INFINITE DISCONTINUOUS GROUPS OF BIRATIONAL TRANSFORMATIONS WHICH LEAVE CERTAIN SURFACES INVARIANT*

BY

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It was shown by SCHWARZ † that no algebraic curve of genus greater than unity can remain invariant under a continuous group of birational transformations. Later HURWITZ ‡ showed that no such curve could belong to any birational group of infinite order.

The corresponding theory for surfaces is by no means complete. While those belonging to continuous groups have been determined, only a few isolated examples are known of surfaces having an infinite discontinuous group. §

All the groups which have been discussed are illustrations of two principles, the first of which refers to quartic surfaces and will be considered in two parts, the latter including the former as a particular case; the second principle is applicable to a much wider category. I propose to discuss these principles and apply the second to the determination of an extended family of new surfaces having an infinite group.

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† *Ueber diejenigen algebraischen Gleichungen zwischen zwei veränderlichen Grössen, welche eine Schaar rationaler eindeutig umkehrbarer Transformationen in sich zulassen*, Crelle's Journal, vol. 87 (1879), pp. 139–146.

‡ *Ueber diejenigen algebraischen Gebilde, welche eindeutige Transformationen in sich zulassen*, Mathematische Annalen, vol. 32 (1888), pp. 290–308, reprinted from the Göttinger Nachrichten of February 7, 1887. See also NOETHER, Mathematische Annalen, vol. 21 (1883).

§ HUMBERT: *Sur la décomposition des fonctions θ en facteurs*, Comptes Rendus, vol. 126, (1898), pp. 394–396; *Sur les fonctions abéliennes singulières*, ibid., pp. 508–510, and Liouville's Journal, ser. 5, vol. 6 (1900), pp. 279-386, see page 372; PAINLEVÉ: *Sur les surfaces qui admettent un groupe infini discontinu de transformations birationnelles*, Comptes Rendus, vol. 126, (1898), pp. 512-514; HUTCHINSON: *The Hessian of the cubic surface II*, Bulletin of the American Mathematical Society, vol. 6 (1900), pp. 328–337, and *On some birational transformations of the Kummer surface into itself*, ibid., vol. 7 (1901), pp. 211-217; FANO: *Sopra alcune superficie del 4° ordine rappresentabili sul piano doppio*, Rendiconti del Istituto Lombardo, vol. 39 (1906), pp. 1071–1086. The first three of these articles are concerned with special forms of the Kummer surface; the treatment is entirely transcendental. The next two treat the general Kummer surface and two others into which it can be transformed; the treatment is partly transcendental and partly algebraic. The last one gives an outline of the theory of those quartic surfaces having a net of hyperelliptic curves; the treatment is purely geometric.

§ 1. *Quartic surfaces that possess a net of hyperelliptic curves of genus two.*

a. The nodal inversion (N).

1. The line joining any point A_1 on a quartic surface F_4 to a conical point N of the surface will meet it again in a point B_1. A (1, 1) correspondence exists between A_1, B_1, defining a non-linear birational transformation of order 2 which will be indicated by (N). If F_4 has two conical points N_1, N_2, these define two such transformations (N_1), (N_2). Every plane section through $N_1 N_2$ will remain invariant under the group generated by (N_1), (N_2). By taking N_1, N_2 as the vertices (0, 0, 0, 1), (0, 0, 1, 0) of the tetrahedron of reference the equation of a plane section may be written in the form

$$x_1^2(ax_2^2+bx_2x_3+cx_3^2)+x_1x_4(a'x_2^2+b'x_2x_3+c'x_3^2)+x_4^2(a''x_2^2+b''x_2x_3+c''x_3^2)=0.$$

The necessary and sufficient condition that the operations (N_1), (N_2) are commutative is the vanishing of the determinant *

$$(a)\qquad \begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix}.$$

If now the binodal quartic be transformed into a cubic, the operations (N_1), (N_2) become ordinary quadric inversions with regard to the polar conics of two ordinary points of the curve. If we still call these points N_1, N_2, the geometric condition that (N_1), (N_2) be commutative is that the tangents at N_1, N_2 meet on the curve, i. e., that N_1, N_2 be conjugate points. In general, the question whether (N_1), (N_2) define a finite group is thus reduced to STEINER's celebrated "Schliessungsproblem." †

In general, if $(N_1N_2)^k=1$, k must satisfy a certain relation which can be most easily expressed in terms of elliptic functions. Since any binodal quartic or non-singular cubic can be birationally derived from a space quartic curve of the first kind c_4, the (2, 2) correspondence between the points in which the lines through N_1, N_2 meet the curve again can be defined by means of the generators of a quadric surface passing through the quartic curve. The quartic curve may be defined by the equations

$$x_1=\rho\wp''(u),\qquad x_2=\rho\wp'(u),\qquad x_3=\rho\wp(u),\qquad x_4=\rho.$$

Four points corresponding to the values u_1, u_2, u_3, u_4 of the parameter u will lie in the same plane if, and only if,

$$u_1+u_2+u_3+u_4\equiv 0\pmod{2\omega_1,\ 2\omega_2}.$$

* STURM: *Die Lehre von den geometrischen Verwandtschaften*, vol. 1 (1908), p. 267.

† J. STEINER: *Geometrische Lehrsätze*, Crelle's Journal, vol. 32 (1845), pp. 182–184. If the cycle is closed for one point, it will be for every point.

Given any two points u and v on c_4. Through the line joining them can be passed a quadric surface of the pencil having c_4 for basis curve. If $u + v \equiv c$, then the parameters of the points on any generator of the second system must satisfy the relation $u + v \equiv -c$. Starting with the point u, we can obtain v by the equation $u + v \equiv c$, then from v we can find u_1 on the other generator by $v + u_1 \equiv -c$. Similarly $u_1 + v_1 \equiv c$, $v_1 + u_2 \equiv -c$, etc. If $u_k = u$, by eliminating the intermediate terms we obtain the condition

$$2kc \equiv 0.$$

For an arbitrary F_4 with two or more double points this condition will not be satisfied for any finite integer k, hence:

Quartic surfaces having m $(1 < m \leqq 16)$ double points exist which are invariant under a noncontinuous group of birational transformations of infinite order.

2. This result is interesting in view of KLEIN's * question regarding operations which leave the 16-nodal Kummer surface invariant. By means of properly chosen operations of the linear G_{16}, any (N_i) can be transformed into any other (N_k). This gives rise to at least thirty-two operations. By duality we have at once 32 more. The question is whether the surface is invariant under any other than these 64 operations. This question was answered in the affirmative by HUMBERT † by means of hyperelliptic theta functions.

On applying condition (a) to the equation of the surface referred to a Göpel tetrad, this question can be answered in the affirmative immediately. Similarly, by making use of the invariants whose vanishing expresses that $(N_i N_k)$ is finite it is seen that the surface has an infinite group, as the condition for finiteness would impose a relation among the three essential constants of the surface. By means of the known G_{32} the generators of this group can now be easily determined.

3. Another interesting case is furnished by the Weddle surface, the locus of the vertex of a quadric cone through six given points. Since this surface can be birationally transformed into the Kummer surface, it must have a G_{32}. This group was shown by BAKER to be defined by the six operations (N_i). ‡

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* *Ueber Konfigurationen, welche der Kummerschen Fläche zugleich ein- und umgeschrieben sind,* Mathematische Annalen, vol. 27 (1885), pp. 106–142. See p. 142.

† *Théorie générale des surfaces hyperelliptiques,* Liouville's Journal, ser. 4, vol. 9 (1893), pp. 29–170 and pp. 361–475. See page 466.

‡ This same result can be obtained from the parameters belonging to two points collinear with N_i as given in HUMBERT's third paper cited above, p. 470, but this fact is not there mentioned. BAKER: *Elementary note on the Weddle quartic surface,* Proceedings of the London Mathematical Society, ser. 2, vol. 1 (1903), pp. 247–261, gives an algebraic proof. A much more extensive discussion of the transformations of both surfaces is given by BAKER: *An introduction to the theory of multiply periodic functions* (1907); see pp. 69–82.

By the preceding principles this theorem can be obtained at once as follows. The equation of the surface may be written in the form

$$\begin{vmatrix} \frac{a_1}{x_1} & x_1 & a_1 & 1 \\ \frac{a_2}{x_2} & x_2 & a_2 & 1 \\ \frac{a_3}{x_3} & x_3 & a_3 & 1 \\ \frac{a_4}{x_4} & x_4 & a_4 & 1 \end{vmatrix} = 0,$$

the six nodes being at the four vertices of the tetrahedron of reference and the two points $(1, 1, 1, 1)$, (a_1, a_2, a_3, a_4). In a plane containing any pair of nodes, as $k_4x_3 = k_3x_4$, the two points on $x_1 = 0$ are $(0, 0, k_3, k_4)$, $[0, k_3(a_2-a_4) + k_4(a_3-a_2), k_3(a_3-a_4), k_4(a_3-a_4)]$. Similarly the two points on $x_2 = 0$ are $(0, 0, k_3, k_4)$, $[k_3(a_1-a_4)+k_4(a_3-a_1), 0, k_3(a_3-a_4), k_4(a_3-a_4)]$. The necessary and sufficient condition that (N_1), (N_2) be commutative is that the point $[k_3(a_1-a_4)+k_4(a_3-a_1), k_3(a_2-a_4)+k_4(a_3-a_2), k_3(a_3-a_4), k_4(a_3-a_4)]$ lie on the surface. By direct substitution we find that it does lie on the surface; similarly for the other pairs of nodes.

4. Moreover the result can be obtained geometrically. Let c_3 be the space cubic curve passing through the six points N_i. It lies on the Weddle surface defined by these points. The quadric cone having its vertex at N_i and passing through the five other points N_k contains c_3 and is the tangent cone at the conical point N_i of the Weddle surface.*

Any plane section through N_iN_k will contain the line N_iN_k and a cubic curve passing through N_iN_k. The tangent to the plane cubic at N_i is the generator of the tangent cone at N_i lying in the given plane and not passing through N_k. Similarly for N_k. Since these generators meet on c_3, it follows that N_i, N_k are conjugate points with regard to every plane cubic section passing through them, hence (N_i), (N_k) are commutative.

It is known that through any given plane quartic curve a Kummer surface can be passed, while all the plane sections of a Weddle surface are restricted, the relation among the constants being expressed by the equation

$$A^2 + 144B = 0,$$

wherein A is the cubic and B the sextic invariant of a quartic curve.†

* HIERHOLZER: *Ueber Kegelschnitte im Raume*, Mathematische Annalen, vol. 2 (1870), pp. 563–586. See page 582.

† F. MORLEY and J. R. CONNER: *Plane sections of a Weddle surface*, American Journal of Mathematics, vol. 31 (1909), pp. 263–270.

The preceding results furnish a new geometric interpretation of this important theorem for the case in which the plane passes through two conical points of the surface.

§ 1*b*. *The Involutions* (I).

5. A quartic surface F_4 of genus one ($p_n = p_g = p = 1$) which contains one curve of genus 2 will contain a net of such curves, any two of which will meet in two points, forming a rational involution I. (FANO, l. c.) The curves of order 4 are plane sections, and the surface has a double point. These curves define the inversions (N). In case the curves are of order 5 they are cut from F_4 by a net of quadrics, all having for residual the same twisted cubic.

Given a curve c_6 of order six and genus 2 on F_4. It is then one of a net σ_1, defining an involution I_1. Each c_6 of σ_1 will be cut in ∞^1 pairs of points of I_1, forming the canonical g_2^1. Through each c_6 pass ∞^2 cubic surfaces F_3. The residual intersection will be another sextic c_6', also of genus 2, and belonging to a net σ_2, distinct from σ_1 provided no F_3 of the net can be found which touches F_4 in every point of c_6. The net σ_2 will define an involution I_2 on F_4.

The ∞^1 lines determined by the pairs of points in g_2^1 on c_6 define a ruled cubic surface R_3 whose double directrix is a quadrisecant of c_6. The other points of intersection of each generator with F_4 will belong to the canonical g_2^1 on the residual c_6'. The curves c_6, c_6' have four points of intersection on the double directrix. Every line joining a pair of points in I_1 will also join a pair of points in I_2. The lines joining any point to its conjugate will therefore define a congruence of order 2, one of the lines joining a given point to its conjugate in I_1 and the other joining the same point to its conjugate in I_2. Starting with any point on the surface we can first find its conjugate in I_1 by a birational transformation of order 2, then the conjugate of the latter as to I_2, also a transformation of order 2. If a line of I_1 be given, there are two lines, one through each of the points of this line belonging to I_1, which connect it with its conjugate in I_2. Hence between the lines of I_1, I_2 there is a (2, 2) correspondence, and the condition for periodicity is reduced to that of the preceding case.*

The determination of these hyperelliptic curves by algebraic processes is in general a very difficult problem.

§ 2. *Systems of bitangents* (T).

6. Let $a \equiv \Sigma a_i x_i = 0$, $b = 0$ be the equations of a straight line, the coefficients a_i, b_i being rational functions of two non-homogeneous parameters κ, τ. If the coördinates of a fixed point y be substituted for x_i, the number of roots in κ, τ defines the order of the congruence T, that is, the number of lines

* FANO, l. c., showed by a different method that the operation ($I_1 I_2$) is in general of infinite order.

belonging to the system which pass through the point. The locus of the point y for which two of the lines coincide is the focal surface of the congruence. Thus, except for particular cases, the line of a given congruence of any order which touches the focal surface F at a given point P can be rationally separated from the other lines passing through P by the process of partial elimination. We shall be concerned only with such congruences as have a single line (counted twice) passing through P on F, and lying in its tangent plane. The other lines of T passing through P have one point of intersection with F at P.*

Congruences of the first order can have no focal surface; for those of the second order there is no residual line, while in the case of cubic congruences points on the focal surface are characterized by one double and one single root in κ, τ.

Every line of T is a bitangent to F, the points of tangency being P_1, P_2. The operation of interchanging P_1, P_2 is a birational transformation of order 2 which will be denoted by (T). If F is the complete focal surface for more than one congruence, we have two or more operations of order 2, and their product will be in general of infinite order. The necessary and sufficient condition that the transformations (T_1), (T_2) be birational is that the two congruences T_1, T_2 be rationally separable.

7. A number of important surfaces having this property can be obtained as the apparent contour of SEGRE's cubic variety Γ in four dimensional space S_4,† quartics by projecting from a point on Γ, and sextics by projecting from any point in S_4. In particular, if Γ contains a plane and can be generated by trilinear systems, the contour will be the focal surface of at least two congruences, and the equations of the transformations (T_1), (T_2) can be determined.

Let

$$\Gamma \equiv \begin{vmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & 0 \\ u_1 & u_2 & u_3 \end{vmatrix} = 0$$

be the equation of a cubic variety containing the plane $x_4 = 0$, $x_5 = 0$. A space $\kappa_2 x_4 - \kappa_1 x_5 = 0$ passing through this plane will cut from Γ a quadric surface, one such quadric passing through any point of Γ.

The two systems of generators are rationally separable, being

$$\frac{u_3}{u_1 k_2 - u_2 k_1} = \frac{x_3}{x_1 k_2 - x_2 k_1} = \tau_1, \qquad \frac{u_3}{x_3} = \frac{u_1 k_2 - u_2 k_1}{x_1 k_2 - x_2 k_1} = \tau_2,$$

* An example of congruences of another kind is furnished by a family of quadrics having one variable parameter, when the equations of the two systems of generators cannot be rationally separated.

† *Sulle varietà cubiche dello spazio a quattro dimensioni e su certi sistemi di rette e certe superficie dello spazio ordinario.* Memorie di Torino, ser. 2, vol. 39 (1889), pp. 1–48.

the second one being defined by the pencil of spaces passing through the other plane $u_3 = 0$, $x_3 = 0$ on Γ.

If now a plane be passed through any line τ_1 and a fixed point K, the section of Γ made by this plane will consist of τ_1 and a conic cutting τ_1 in two points P', P''. The locus of the points P_1, P_2 in which the lines KP', KP'' are cut by an S_3 not passing through K is the focal surface of the congruence T_1, the points P_1, P_2 being the points of contact. If $k \equiv (c_1, c_2, \cdots)$, and S_3 be defined by $\Sigma e_i x_i = 0$, the equations take the form

$$\frac{u_3 - \tau_1(u_1 k_2 - u_2 k_1)}{u_3(c) - \tau_1[u_1(c)k_2 - u_2(c)k_1]} = \frac{x_3 - \tau_1(x_1 k_2 - x_2 k_1)}{c_3 - \tau_1(c_1 k_2 - c_2 k_1)} = \frac{k_2 x_4 - k_1 x_5}{k_2 c_4 - k_1 c_5}$$

from which x_5, for example, can be eliminated by means of $\Sigma e_i x_i = 0$. If τ_1 be eliminated, a cubic in $k_1 : k_2$ results, whose envelope is the focal surface.

If then we start with a point $(x_1', \cdots, x_5')$ or x' on Γ, the associated S_3 and the line τ_1 are known and the points $P' \equiv (x')$, $P'' \equiv (x'')$ are at once defined as the roots of a quadratic equation, one of which is known. The other root is expressed as a rational function of the given one, and the equations of (T_1) are determined. In the same way we determine (T_2).

If the system T_1 belongs to a linear complex, the points P_1, P_2 are poles of the tangent planes at P_2, P_1 respectively, so that (T_1) may be regarded as the product of the two commutative operations, duality as to the surface and duality as to the complex. When T_1, T_2 both belong to linear complexes, the group generated by (T_1), (T_2) may be finite. In particular, the necessary and sufficient condition that (T_1), (T_2) are commutative is that the complexes to which T_1, T_2 belong are in involution.

The lines of the congruence T_1 can be arranged on a system of ∞^1 quadric surfaces, the congruence T_2 being composed of the other system of generators of these quadrics. Any line l of T_1 will therefore determine two lines m_1, m_2 of T_2, so that a (2, 2) correspondence exists between l, m. Thus the locus of P_1, P_2 corresponding to the quadric defined by l, m is a space curve c_4 of order 4 of the first kind. By the operations (T_1), (T_2) the curve c_4 remains invariant, so that each quadric of the system goes into itself. The points P_1, P_2 define a (2, 2) correspondence upon c_4, hence the discussion of the periodicity of the operation $(T_1 T_2)$ can be reduced to that of the preceding case.*

When the focal surface is of order 6 the residual section made by any quadric of bitangents is another c_4 which is also the curve of contact of another quadric. If the line $P_1 P_2$ cuts F_6 in Q_1, Q_2 the operation of interchanging Q_1, Q_2 is also birational, since only a single line of each congruence passes through Q_1

* For the literature concerning the (2, 2) correspondence on c_4, see O. STAUDE : *Flächen zweiten Grades und ihre Systeme und Durchdringungskurven*, Encyklopädie der mathematischen Wissenschaften, III C2., no. 123 ; in particular, footnote 513.

apart from the one (counted twice) which touches F_6 at Q_1. All the theorems regarding periodicity can be applied immediately to this case also.

8. An interesting illustration is furnished by the variety Γ having ten nodes. The contour from any point on Γ is the general Kummer surface. The six systems of bitangents belong to six linear complexes mutually in involution, hence the operations (T_i) generate the well-known G_{32} first found by Klein.*

The product of any two operations (T_i), (T_k) is a linear transformation, and thus the linear group of order 16 is defined. Many of the properties of the Kummer surface and of its systems of bitangents can be deduced readily from this starting point.

9. An illustration of a different kind, wherein the group generated by the $(T_i)(T_k)$ is finite, is obtained from the variety

$$\Gamma \equiv x_1 x_2 x_3 + \lambda x_4 x_5 x_6 = 0, \qquad \Sigma x_i = 1, \qquad \lambda \neq 1, \tag{1}$$

the center of projection being $P \equiv (1, 1, 1, -1, -1, -1)$. The details of this case will sufficiently illustrate the general procedure. The line of the system† I which passes through the point x' may be expressed by the equations

$$\frac{x_1' x_4 - x_4' x_1}{x_1' + x_4'} = \frac{x_2' x_5 - x_5' x_2}{x_2' + x_5'} = \frac{x_3' x_6 - x_6' x_3}{x_3' + x_6'} = l. \tag{2}$$

Solve these equations for x_4, x_5, x_6 in terms of l and substitute the results in (1). The equations of the conic and of the line may be written in the form

$$x_5' x_6' (x_1' + x_4') x_2 x_3 + x_4' x_6' (x_2' + x_5') x_1 x_3 + x_4' x_5' (x_3' + x_6') x_1 x_2 = 0, \tag{3}$$

$$x_2' x_3' (x_1' + x_4') x_1 + x_1' x_3' (x_2' + x_5') x_2 + x_1' x_2' (x_3' + x_6') x_3 = 0. \tag{4}$$

If now x' be taken on the first polar of Γ as to P,

$$H \equiv x_1 x_2 + x_1 x_3 + x_2 x_3 - \lambda (x_4 x_5 + x_5 x_6 + x_6 x_4) = 0, \tag{5}$$

it will be one of the points of intersection of the line and the conic. Making (3), (4) simultaneous and making use of (2), we obtain the coördinates of x'', the second point of intersection. The results are

$$x_1'' = \frac{1}{x_4'}, \qquad x_2'' = \frac{1}{x_5'}, \qquad x_3'' = \frac{1}{x_6'}, \qquad x_4'' = \frac{1}{x_1'}, \qquad x_5'' = \frac{1}{x_2'}, \qquad x_6'' = \frac{1}{x_3'}. \tag{6}$$

* *Zur Theorie der Liniencomplexe des ersten und zweiten Grades*, Mathematische Annalen, vol. 2 (1870), pp. 201–226.

† For this notation and the discussion of this variety, see my paper: *Surfaces derived from the cubic variety having nine double points in four dimensional space*, these Transactions, vol. 10 (1909), pp. 71–78.

Let the lines joining x' to P be cut by the polar S_3 of P as to Γ

$$(7)\qquad x_1 + x_2 + x_3 + \lambda(x_4 + x_5 + x_6) = 0.$$

To distinguish between points x on Γ and points on the S_3 defined by (7), coördinates in the latter will be denoted by y_i. It is defined by

$$(8)\qquad y_1 + y_2 + y_3 = 0, \qquad y_4 + y_5 + y_6 = 0.$$

From a point x on $\Gamma(x)$ we obtain the corresponding point in S_3 by means of the equations

$$(9)\qquad \begin{aligned} y_1 &= x_2 + x_3 - 2x_1, & y_4 &= x_5 + x_6 - 2x_4,\\ y_2 &= x_1 + x_3 - 2x_2, & y_5 &= x_4 + x_6 - 2x_5,\\ y_3 &= x_1 + x_2 - 2x_3, & y_6 &= x_4 + x_5 - 2x_6, \end{aligned}$$

and the reciprocal relations, giving x when y is known, are

$$(10)\qquad \begin{aligned} x_1 &= 2y_1H(y) - 3\Gamma(y), & x_4 &= 2y_4H(y) + 3\Gamma(y),\\ x_2 &= 2y_2H(y) - 3\Gamma(y), & x_5 &= 2y_5H(y) + 3\Gamma(y),\\ x_3 &= 2y_3H(y) - 3\Gamma(y), & x_6 &= 2y_6H(y) + 3\Gamma(y). \end{aligned}$$

Now by means of equations (6), (9), (10) we can obtain the equations of the birational transformation in S_3. The results are

$$(11)\qquad \begin{aligned} \rho y_1' &= -\lambda\,[2H(y)(y_4y_5 + y_5y_6 - 2y_4y_6) + 3\Gamma(y)(y_5 - 2y_4 - 2y_6)],\\ \rho y_2' &= -\lambda\,[2H(y)(y_4y_6 + y_5y_6 - 2y_4y_5) + 3\Gamma(y)(y_6 - 2y_4 - 2y_5)],\\ \rho y_3' &= -\lambda\,[2H(y)(y_4y_6 + y_4y_5 - 2y_5y_6) + 3\Gamma(y)(y_4 - 2y_5 - 2y_6)],\\ \rho y_4' &= 2H(y)(y_1y_2 + y_2y_3 - 2y_1y_3) - 3\Gamma(y)(y_2 - 2y_1 - 2y_3),\\ \rho y_5' &= 2H(y)(y_1y_3 + y_2y_3 - 2y_1y_2) - 3\Gamma(y)(y_3 - 2y_1 - 2y_2),\\ \rho y_6' &= 2H(y)(y_1y_2 + y_1y_3 - 2y_2y_3) - 3\Gamma(y)(y_1 - 2y_2 - 2y_3), \end{aligned}$$

the equations of the surface being

$$y_1 + y_2 + y_3 = 0, \qquad y_4 + y_5 + y_6 = 0, \qquad 27(1-\lambda)\Gamma^2(y) + 4H^3(y) = 0.$$

The transformation defined by (11) is not a Cremona transformation, being birational for points of the surface only.

In the same manner we may obtain the systems (T_2), (T_3), defined by II, III, $\cdots$. If we use the notation $(x_ix_kx_l)$ to indicate the cyclic substitution

$x_i = \rho x'_k$, etc., the results may be written in the form

$$T_2 = T_1 l_2, \quad T_3 = T_1 l_2^2, \quad T_4 = T_1 l_4, \quad T_5 = T_1 l_5, \quad T_6 = T_1 l_4 l_5 l_4,$$

wherein

$$l_2 = (y_1 y_2 y_3)(y_4 y_6 y_5); \; l_4 = (y_2 y_3)(y_5 y_6); \; l_5 = (y_1 y_2)(y_4 y_5).$$

Moreover

$$T_1 l_2 = l_2 T_1, \quad T_1 l_4 = l_4 T_1, \quad T_1 l_5 = l_5 T_1,$$

from which it follows that $T_1, \cdots, T_6$ generate a group of order 36 which contains a linear subgroup of order 18. The surface is also invariant under the odd substitutions $t = (y_2 y_3)$ etc., making another linear group of order 18. The operations (T_i) combine with these in the same manner as with the preceding, making a total group of order 72.

10. From any point P_1 on F_6 can be drawn just one line of I touching F_6 at P_1. Let P_2 be the second point of contact of this line. Similarly, the line $P_2 P_3$ belongs to IV and touches F_6 at P_2 and at P_3. The line $P_3 P_4$ belongs to I. Since $(T_1 T_4)^2 = 1$ it follows that $P_4 P_1$ belongs to IV and has its points of contact at P_4 and at P_1. The vertices P_1, P_2, P_3, P_4 define a tetrahedron inscribed in F_6 and the planes $P_1 P_2 P_3$, etc. are all tangent planes. The transformation $(T_1 T_4)$ transforms P_1 into P_3 and P_2 into P_4; it is the axial involution l_4. We have the following theorem:

There are nine systems of ∞^2 tetrahedra which are inscribed in and circumscribed about F_6. Two opposite edges of the tetrahedra of each system always meet two fixed lines.

11. When the line σ touches the residual conic, section of Γ, $P_1 = P_2$ and the corresponding line in S_3 has four coincident points in common with F_6. The locus of the point of contact in I is a curve defined as the intersection of $\Gamma(x)$ with $H(x)$ and the variety defined by the equation

$$x_1 x_2 x_3 x_4 x_5 x_6 \left[\frac{(x_1 + x_4)^4}{x_1^2 x_4^2} + \frac{(x_2 + x_5)^4}{x_2^2 x_5^2} + \frac{(x_3 + x_6)^4}{x_3^2 x_6^2} \right]$$
$$-2(x_1+x_4)^2(x_2+x_5)^2(x_3+x_6)^2 \left[\frac{x_1 x_4}{(x_1 + x_4)^2} + \frac{x_2 x_5}{(x_2 + x_5)^2} + \frac{x_3 x_6}{(x_3 + x_6)^2} \right] = 0.$$

By means of equations (6), (9), (10) the equations of the curve in S_3 can be obtained. Let C_1 denote the curve at the points of which the lines of I have four point contact. From any point P_1 of this curve draw the line of IV. From the preceding theorem the second point of contact must also lie on C_1. The ruled surfaces belonging to IV, V, VI which have C_1 for directrix curve are such that every generator of each is a bisecant of the curve. If the line of II touching F_6 at P_1 also touches it at P', and the line $P'P''$ belongs to I, then P'' is a point of C_2. *The six curves C_i are birationally equivalent.*

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