad - \rho_2 H \cdot \lambda_1^2(\lambda_1^{10} + 66\lambda_1^5\lambda_2^5 - 11\lambda_2^{10}). \end{aligned}$$

That this is a quadratic system of curves follows at once from the definition of the parameter λ as one of the two parameters λ and λ' in (4). The order of the curves is, in general, 32. But when $\rho_1 = 0$, the order is 12; when $\rho_2 = 0$, the order is 20. Also when $\rho_1 = \rho_2$, the order is 2. For then $a_0 = \lambda_1\lambda_2 T$, $a_1 = -\lambda_2^2 T$, $a_2 = \lambda_1^2 T$. The curve is $a = a_0^2 + a_1 a_2$. This is seen at once by forming $4(a_0^2 + a_1 a_2)$ in (6) and making use of Euler's theorem. We find

$$(8) \quad 4(a_0^2 + a_1 a_2) = 3600f^2 H^2 (\rho_1 - \rho_2)^2.$$

The equation of the general curve U is quadratic in ρ_1, ρ_2 . For a fixed point $a_0 : a_1 : a_2$, the two values of ρ_1/ρ_2 are obtained from (5), as it stands, and from interchanging λ and λ' in (5). Denote by \bar{g} the value that g takes when λ and λ' are interchanged. Then

$$\rho_1 \bar{\rho}_1 = \frac{r\bar{r}}{9f\bar{f}}, \quad \rho_2 \bar{\rho}_2 = \frac{s\bar{s}}{25H\bar{H}}, \quad \rho_1 \bar{\rho}_2 + \rho_2 \bar{\rho}_1 = \frac{Hr\bar{f}\bar{s} + \bar{H}\bar{r}f s}{15f\bar{f}H\bar{H}},$$

and ρ_1, ρ_2 satisfy the equation

$$9f\bar{f} \cdot s\bar{s}\rho_1^2 - 15(Hr\bar{f}\bar{s} + \bar{H}\bar{r}f s)\rho_1\rho_2 + 25H\bar{H} \cdot r\bar{r}\rho_2^2 = 0.$$

The coefficients being symmetric in λ and λ' of degree 32 in each are functions of a_0, a_1, a_2 of degree 32. Being invariant as well they are functions of a, b, c . The equation can be made more symmetrical by obtaining a new middle coefficient. When $\rho_1 = \rho_2$ the equation must reduce to $k \cdot T\bar{T} \cdot a = 0$. By equating the coefficients of $\lambda_1^{32}\lambda_2^{32}$, k is found to be -25 .

The new equation reads

$$(9) \quad 9f\bar{f} \cdot s\bar{s} \cdot \rho_1^2 - (9f\bar{f} \cdot s\bar{s} + 25H\bar{H} \cdot r\bar{r} + 25T\bar{T} \cdot a)\rho_1\rho_2 + 25H\bar{H} \cdot r\bar{r}\rho_2^2 = 0.$$

We have therefore to calculate $f\bar{f}$, $H\bar{H}$, $T\bar{T}$, $r\bar{r}$, $s\bar{s}$, and solve the equation (9) to obtain a value of $\rho = \rho_1/\rho_2$. We have further to find $Z_{(\rho)}$, the ikosahedral parameter on the curve $U_{(\rho)}$ fixed by ρ . Denote by $a_{(\rho)}, b_{(\rho)}, c_{(\rho)}$, the values of a, b, c , on the curve $U_{(\rho)}$. The curves $r\bar{r} = 0$ and $s\bar{s} = 0$ are $U_{(0)}$ and $U_{(\infty)}$ respectively. For $U_{(0)}$ we find

$$\begin{aligned} a_{(0)} &= 36f^2, \\ (10) \quad b_{(0)} &= f(12^3 \cdot 32f^5 - 5H^3), \\ c_{(0)} &= (16 \cdot 12^3 f^5)^2 - 11(16 \cdot 12^3 f^5)H^3 + H^6, \end{aligned}$$

Therefore

$$\frac{b_{(0)}}{a_{(0)}^3} = \frac{1}{27} (32 - 5Z_{(0)}), \quad \frac{3^4}{4} \frac{c_{(0)}}{a_{(0)}^5} = 2^8 - 11 \cdot 2^4 Z_{(0)} + Z_{(0)}^2.$$

Solving for $Z_{(0)}$, then eliminating $Z_{(0)}$ to obtain the equation of the curve we have

$$\begin{aligned} (11) \quad Z_{(0)} &= \frac{32a^3 - 27b}{5a^3}, \\ 16r\bar{r} &= 4(9b^2 + 16 \cdot 17a^3b - 16^2a^6) - 25ac. \end{aligned}$$

The coefficient of $r\bar{r}$, 16, is determined from the highest term. Similarly for $U_{(\infty)}$, we find

$$\begin{aligned} (12) \quad a_{(\infty)} &= 3 \cdot 12H^2, \\ b_{(\infty)} &= 12^6 f^5 (28H^3 - 12^3 f^5), \\ c_{(\infty)} &= \frac{1}{4} 12^{10} \cdot f^5 H (-16H^6 + 40H^3 \cdot 12^3 f^5 + 3 \cdot 12^6 f^{10}), \end{aligned}$$

Therefore

$$\frac{b_{(\infty)}}{a_{(\infty)}^3} = \frac{1}{27Z_{(\infty)}} \left(28 - \frac{1}{Z_{(\infty)}} \right), \quad \frac{c_{(\infty)}}{a_{(\infty)}^5} = \frac{4}{27Z_{(\infty)}} \left(-16 + \frac{40}{Z_{(\infty)}} + \frac{3}{Z_{(\infty)}^2} \right);$$

$$(13) \quad Z_{(\infty)} = \frac{4a^2(128a^3 - 3b)}{c + 496a^2b},$$

$$s\bar{s} = 2^7 \cdot 5^2(c^2 + 2^4 \cdot 8^3ca^2b + 2^4 \cdot 3^5ab^3 - 2^{11} \cdot 7ca^5 + 2^{10} \cdot 79a^4b^2 - 2^{15} \cdot a^7b).$$

The value of $Z_{(1)}$ is given by KLEIN (*Ikosaeder*, p. 221),

$$(14) \quad Z_{(1)} = \frac{c^3}{12^3 b^5}.$$

To every value of ρ corresponds a definite U_ρ and on it a definite $Z_{(\rho)}$. Hence $Z_{(\rho)}$ is a rational function of ρ . Since ρ satisfies the quadratic equation (9), there is no loss in assuming $Z_{(\rho)}$ to be a linear function of ρ . And the values of $Z_{(\rho)}$ being known for three values of ρ we have in general:

$$(15) \quad Z_{(\rho)} = \frac{Z_{(\infty)}(Z_{(0)} - Z_{(1)})\rho + Z_{(0)}(Z_{(1)} - Z_{(\infty)})}{(Z_{(0)} - Z_{(1)})\rho + (Z_{(1)} - Z_{(\infty)})}.$$

There remains only the calculation of $f\bar{f}$, $H\bar{H}$, $T\bar{T}$, in terms of a , b , c . They are respectively the products of the 12, 20, 30 tangents at the f , H , and T points on a . By direct calculation we find

$$f\bar{f} = (115b + 128a^3)b,$$

$$(16) \quad H\bar{H} = c^2 + 2^{20}a^{10} - 2^{17} \cdot 5a^7b + 2^9 \cdot 5 \cdot 241a^4b^2$$

$$- 2^9 \cdot 101a^5c - 2^5 \cdot 131cba^2 - 2^5 \cdot 3359ab^3.$$

The quickest way to get $T\bar{T}$ is to suppose $2a_0 = i(a_1 + a_2)$, as this is one of the lines of the curve. Writing $a_1 + a_2 = 2\alpha$, $a_1 - a_2 = 2\beta$; a , b , c reduce to functions of α^2 and β^2 . Finally putting $(1 - 2i)\alpha^2 = k$, $\beta^2 = l$; we find for α , b , and c the values

$$a = -l, \quad b = -k^3 + 5k^2l + 5kl^2 - l^3,$$

$$c = 4(-3k^5 - 45k^4l + 10k^3l^2 - 50k^2l^3 + 25kl^4 - k^5).$$

The identity of degree 15 connecting these forms is to be found. This gives

$$(17) \quad T\bar{T} = c^3 - 12^3b^5 - \frac{1}{2}3 \cdot 11 \cdot 17c^2a'^5 - 2 \cdot 3 \cdot 5 \cdot 11c^2ba'^2 + 2 \cdot 3 \cdot 23ca'^{10}$$

$$- 2 \cdot 3 \cdot 5^2 \cdot 17cba'^7 + 2^4 \cdot 3 \cdot 5 \cdot 11cb^2a'^4 - 2^3 \cdot 3^2 \cdot 5 \cdot 7cb^3a'$$

$$- 2^3 \cdot 3^2 \cdot 5 \cdot 349b^4a'^3 + 2^3 \cdot 5 \cdot 691b^3a'^6 - 2^2 \cdot 3 \cdot 5 \cdot 37b^2a'^9$$

$$- 2^2 \cdot 3 \cdot 5ba'^{13} + a'^{15}$$

where $a' = 4a$.

Briefly, then, to solve the problem of the a 's we find, from formulas (11), (13), (14), (16), (17), the values of ff , $H\bar{H}$, $T\bar{T}$, $r\bar{r}$, $s\bar{s}$, $Z_{(0)}$, $Z_{(\infty)}$, $Z_{(1)}$ from the given values of a , b , c . By the extraction of a square root, a value of ρ is found in (9). The value of $Z_{(\rho)}$ is given by (15), and from it the ikosahedral irrationality λ_1/λ_2 is calculated. Finally the values of $a_0 : a_1 : a_2$ are obtained from (7).*

§ 6. *The solution of the quintic equation.*

We have now all the material at hand to outline a series of operations that will effect the solution of the general quintic equation, $Q(z) = 0$.

1°. The rational invariants A , D , C , and R are calculated.

We suppose $D \neq 0$, otherwise the double root can be calculated rationally and the quintic reduced to a cubic; also that $R \neq 0$, for otherwise one root is rationally known, the linear covariant of fifth order, α_z ; and α_z vanishes identically only for special types of quintics whose solutions can be obtained by elementary methods.

2°. The square root of the discriminant, \sqrt{D} , is adjoined.

This determines B_1 , B_2 , C , the known quantities in the I '-form problem.

3°. The I '-form problem is solved to obtain the 60 points $y_1 : y_2 : y_3$.

This solution is discussed in § 4, account being taken of the special cases $S_1 = 0 \{ S_2 \neq 0, S_2 = 0 \}$.

4°. One of the 60 points determines, by (2) § 2, a quintic, $Q'(s)$, whose roots, ∞, s_1, \dots, s_4 , are projective to the roots of $Q(z) = 0$.

5°. If α_z and δ_z are the linear covariants of degrees 5 and 13 of $Q(z)$ and D its discriminant; and if α'_z , δ'_z , D' are similar comitants of $Q'(s)$, then the linear transformation

$$(1) \quad \frac{D \cdot \alpha_z}{\delta_z} = \frac{D' \cdot \alpha'_z}{\delta'_z}$$

transforms ∞, s_1, \dots, s_4 into z_0, \dots, z_4 , the required roots of $Q(z) = 0$.

Since $R \neq 0$, α and δ are distinct, non-vanishing covariants and the transformation (1) is perfectly definite.

§ 7. *Remarks on the above solution of the quintic.*

The problem of the quintic involves five parameters, the given coefficients. The process sketched above (steps 1°, 2°, 4°, 5°) by which the transition is made from this five parameter problem to the I '-form problem containing two

* The calculations necessary in the above solution seem simpler than those required in the method proposed by KLEIN (*Ikosaeder*, p. 237). The most difficult one, the calculation of $T\bar{T}$ is shorter than several there indicated. In this connection it may be noted that a similar set of three degenerate curves occurs in the linear system of rational curves employed to solve the principal equation (see pp. 190-1 and p. 204, footnote).

parameters, and *vice versa*, leaves little to be desired. The full effect of the general linear transformation is obtained and the analytical processes are entirely those of the well-known theory of binary forms. But the same can not be said of step 3°, the suggested solution of the *I'*-form problem. This is obtained *indirectly*, being made to depend on the problem of the *a*'s which involves the same number of parameters and a group of the same order and which is, therefore, not essentially a simpler problem. The solution of either should involve that of the other. An independent reduction of the *I'*-form problem to a determination of the ikosahedral irrationality is desirable. It ought to be possible to effect this, as in the other problems, by means of the rational invariant curves. Since a rational curve can admit a group of order 60 at most, it is necessarily a function of B_1, B_2, C , unsymmetric in B_1 and B_2 . The totality of such curves can be thus obtained: If α_0 and β_0 are two five-valued functions of λ , invariant under the same subgroup, γ_{12} , of γ_{60} ; and α_i and β_i their conjugate values, the quintic whose roots are α_i/β_i will be represented by a point y whose coördinates are functions of λ , [(6), § 2]. On the rational curve thus obtained, the group G_{60} of y and the group γ_{60} of λ are identical. One of the two simplest rational curves, B_1 itself, is thus obtained from the simplest five valued function, t . Writing

$$t_i = \epsilon^{2i} \lambda_1 (\lambda_1^5 - 2\lambda_2^5) + \epsilon^{2i} \lambda_2 (2\lambda_1^5 - \lambda_2^5) - 5\lambda_1^2 \lambda_2^2 (\epsilon^i \lambda_1^2 + \epsilon^{4i} \lambda_2^2)$$

and indicating the quadratic factors of H by the notation

$$\begin{aligned} [2 + i, 3 + i] &= (\epsilon - \epsilon^4)(\epsilon^i \lambda_1^2 - \delta_1^2 \lambda_1 \lambda_2 - \epsilon^{4i} \lambda_2^2), \\ [4 + i, 1 + i] &= (\epsilon^2 - \epsilon^3)(\epsilon^i \lambda_1^2 - \delta_2^2 \lambda_1 \lambda_2 - \epsilon^{4i} \lambda_2^2), \end{aligned}$$

we have for the differences of the t 's, the values

$$\begin{aligned} -\sqrt{5}(t_{2+i} - t_{3+i}) &= [0 + i, 4 + i][4 + i, 1 + i][1 + i, 0 + i], \\ -\sqrt{5}(t_{4+i} - t_{1+i}) &= [0 + i, 2 + i][2 + i, 3 + i][3 + i, 0 + i]. \end{aligned}$$

Substituting these differences in (6) § 2, and taking out the common factor $\pi[i, k], (i, k = 1, 2; \dots; 3, 4)$ we have the rational representation of the sextic B_1 in the same form as (6) except that $t_i - t_k$ is replaced by the quadratic $[i, k]$. The author hopes to discuss these curves more fully in a later paper.

It may be mentioned here that by formulas (4), (5), § 4, a point a is determined by five points y , since a pencil of cubics in the net of C_1, C_2, C_3 , has five movable base points. The separation of the five sets of 60 conjugate points y thus determined by a set of 60 conjugate points a can be accomplished by the general "diagonal" quintic equation. For if in equations (12), § 4, we put $B_1/B_2 = (u - 5a)/20a$ and eliminate B_1, B_2, C , we have the result

$$4cu^5 - 20bu^3 + 5au - 1 = 0.$$

Here a , b , c , being given and u determined, we have the values of B_1/B_2 and from b/a^3 , the value of C/B_2^3 .

§ 8. *Generalizations of the two forms of the Cremona G_{120} .*

The I -form-problem and its accompanying Cremona group, as discussed in § 2, admits of an immediate generalization to apply to the equation of the n th degree. In a space of $n - 3$ dimensions, S_{n-3} , a general set of $n - 1$ points is chosen, the points $1, 2, \dots, n - 1$. Through these points are ∞^{n-4} rational norm-curves of order $n - 3$, R ; through each point, P_1 , of S_{n-3} a single curve R . At P there is a definite S_{n-5} having $(n - 4)$ -point contact with R . This S_{n-5} and the $n - 1$ points determine a pencil of spaces S_{n-4} , which pencil also includes the space having $(n - 3)$ -point contact with R at P . In this pencil the $(n - 1)$ spaces determined by the $n - 1$ points and the hyper-osculating space fix a system of projective binary n -ics. Conversely a given binary n -ic fixes a set of $n!$ points P according to the order in which $n - 1$ of its roots are associated with the $n - 1$ fixed points. A single curve R is picked out by requiring the parameters of the fixed points to be projective to the selected $n - 1$ roots. On this curve there is a single point P whose parameter is at the same time projective to the n -th root.

The $n!$ points P determined by a binary n -ic are a set of points conjugate under a group $G_{n!}$ of Cremona point transformations. This group is generated by the collineation group of order $(n - 1)!$ which permutes the $n - 1$ fixed points and by the Cremona transformation of order $n - 3$ which has $n - 2$ of the points as a "principal basis" and the remaining point as a fixed point, a transformation of the type $x_i = 1/x_i$. It consists of $(n - 1)!$ collineations, and $(n - 1)!(n - 1)$ Cremona transformations of order $n - 3$ with $n - 2$ of the fixed points as principal points of order $n - 4$. To every invariant of the binary n -ic corresponds a locus of points P for which the corresponding invariant of the n spaces S_{n-4} determined at P vanishes. The locus I is of dimensions $n - 4$ and is an invariant of the group. Conversely to every invariant spread of dimensions $n - 4$ corresponds an invariant of the n -ic. To the discriminant D corresponds the square of the $\frac{1}{2}(n - 1)(n - 2)$ spaces through $n - 3$ of the fixed points.

The $n - 1$ points being given in any convenient coördinate system, the equations of the invariant spreads are obtained as before by means of the typical representation of the binary n -ic with one reference point at a zero of the n -ic and the other at the linear polar of this one with regard to the remaining $n - 1$. This requires a calculation of the invariants of the n -ic in the typical form. If I_1, I_2, \dots, I_{n-2} , are $n - 2$ independent invariant spreads, i. e., the corresponding invariants are connected by no syzygy, we can state again the I -form problem:

Given the numerical values of I_1, \dots, I_{n-2} , or of the $n - 3$ absolute invariants formed from them, to calculate the $n - 3$ ratios of the coördinates of one of a movable set of $n!$ conjugate points for which the spreads take the assigned values.

If \sqrt{D} be adjoined as an invariant curve, the order of the group reduces to $\frac{1}{2}n!$ A new set of $n - 3$ absolute invariants can now be chosen and a statement of the I' -form problem made as before. The solution of the I' -form problem gives the roots of an n -ic projective to the original n -ic. In the case of n an even number the transformation by which the transition to the given n -ic is made involves certain square roots (see CLEBSCH, *Binäre Formen*, pp. 425-7).

As a special case of the above, take $n = 6$. The binary sextic has 4 independent invariants, A, B, C , and D the discriminant, of degrees 2, 4, 6, and 10 in the coefficients, and one skew invariant R of degree 15 whose square is rational in the other four. The corresponding surfaces, invariant under the Cremona G_{720} , determined by five points in space, are of orders 4, 8, 12, 20, and 30 with 2-, 4-, 6-, 12-, and 15-fold points at the five points. Some of these invariants are determined by the transformations of period 2. D is the surface of fixed points under a transformation of the type (12). R is the product of 15 quadrics each a locus of fixed points under a transformation of the type (01) (23) (45). The 45 transformations of the type (12) (34) determine 15 line pairs and 30 conic pairs as their locus of fixed points.

The parameters in the I -form problem can be chosen to be $B/A^2, C/A^3, D/A^5$. In the I' -form problem occurs \sqrt{D} , a surface of order 10. Since \sqrt{D} and R are the only two surfaces whose orders are not divisible by 4 we have to take as the simplest set of parameters $B/A^2, C/A^3$ and $R/(\sqrt{D})^3$. The sets of 360 conjugate points then occur as the movable meets of a doubly infinite system of curves of order 96, invariant under G_{720} , and a simply infinite system of surfaces of order 30, invariant under G_{360} only.

A method of attack that is suggested in connection with this problem is to find a system of ∞^1 "homaloidal" surfaces invariant under G_{360} . Through any set of 360 conjugate points will pass a certain number of surfaces and one of this number can be singled out by the adjunction of an accessory irrationality. On the surface so obtained, which can be rationally represented by means of three homogeneous variables, the Cremona G_{360} is represented by WIMAN's group of 360 collineations on the three variables. Thus the solution of the I' -form problem, which contains 3 parameters, would be made to depend, by means of an accessory irrationality, on the solution of the form problem associated with WIMAN's group which depends on only two parameters. As in the case of the quintic, where the ikosahedral binary form problem is the final one, owing to the non-existence of a unary g_{60} , so in the case of the sextic, the WIMAN form-problem is final, owing to the non-existence of a binary g_{360} . Homaloidal sur-

faces of the desired sort can be obtained from the six values of the ratios of any two six-valued functions under g_{360} , i. e., any rational functions of a, b, c, d (notation of §§ 4, 5).

The G_{120} , discussed in § 1, is generalized in a very different way from that of § 2. In the plane we find geometrical analogues of the permutation groups of four and five things determined by four and five points respectively. *That the G_{24} of the four point is a collineation rather than a Cremona group, is accidental.* The analogue of the ∞^1 projectively distinct quartics is the pencil of conics through the four points. Owing to the presence of the invariant subgroup G_4 which transforms each conic into itself, a binary quartic is represented by only six members of the pencil. To extend this representation to the sextic we fix six points p_0, \dots, p_5 in a space S_3 . Through these six points pass ∞^3 quadric surfaces Q . A line L of the space is contained on a definite one of the Q and on it is, say, an h -generator. The six planes through L and the points p_0, \dots, p_5 determine projectively a sextic. The same sextic is determined by the parameters of the six h -generators through the point p_i and is therefore independent of the originally chosen h -generator L . Thus the lines of a regulus through the six points determine a system of projective sextics. Conversely a binary sextic determines a definite regulus through the six points. For five of the roots fix a cubic curve through the five points p_0, \dots, p_4 and the sixth fixes a point q_5 on this curve. The pencil of planes on the line p_5q_5 cut the cubic in two variable points whose join lies on a regulus through the six points. The only case of exception to the correspondence of sextic and regulus occurs when the sextic is projective to the parameters of the six points on the cubic curve through them. This system of sextics is represented by the ∞^2 chords of the cubic curve. The invariants of the sextic are represented by certain line complexes containing ∞^2 reguli.

Again with reference to seven given points in space, p_0, \dots, p_6 a line L fixes a system of projective binary septimics, and a given septimic, with ordered roots, will determine in general a definite line. Exceptions occur for the reguli through the seven points all lines of which determine the same septimics. In both the above cases the complexes of lines corresponding to invariants of the sextic and septimic can be obtained by CLEBSCH'S principle of transference. These complexes are invariant under Cremona groups of "line transformations," i. e., Cremona groups in six variables p_{ik} which have an invariant quadric. Owing to this identity among the variables the form problem of the septimic would involve only four parameters. Practically however this is not simpler than a form problem in five dimensions with five parameters. The large increase in the identities among the variables would seem to render useless further generalizations along this line by means of which octavics and nonics are represented by planes in four dimensions with references to 8 and 9 fixed points in the space.

We have in the above various methods for basing the solution of an equation of degree n , which contains n parameters, on the solution of a form problem which contains $n - 3$ parameters. The underlying group of the form problem is, to be sure, a Cremona group, more complicated therefore than the collineations groups heretofore employed. But the proof of WIMAN that no collineation group, isomorphic with the even permutation group of eight things, exists in less than six dimensions shows that the octavic can only be reduced to the solution of a collineation form problem containing not less than six parameters. On the other hand, the Cremona group form problem outlined at the beginning of this paragraph contains only five parameters. And there is no reason as yet to suppose that isomorphic Cremona groups do not exist in spaces of fewer dimensions. It is thus important to investigate the occurrence or non-occurrence of Cremona groups, isomorphic with the even permutation groups of k things, in the spaces of a moderate dimension.

In the case of the quintic we have seen that the adjunction of an irrationality (ρ in § 5 or \sqrt{A} of KLEIN) permits of the solution of a form problem with two parameters in terms of the solution of a form problem with only one ($z = H^3/12^3 f^5$). In connection with the form problems associated with a given equation three kinds of irrationalities can be defined: natural, accessory, and functional.

A natural irrationality is one whose adjunction permits of reducing a form problem with a certain number of parameters and a group of a certain order to another form problem with the same number of parameters but with a group of lower order.

Thus the adjunction of \sqrt{D} in § 3 lowers the order of the group from 120 to 60 while the number of parameters is the same.

An accessory irrationality is one whose adjunction permits of reducing the given form problem to another whose group has the same order but which depends on a smaller number of parameters. E. g., the adjunction of ρ in § 5 reduces the a -problem to the Z -problem, both having a group of order 60, but the first depending on two parameters, the second on one.

A functional irrationality is one whose adjunction permits of reducing the given form problem to another whose group has a lower order and which contains a smaller number of parameters.

In the case of an equation of degree n , one root of the equation is such an irrationality. So also is λ , the ikosahedral irrationality, whose adjunction reduces g_{60} to g_1 and reduces the number of parameters to zero.

The plan of procedure in solving a general equation of degree n suggested by the above is:

(1) To obtain the values of the $n - 3$ parameters in any I -form problem, whose group is $G_{n,1}$.

(2) To obtain, after the adjunction of the natural irrationality $\sqrt[n]{D}$, the $n - 3$ parameters in the I' -form problem whose group is $G_{\frac{1}{2}n!}$.

(3) To reduce, by the adjunction of successive accessory irrationalities, the number of parameters as far as possible, i. e., until the group, collineation or Cremona, isomorphic with $G_{\frac{1}{2}n!}$, in the smallest number of variables, has been attained.

(4) To adjoin, as a functional irrationality, the solution of this final form problem.

(5) To obtain from this solution, by rational processes (or also by the irrationalities involved in a typical representation of the n -ic) the roots of the given equation.

The writer hopes to apply the process here sketched to the sextic in a later paper.

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