ON CANTOR'S THEOREM CONCERNING THE COEFFICIENTS OF A CONVERGENT TRIGONOMETRIC SERIES, WITH GENERALIZATIONS*

BY

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§ 1. CANTOR'S THEOREM.

CANTOR† has shown that if the trigonometric series

$$\sum_{n=0}^{\infty}(a_n \cos nx + b_n \sin nx)$$

converges for all values of x in any interval $a \leqq x \leqq b$, however short, then

$$\lim_{n=\infty} a_n = 0 \qquad \text{and} \qquad \lim_{n=\infty} b_n = 0.$$

More generally, the conclusion holds if merely

$$\lim_{n=\infty}(a_n \cos nx + b_n \sin nx) = 0$$

for all points of the interval.

His proof is not simple, being based on arithmetic considerations of an intricate character. The following proof has the advantage of transparency as well as of rigor, and it admits of extension to cases to which CANTOR's methods do not apply.

Let A_n and γ_n be so chosen that

$$a_n \cos nx + b_n \sin nx = A_n \sin(nx + \gamma_n),$$

where $A_n \geqq 0$. Then

$$A_n = \sqrt{a_n^2 + b_n^2}.$$

* Presented to the Society April 24, 1909.

† G. CANTOR, *Ueber einen die trigonometrischen Reihen betreffenden Lehrsatz*, Journal für Mathematik, vol. 72 (1870), p. 130; *Sur les séries trigonométriques*, Acta Mathematica, vol. 2 (1883), p. 329. For other proofs cf. HARNACK, Mathematische Annalen, vol. 19 (1881), p. 251 and p. 524; LEBESGUE, *Leçons sur les séries trigonométriques*, 1906, p. 110.

Obviously a necessary and sufficient condition that

$$\lim_{n=\infty} a_n = 0 \qquad \text{and} \qquad \lim_{n=\infty} b_n = 0$$

is that

$$\lim_{n=\infty} A_n = 0.$$

Hence it is enough for the proof of the theorem to show that *from the hypothesis*

$$\lim_{n=\infty} A_n \sin(nx + \gamma_n) = 0$$

follows that

$$\lim_{n=\infty} A_n = 0.$$

Suppose the conclusion were false. Then there would exist a positive constant h such that

$$A_n \geqq h$$

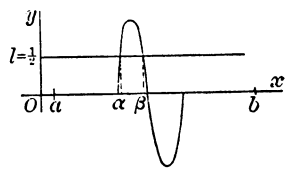
for an infinite set of values of the index, $n = n_1, n_2, \cdots$. Thus

$$A_{n_i} \geqq h \qquad (i = 1, 2, \cdots).$$

The function

$$\sin(n_i x + \gamma)$$

— we will drop the index of γ_n, since the value of the latter quantity is immaterial for our proof — has the period



$$\omega_i = \frac{2\pi}{n_i}.$$

Choose $i = i_1$ so that

$$\omega_{i_1} < \tfrac{1}{2}(b - a).$$

Then the above function has a complete arch above the axis of x, the base of the arch being enclosed in the interval (a, b). Hence throughout a certain subinterval $\alpha_1 \leqq x \leqq \beta_1$, where $a \leqq \alpha_1 < \beta_1 \leqq b$,

$$\sin(n_{i_1} x + \gamma) \geqq \tfrac{1}{2}.$$

Thus we have

$$A_n \sin(nx + \gamma) \geqq \tfrac{1}{2}h$$

when

$$n = n_{i_1} \qquad \text{and} \qquad \alpha_1 \leqq x \leqq \beta_1.$$

Proceeding next to the interval (α_1, β_1) we repeat the foregoing reasoning, choosing $i = i_2 > i_1$ so that

$$\omega_{i_2} < \tfrac{1}{2}(\beta_1 - \alpha_1),$$

and thus obtaining a second arch above the axis of x, its base being included

in the interval (α_1, β_1). Thus we get an interval $\alpha_2 \leqq x \leqq \beta_2$, where $\alpha_1 \leqq \alpha_2 < \beta_2 \leqq \beta_1$, in which

$$\sin(n_{i_2}x + \gamma) \geqq \tfrac{1}{2}.$$

Hence we have

$$A_n \sin(nx + \gamma) \geqq \tfrac{1}{2}h$$

when

$$n = n_{i_2} \qquad \text{and} \qquad \alpha_2 \leqq x \leqq \beta_2.$$

A continued repetition of the process leads to an infinite sequence of intervals (α_k, β_k), each lying in its predecessor, whose lengths approach 0 as their limit and which are such that

$$A_n \sin(nx + \gamma) \geqq \tfrac{1}{2}h$$

when

$$n = n_{i_k} \qquad \text{and} \qquad \alpha_k \leqq x \leqq \beta_k.$$

The extremities of these intervals determine a point

$$\lim_{k=\infty} \alpha_k = \lim_{k=\infty} \beta_k = X$$

pertaining to all of them. For this point

$$A_n \sin(nX + \gamma) \geqq \tfrac{1}{2}h \qquad (n = n_{i_1}, n_{i_2}, \cdots).$$

This is contrary to the hypothesis that

$$\lim_{n=\infty} A_n \sin(nx + \gamma) = 0$$

for every point of the interval (a, b). Hence the theorem is true.

A more general form of CANTOR's theorem, to which both his proof and the one here given apply, is obtained by replacing the expression

$$a_n \cos nx + b_n \sin nx \qquad \text{by} \qquad a_n \cos \rho_n x + b_n \sin \rho_n x,$$

where ρ_n is any function of n which merely becomes infinite with n:

$$\lim_{n=\infty} |\rho_n| = \infty.$$

§ 2. A GENERALIZATION.

THEOREM. *Let $f_n(x)$ be a continuous function of x in the interval $a \leqq x \leqq b$; let l be a certain positive constant, and let f_n have the following property. To every subinterval of (a, b), however short, there corresponds a positive integer m such that, if $n \geqq m$ be chosen at pleasure,*

$$|f_n(x)| > l$$

at some point of the latter interval.

If under these conditions the series

$$\sum_{n=1}^{\infty} A_n f_n(x)$$

converges for every value of x in the interval (a, b), then

$$\lim_{n=\infty} A_n = 0.$$

The foregoing proof applies with slight modifications to this more general case. Suppose the theorem false. Then there exists a positive constant h such that

$$|A_{n_i}| \geqq h \qquad (i=1, 2, \cdots).$$

Determine m_1 so that, for any given $n \geqq m_1$,

$$|f_n(x)| > l$$

at some point of the interval (a, b). Denote by i_1 a value of i for which $n_i \geqq m_1$. Then there will be a subinterval $\alpha_1 \leqq x \leqq \beta_1$, where $a \leqq \alpha_1 < \beta_1 \leqq b$, in every point of which

$$|f_n(x)| \geqq l \qquad (n = n_{i_1}).$$

Hence

$$|A_n f_n(x)| \geqq hl$$

when

$$n = n_{i_1} \qquad \text{and} \qquad \alpha_1 \leqq x \leqq \beta_1.$$

A continued repetition of the process leads to a set of intervals (α_k, β_k) like those of § 1, for which

$$|A_n f_n(x)| \geqq hl$$

where

$$n = n_{i_k} \qquad \text{and} \qquad \alpha_k \leqq x \leqq \beta_k.$$

Their extremities determine a point

$$\lim_{k=\infty} \alpha_k = \lim_{k=\infty} \beta_k = X$$

pertaining to all of them. For this point

$$|A_n f_n(X)| \geqq hl \qquad (n = n_{i_1}, n_{i_2}, \cdots),$$

and thus the theorem is proven.

Series of Functions of Several Variables. The theorem admits of immediate extension to functions of several variables. Thus, if $f_n(x, y)$ is continuous throughout a region S of the (x, y)-plane, we shall demand that to every subregion Σ of S, no matter how small, there correspond a positive integer m such

that, if $n \geqq m$ be chosen at pleasure,

$$|f_n(x, y)| > l$$

at some point of Σ, l denoting as before a positive constant. It is immaterial whether S include its boundary, just as it was unimportant that the interval (a, b) include its extremities.

The proof follows precisely the same lines as in the earlier case, a succession of squares, for example,—$\Sigma_1, \Sigma_2, \cdots, \Sigma_k, \cdots$,—each lying in its predecessor, replacing the intervals (α_k, β_k).

§ 3. Applications. Zonal Harmonics. Bessel's Functions.

Zonal Harmonics. The zonal harmonics $P_n(x)$ have precisely the property of the functions $f_n(x)$ in § 2, and thus we have the

THEOREM. *If a series of zonal harmonics,*

$$A_0 P_0(x) + A_1 P_1(x) + \cdots,$$

converges at all points of an interval $a \leqq x \leqq b$, however short, lying in the interval $-1 < x < 1$, then the coefficients A_n approach 0 as their limit.

Bessel's Functions. If we write

$$\sqrt{x} J_\nu(x) = \sqrt{\frac{2}{\pi}} \cos(x - \gamma) + \zeta,$$

where $\gamma = (\nu + \frac{1}{2})\pi/2$, it is well known that

$$\lim_{x=\infty} \zeta = 0.$$

We are thus enabled to state the following theorem regarding the behavior of the coefficients in a convergent series of Bessel's functions.

THEOREM. *If the series*

$$A_1 J_\nu(\lambda_1 x) + A_2 J_\nu(\lambda_2 x) + \cdots,$$

where $\lambda_n > 0$ does not depend on x and $\lim_{n=\infty} \lambda_n = \infty$ converges at all points of an interval $0 < a \leqq x \leqq b$, however short, then

$$\lim_{n=\infty} \frac{A_n}{\sqrt{\lambda_n}} = 0, \qquad \text{or} \qquad |A_n| < \epsilon\sqrt{\lambda_n},$$

where ϵ is an arbitrarily small positive quantity and $n \geqq m$.

For the general term can be written in the form

$$A_n J_\nu(\lambda_n x) = \frac{A_n}{\sqrt{\lambda_n}} \sqrt{\lambda_n} J_\nu(\lambda_n x),$$

and if we set

$$f_n(x) = \sqrt{\lambda_n} J_\nu(\lambda_n x),$$

then

$$|f_n(x)| = \frac{1}{\sqrt{x}} \sqrt{\lambda_n x}\, |J_\nu(\lambda_n x)| \geqq \frac{1}{\sqrt{b}} |\sqrt{\lambda_n x} J_\nu(\lambda_n x)|.$$

Hence the series

$$\sum_{n=1}^{\infty} \left(\frac{A_n}{\sqrt{\lambda_n}} \right) f_n(x)$$

satisfies the conditions of § 2, and this proves the theorem.

We have cited the asymptotic expansion of $J_\nu(x)$; but all that we need for our proof is the fact that the continuous function $\sqrt{x} J_\nu(x)$ takes on values numerically greater than a certain positive constant l in every interval whose length exceeds a certain fixed quantity ω and which lies beyond a definite point $x = \bar{x}$. By the aid of the asymptotic expansions the theorems of this paragraph can also be established by Cantor's methods.

§ 4. Developments in Multiple Series.

The theorem of § 2 can readily be extended to developments in multiple series of functions of several variables, like

$$\sum_{m,n} A_{m,n} \sin mx \cos ny.$$

The multiple series here considered are said to converge if every simple series formed out of the terms of the multiple series converges. Thus if such a multiple series converges at all, it converges absolutely.

Let us plot the points (m, n) in a plane, and let $R_1, R_2, \cdots$ be a sequence of regions each lying in its predecessor and each containing an infinite number of points (m, n). Examples: If (ξ, η) is an arbitrary point of the plane, we may take as R_N the points (ξ, η) for which

$$(a) \qquad \xi \geqq N, \qquad \eta \geqq N;$$

or, again, those for which

$$(b) \qquad \xi + \eta \geqq N, \qquad \xi \geqq 0, \qquad \eta \geqq 0.$$

Theorem. *Let $f_{m,n}(x, y)$ be a continuous function of the two independent variables x, y in a region S of the (x, y)-plane; let l be a certain positive constant, and let f have the following property. To every subregion Σ of S, no matter how small, there corresponds a region R_N such that, if the point (m, n) be chosen at pleasure in R_N,*

$$(1) \qquad |f_{m,n}(x, y)| > l$$

at some point of Σ.

If under these conditions the series

$$\sum_{m,n} A_{m,n} f_{m,n}(x, y)$$

converges at every point (x, y) *of* S, *then to an arbitrarily small positive number* ϵ *there corresponds a positive integer* p *such that, for every point* (m, n) *in* R_p,

$$|A_{m,n}| < \epsilon.$$

If, in particular, the regions R_k *are those of Example* (b), *then*

$$\lim_{m=\infty,\, n=\infty} A_{m,n} = 0.$$

The proof is similar to that given in § 2. Suppose the theorem false. Then there must be a point (m, n) in each R_k for which

$$|A_{m,n}| \geqq h,$$

where h denotes a certain positive constant. Thus we are led to a sequence of points (m_i, n_i), for which

$$|A_{m_i, n_i}| \geqq h \qquad (i = 1, 2, \cdots),$$

and such that, for any preassigned R_k, all those points for which $i \geqq q$ lie in that R_k.

Now choose R_{N_1} so that, for every point (m, n) in R_{N_1}, relation (1) holds at some interior point of S, and let i_1 be a value of i for which (m_{i_1}, n_{i_1}) lies in R_{N_1}. Then there will be an interior point (x_1, y_1) of S, for which (1) holds when $m = m_{i_1}$ and $n = n_{i_1}$. Surround this point by a square Σ_1 lying wholly in S and such that

$$|f_{m,n}(x, y)| \geqq l \tag{2}$$

when $m = m_{i_1}$, $n = n_{i_1}$ and the point (x, y) lies within or on the boundary of Σ_1.

The reasoning is now repeated, a smaller square Σ_2 lying in Σ_1 being obtained, throughout which relation (2) holds if $m = m_{i_2}$, $n = n_{i_2}$ and the point (x, y) lies in Σ_2. And so on. Thus a point (X, Y) is found, pertaining to all the squares and such that

$$|f_{m,n}(X, Y)| \geqq l$$

when (m, n) denotes successively the points

$$(m_{i_k}, n_{i_k}) \qquad (k = 1, 2, \cdots).$$

Hence for these values of (m, n)

$$|A_{m,n} f_{m,n}(X, Y)| \geqq hl,$$

and the theorem is proven.

The extension of both theorem and proof to r-fold series of functions of s independent variables presents no difficulty.

By combining the above theorem and that of § 2 we are enabled to state a further theorem.

THEOREM. *If the series*

$$\sum_{m,n} A_{m,n} f_{m,n}(x, y)$$

satisfies the conditions of the foregoing theorem, the regions R_k being those of Example (a); and if, furthermore, every simple series formed out of the double series by holding one of the indices m, n fast satisfies the conditions of the theorem of § 2, the l of this latter theorem, however, being at liberty to vary from series to series, then

$$\lim_{m=\infty,\, n=\infty} A_{m,n} = 0.$$

We wish to show that, under the above hypotheses, a positive ϵ being chosen at pleasure, there exists a positive integer M such that

$$|A_{m,n}| < \epsilon$$

for all values of m and n for which

$$m + n \geqq M.$$

To do this, choose k so that for all points (m, n) of R_k

$$|A_{m,n}| < \epsilon.$$

Next, consider the series

$$\sum_{m=1}^{\infty} A_{m,n'} f_{m,n'}(x, y),$$

where $1 \leqq n' \leqq k-1$. By the theorem of § 2 its coefficients approach the limit 0, and hence

$$|A_{m,n'}| < \epsilon \qquad (m \geqq \mu').$$

Let μ be the largest of the $k-1$ numbers μ' corresponding to $n' = 1, 2, \cdots k-1$. Then

$$|A_{m,n}| < \epsilon \qquad (m \geqq \mu,\ 1 \leqq n \leqq k-1).$$

In like manner consider the series

$$\sum_{n=1}^{\infty} A_{m',n} f_{m',n}(x, y),$$

where $1 \leqq m' \leqq k-1$. Its coefficients satisfy the condition

$$|A_{m',n}| < \epsilon \qquad (n \geqq \nu')$$

Let ν be the largest of the $k-1$ numbers ν' corresponding to $m'=1, 2, \cdots k-1$. Then

$$|A_{m,n}| < \epsilon \qquad (1 \leqq m \leqq k-1,\ n \geqq \nu).$$

Thus we see that

$$|A_{m,n}| < \epsilon$$

for all but a finite number of points (m, n), and hence, in particular, for all points (m, n) of a suitably chosen R_M of Example (b):

$$m+n \geqq M.$$

The conditions of the last part of the foregoing theorem are herewith seen to be fulfilled, and the present theorem is thus proven.

This theorem also admits of extension to multiple series of higher order than the second. In particular, for triple series of functions of three independent variables it can be formulated as follows.

THEOREM. *Let*

$$\sum_{m,n,p} A_{m,n,p} f_{m,n,p}(x, y, z)$$

be a triple series satisfying the conditions of the first theorem of this paragraph, the regions R_N being those of Example (a):

$$R_N: \qquad m \geqq N, \qquad n \geqq N, \qquad p \geqq N.$$

Furthermore, let every double series formed out of the triple series by holding one of the three indices m, n, p fast satisfy the conditions of the foregoing theorem. Then

$$\lim_{m=\infty,\, n=\infty,\, p=\infty} A_{m,n,p} = 0.$$

§ 5. Application to Multiple Trigonometric Series.

Consider any one of the double trigonometric series

$$\sum_{m,n} A_{m,n} \sin mx \sin ny,$$

where either or both of the sine factors may be replaced by the corresponding cosines. Let such a series converge at all points of a two dimensional region S of the (x, y)-plane. Then it can be shown exactly as in § 1 that this series satisfies the conditions of the first theorem in § 4, the regions R being those of Example (a).

Next, hold n fast and give y a constant value for which $\sin ny$ (or $\cos ny$) does not vanish. Thus the conditions of the theorem of § 2 are seen to be ful-

filled. Hence all the conditions of the second theorem in § 4 are satisfied and we have

$$\lim_{m=\infty,\, n=\infty} A_{m,n} = 0 .$$

Similarly it can be shown that if any one of the triply infinite series

$$\sum_{m,n,p} A_{m,n,p} \sin mx \sin ny \sin pz ,$$

where the sines may be replaced by cosines at pleasure, converges at all points of a three dimensional region V of space, its coefficients must approach the limit zero.

HARVARD UNIVERSITY,
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