THE INTRODUCTION OF IDEAL ELEMENTS AND A NEW DEFINITION OF PROJECTIVE n-SPACE*

BY

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Introduction.

The ideal elements of projective geometry are usually introduced by means of the parallel and congruence axioms. The idea of defining the ideal elements without the assumption of the parallel axiom is due to KLEIN.† It was developed by PASCH;‡ by SCHUR;§ by BONOLA;‖ and by VEBLEN.¶ In all of these developments a three-dimensional geometry is assumed. The problem of defining the ideal elements in a plane geometry satisfying only order relations is closely connected with the problem of finding the necessary and sufficient condition that a plane may be a part of a three-space in which the axioms of order are satisfied. This condition is stated by HILBERT** to be the validity of the Desargues theorem in the plane. The Desargues theorem may be proved in a three-dimensional space satisfying only order relations, but can not be proved in the corresponding plane geometry, without additional assumptions, e. g., of the parallel and congruence axioms.

In this paper plane axioms of order will be assumed in the form given them by VEBLEN, the undefined elements being taken as the *point*, and a relation among points, called *order*. The first eight axioms are identical, except for notation, with his. Another axiom, one of closure, is then introduced, limiting the set of *points* considered to a plane. Two more axioms are then introduced, forms of the Desargues theorem, and of its converse, in terms of the set of points satisfying only order relations.

* Presented to the Society (Chicago) under a different title, April 22, 1905.

† *Ueber die sogenannte Nicht-Euklidische Geometrie*, Mathematische Annalen, vol. 6 (1872), p. 132.

‡ *Vorlesungen über neuere Geometrie*, pp. 40–72.

§ *Ueber die Einführung der sogenannten idealen Elemente in die projective Geometrie*, Mathematische Annalen, vol. 39 (1891), pp. 113–128.

‖ *Sulla introduzione degli enti improprii in Geometria projettiva*, Giornale di Matematiche, vol. 38 (1900), p. 105.

¶ *A system of axioms for geometry*, these Transactions, vol. 5 (1904), pp. 343–384.

** *Grundlagen der Geometrie, Festschrift* (1899), § 30.

On the basis of these eleven axioms, a new set of elements, called *pencils*, including in particular the pencils of lines centered at points, are defined, and their properties deduced. This set of pencils is shown to satisfy the incidence axioms and the general Desargues theorem of projective geometry, but not necessarily the Pascal theorem, well known * to be independent of the Desargues theorem, and even stronger assumptions. It is also shown that if we leave out of this set of pencils a subset called a *range*, the remaining elements satisfy the original axioms, I to XI.

In Section IV, a new set of elements called three-points, including in particular the projective points (i. e. pencils) of the plane, is defined, and this set is shown to form a projective three-space containing the plane set as a subset.

In Section V, an extension is made to n-space by generalizing the method of Section IV.

The notation used, though sometimes cumbersome, is very convenient in giving a concise expression for the proofs; and by displaying the configurational character of many of the figures used, it has been found of value in devising the proofs of the theorems. The *points* which are characterized by Axioms I–XI will be denoted by the symbol P_k, the P simply indicating that the element belongs to the class of points, and the subscript being a mark, not necessarily a single digit or letter, and the order of the digits or letters in the subscript immaterial, used to distinguish the various points; for instance, conveniently, the notation P_{xy}, $x, y = 1, \cdots, 5$, $x \neq y$, and $P_{xy} = P_{yx}$, will be used for the configuration of ten points called the Desargues configuration; the notation l_k will be used for lines, the k again being a mark of one or more digits or letters, the order of which is immaterial; the notation l_{123} will be used where convenient for a line containing points P_{12}, P_{13}, P_{23}. Such implications of the notation, while useful, are avoided in the statement of the proofs, as far as possible, but the notation is used consistently, even the axioms themselves being stated in it, in order to accustom the reader to it. Other forms of notation will be explained, as they occur.

The independence of the Axioms X and XI from Axioms I–IX may be shown from the Non-Desarguesian geometry of MOULTON † if the "line of break" of this geometry is a line of our points. The relation of Axioms X and XI on the basis of Axioms I–IX has not been determined.

This problem was first suggested to the author by Professor O. VEBLEN, to whom, and to Professor E. H. MOORE, he tenders his grateful acknowledgment for many helpful suggestions and criticisms.

* *Grundlagen der Geometrie, Festschrift*, § 31. The form of Pascal's theorem here referred to is, of course, that for the degenerate conic. (Theorem of Pappus.)

† *A simple Non-Desarguesian geometry*, Transactions of the American Mathematical Society, vol. 3 (1902), pp. 192–195.

The Axioms and Definitions.

AXIOM I. There exist at least two distinct points.

AXIOM II. If points P_1, P_2, P_3 are in the order $P_1P_2P_3$, they are in the order $P_3P_2P_1$.

AXIOM III. If points P_1, P_2, P_3 are in the order $P_1P_2P_3$, they are not in the order $P_2P_3P_1$.

AXIOM IV. If points P_1, P_2, P_3 are in the order $P_1P_2P_3$, then P_1 is distinct from P_3.

AXIOM V. If P_1 and P_2 are any two distinct points, there exists a point P_3, such that P_1, P_2, P_3 are in the order $P_1P_2P_3$.

If P_1 and P_2 stand for the same points, we write $P_1 = P_2$; if for distinct points, $P_1 \neq P_2$.

DEFINITION 1. The line P_1P_2, or conveniently, l_{12}, consists of P_1 and P_2 and all points P_k in any of the orders, $P_1P_kP_2$, $P_1P_2P_k$, $P_kP_1P_2$. The subscripts 1, 2 in the notation l_{12} refer to the points by which the line is determined.

The points P_k in the order $P_1P_kP_2$ constitute the *segment* P_1P_2.

AXIOM VI. If points P_3 and $P_4 (P_3 \neq P_4)$ lie on the line l_{12} (i. e., P_1P_2) then P_1 lies on the line l_{34} (i. e., P_3P_4).

AXIOM VII. If there exist three distinct points, there exist three points P_1, P_2, P_3 not in any one of the orders $P_1P_2P_3$, $P_1P_3P_2$, or $P_3P_1P_2$.

DEFINITION 2. Points of the same line are *collinear*. Three distinct non-collinear points, P_1, P_2, P_3 are the *vertices* of a *triangle* $P_1P_2P_3 = T_{123}$, whose *sides* are the segments P_1P_2, P_1P_3, P_2P_3, and whose *boundary* consists of its vertices and the points on its sides. Two triangles having the same vertices are identical.

AXIOM VIII. (Triangle transversal axiom.) If three distinct points $P_1P_2P_3$ do not lie on the same line, and P_4 and P_5 are two points in the orders $P_2P_3P_4$ and $P_3P_5P_1$, respectively, then a point P_6 exists in the order $P_1P_6P_2$, and such that P_4, P_5, P_6 lie on the same line.

DEFINITION 3. If P_1, P_2, P_3 are three non-collinear points, the *plane* $P_1P_2P_3$ consists of all the points collinear with any two points of the sides of the triangle $P_1P_2P_3$.

AXIOM IX. The set of points P_k here considered constitutes a plane.

DEFINITION 4. A triangle is said *to lie on three lines* if it has a vertex on each of these lines.

DEFINITION 5. Three distinct lines l_{340}, l_{341}, l_{342} are *in pencil* if there exists on these lines a pair of triangles having no side common $P_{30}P_{31}P_{32}$ and $P_{40}P_{41}P_{42}$, P_{30} and P_{40}, $P_{30} \neq P_{40}$, being points of the line l_{340}, etc., such that the lines $P_{30}P_{31}$ and $P_{40}P_{41}$ meet* in a point P_{01}, the lines $P_{31}P_{32}$ and

* Two lines *meet* only if distinct.

$P_{41}P_{42}$ meet in a point P_{12}, and the lines $P_{30}P_{32}$ and $P_{40}P_{42}$ meet in a point P_{02}, and such that the points P_{01}, P_{02}, P_{12} are collinear.

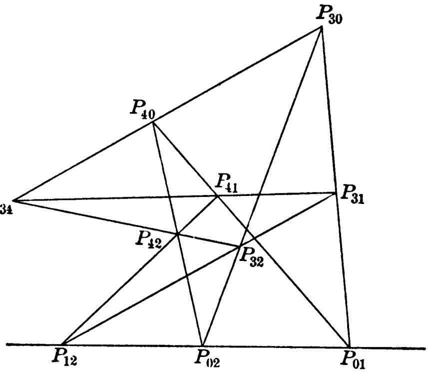

FIG. 1.

The figure of Definition 5 (see Fig. 1) will be called a Desargues configuration, and the notation

$$(1)\qquad \begin{array}{ccc} P_{12} & P_{02} & P_{01} \\ \hline P_{40} & P_{41} & P_{42} \\ P_{30} & P_{31} & P_{32} \\ & Q_{34} & \end{array}$$

will be used when by hypothesis the triads of points, $P_{12}P_{02}P_{01}$, $P_{12}P_{41}P_{42}$, $P_{12}P_{31}P_{32}$, $P_{02}P_{40}P_{42}$, $P_{02}P_{30}P_{32}$, $P_{01}P_{40}P_{41}$, $P_{01}P_{30}P_{31}$ are collinear, and the triads $P_{30}P_{31}P_{32}$ and $P_{40}P_{41}P_{42}$ are vertices of triangles; by the above definition, then, the lines $P_{40}P_{30}$, $P_{41}P_{31}$, and $P_{42}P_{32}$ will be *in pencil*, Q_{34}.

DEFINITION 6. The set of all lines *in pencil* with a given pair, together with this pair, constitute a pencil, determined by the two lines, and by Theorem VI, determinable by *any* two lines in it. The notation Q_k will be used for pencils, the mark k, as before, being any convenient symbol of one, two, or more digits or letters, these usually referring to the lines by which it is determined, and the order of the digits of the subscript being immaterial as far as denoting the pencil is concerned. A line is said to *belong* to a pencil if it is one of the lines of the pencil.

AXIOM X. If l_{340}, l_{341}, l_{342} are three lines in a pencil, and $P_{30}P_{31}P_{32}$ and $P_{40}P_{41}P_{42}$ are any two triangles on these lines, P_{30} and P_{40} lying on l_{340} (but not necessarily distinct), etc., such that the lines

$$\begin{aligned} P_{30}P_{31} &= l_{301} \quad\text{and}\quad P_{40}P_{41} = l_{401} \text{ meet in a point } P_{01}, \\ P_{31}P_{32} &= l_{312} \quad\text{and}\quad P_{41}P_{42} = l_{412} \text{ meet in a point } P_{12}, \\ P_{30}P_{32} &= l_{302} \quad\text{and}\quad P_{40}P_{42} = l_{402} \text{ meet in a point } P_{02}, \end{aligned}$$

and P_{01}, P_{12}, P_{02} are distinct and determinate, then P_{01}, P_{12}, and P_{02} are collinear.

The figure of Axiom X (see Fig. 2) will be represented by the notation

$$(2)\qquad \begin{array}{ccc} & Q_{34} & \\ P_{30} & P_{31} & P_{32} \\ P_{40} & P_{41} & P_{42} \\ \hline P_{12} & P_{02} & P_{01} \end{array}$$

which indicates that by hypothesis the lines $P_{30}P_{40}$, $P_{31}P_{41}$, $P_{32}P_{42}$ are in pencil Q_{34}, and the triads of points $P_{30}P_{31}P_{32}$ and $P_{40}P_{41}P_{42}$ are the vertices of triangles on these lines, and the triads of points $P_{30}P_{31}P_{01}$, $P_{40}P_{41}P_{01}$, $P_{30}P_{32}P_{02}$, $P_{40}P_{42}P_{02}$, $P_{31}P_{32}P_{12}$, $P_{41}P_{42}P_{12}$, exist and are respectively collinear. Axiom X states, then, that the points $P_{12}P_{02}P_{01}$ are collinear.

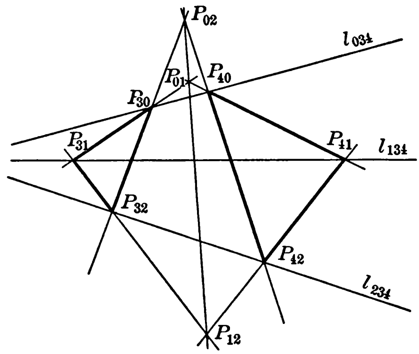

FIG. 2.

DEFINITION 7. Three distinct pencils Q_{01}, Q_{12}, Q_{02} are *in range*, if there exist two triangles, $P_{30}P_{31}P_{32}$ and $P_{40}P_{41}P_{42}$, whose sides are all distinct and such that $P_{30}P_{31}$ and $P_{40}P_{41}$ belong to Q_{01}, $P_{31}P_{32}$ and $P_{41}P_{42}$ belong to Q_{12}, and $P_{30}P_{32}$ and $P_{40}P_{42}$ belong to Q_{02}; and such that the lines $P_{30}P_{40}$, $P_{31}P_{41}$, and $P_{32}P_{42}$ are in a pencil Q_{34}.

The notation used for the figure of Definition 7 will be identical with that of Axiom X, except that for P_{12}, P_{02}, P_{01} corresponding symbols Q_{12}, Q_{02}, Q_{01} will be used, thus:

$$\begin{array}{ccc} & Q_{34} & \\ P_{30} & P_{31} & P_{32} \\ P_{40} & P_{41} & P_{42} \\ \hline Q_{12} & Q_{20} & Q_{01} \end{array}$$

It will be observed that whenever the hypothesis of Axiom X is fulfilled, so is the hypothesis in Definition 7, and to the points $P_{12}P_{02}P_{01}$ of Axiom X correspond pencils $Q_{12}Q_{02}Q_{01}$ which by Definition 7 are in range; for whether the pairs of lines $P_{30}P_{32}$ and $P_{40}P_{42}$, $P_{31}P_{32}$ and $P_{41}P_{42}$, and $P_{30}P_{31}$ and $P_{40}P_{41}$, determine points or not, they *do* determine *pencils* (cf. Corollary to Theorem VII).

AXIOM XI. If three pencils Q_{01}, Q_{02}, Q_{12} are in range, *every* pair of triangles $P_{30}P_{31}P_{32}$ and $P_{40}P_{41}P_{42}$, such that

lines $P_{30}P_{31}$ and $P_{40}P_{41}$ belong to Q_{01},
lines $P_{31}P_{32}$ and $P_{41}P_{42}$ belong to Q_{12},
lines $P_{30}P_{32}$ and $P_{40}P_{42}$ belong to Q_{02},

has the property that the lines $P_{30}P_{40}$, $P_{31}P_{41}$ and $P_{32}P_{42}$ are in a pencil Q_{34}.

The figure of Axiom XI will be denoted by the notation:

$$\begin{array}{ccc} Q_{12} & Q_{20} & Q_{01} \\ \hline P_{30} & P_{31} & P_{32} \\ P_{40} & P_{41} & P_{42} \\ & Q_{34} & \end{array}$$

The inversion of the notational form is to call attention to the change in the hypothesis. That Definitions 6 and 7 and Axioms X and XI all involve the Desargues configuration explains the similarity of the notation.

DEFINITION 8. The set of all pencils in range with a given pair of ranges, together with this pair, constitutes a *range*. The notation r_k will be used for ranges, the mark k being again any convenient digit or letter or combination of them in which the order is immaterial, so far as denoting the range is concerned.

From Axioms I–IX, such theorems as the following may be deduced (cf. VEBLEN, loc. cit.):

a) Between any two distinct points of a line there is another point (P_1 is between P_2 and P_3 if they are in the order $P_2P_1P_3$).

b) If a line, not a side of a triangle, meets the perimeter of the triangle in one point not a vertex, then it meets the perimeter in another distinct point.

c) To any four points of a line the notation P_1, P_2, P_3, P_4, can be so assigned that they are in the order, $P_1P_2P_3P_4$, i. e. in the orders, $P_1P_2P_3$, $P_1P_3P_4$, $P_1P_2P_4$, and $P_2P_3P_4$.

d) Any line divides the plane into two regions which may be denoted by $+$ and $-$, and have the property that any segment joining a point of $+$ to a point $-$ contains a point of the line.

e) Two non-intersecting lines divide a plane into three regions, which can be conveniently denoted by $++$, $+-$, $--$.

f) Two intersecting lines divide a plane into four regions, which may be denoted by $++$, $+-$, $-+$, $--$.

g) Three lines forming an actual triangle divide the plane into seven regions, which may be denoted by $+++$, $++-$, $+-+$, $+--$, $--+$, $-++$, $-+-$.

SECTION I.

Consequences of Axioms I–X.

Our main object in Section I is to show that a line is uniquely determined by a point and the fact that it is in pencil with two other given lines neither of which passes through the point. This is done without any use of Axiom XI.

THEOREM I. If the lines l_{034}, l_{134}, l_{234} are in pencil Q_{34}, and l_{034} and l_{134} meet in a point P_{30}, then l_{234} passes through P_{30}. (Fig. 3.)

Proof. Suppose the theorem untrue. Take P_{32}, any point of l_{234} (P_{32} not a point of l_{034} or l_{134}). Take on l_{134} points P_{71} and P_{61} each distinct from P_{30}, not on l_{234}, in the order $P_{71}P_{30}P_{61}$, and such there is no point of l_{234} on the segment $P_{61}P_{71}$. The line l_{034} cuts the side $P_{71}P_{61}$ of the triangle $P_{71}P_{61}P_{32}$. Hence it cuts either the segment $P_{32}P_{71}$ or the segment $P_{32}P_{61}$ in a point P_t. Call the side cut $P_{32}P_{31}$, so that either P_{71} or P_{61} is P_{31}, and call the other P_{81}. Choose a point P_{02} in the order $P_{32}P_{30}P_{02}$, and a point P_y in the order

$P_{31}P_{32}P_y$. The segment $P_{02}P_y$ contains a point of the segment $P_{32}P_{81}$ or does not. If it does, use P_y as P_{01} below. If not, take a point P_r on the segment $P_{32}P_{81}$, and the line $P_{02}P_r$ will meet the segment $P_{32}P_y$ in a point

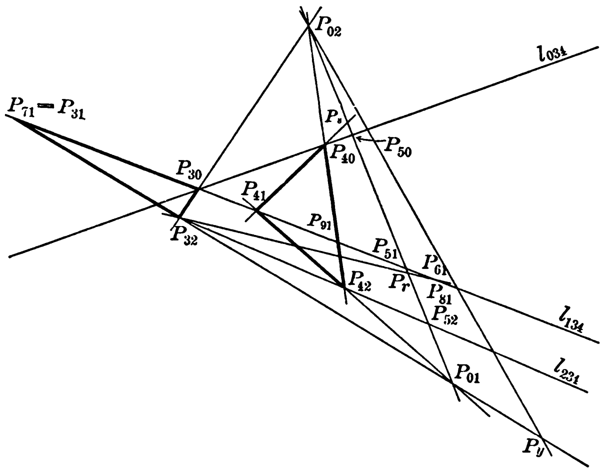

FIG. 3.

P_{01}, which is in the order $P_{02}P_rP_{01}$. Then the segment $P_{02}P_{01}$ will meet l_{034}, l_{134}, and l_{234} in points P_{50}, P_{51}, P_{52}, respectively, in the order $P_{02}P_{50}P_{51}P_rP_{52}P_{01}$. Take P_{41} in the order $P_{30}P_{41}P_{51}$. Then the segment $P_{41}P_{01}$ meets l_{234} in a point P_{42} of the segment $P_{32}P_{52}$. The segment $P_{42}P_{02}$ contains points, P_{40} on the line l_{034}, and P_{91} on the line l_{134}, in the order $P_{02}P_{40}P_{91}P_{42}$, and P_{91} is in the order $P_{30}P_{41}P_{91}P_{51}$. Therefore lines $P_{41}P_{40}$ and $P_{02}P_{01}$ determine a point P_s in the orders $P_{41}P_{40}P_s$ and $P_{02}P_sP_{50}$. We have then the Desargues configuration

$$\begin{array}{ccc} & Q_{34} & \\ P_{30} & P_{31} & P_{32} \\ P_{40} & P_{41} & P_{42} \\ \hline P_{01} & P_{02} & P_{41}, \end{array}$$

in which the hypothesis of Axiom X is satisfied, but P_{01}, P_{02}, P_{41} are not collinear. Hence the assumption that l_{234} does not pass through P_{02} is contradictory to Axiom X. Hence the theorem is true.

THEOREM II. Any three lines l_{034}, l_{134}, l_{234} incident at a point P_{34} are in a pencil Q_{34}. (Fig. 4.)

Proof. Choose $P_{30} \neq P_{34}$ on l_{034} and $P_{61} \neq P_{34}$ on l_{134}, and $P_{32} \neq P_{34}$ on l_{234}, in the order $P_{32}P_{61}P_{30}$. Then choose P_{02} in the order $P_{32}P_{61}P_{30}P_{02}$; P_{31} in order $P_{34}P_{61}P_{31}$; and P_{12} in the order $P_{31}P_{32}P_{12}$, and such that segment $P_{32}P_{12}$ contains no point of line $P_{34}P_{02}$. Then, by consideration of triangle $P_{02}P_{32}P_{12}$, $P_{02}P_{12}$ contains points P_{50}, P_{51}, in order $P_{02}P_{50}P_{51}P_{12}$, and P_{50} is on the segment $P_{34}P_{30}$, and P_{51} is on the segment $P_{34}P_{61}$. $P_{31}P_{30}$ meets $P_{02}P_{12}$ in a point P_{01}, in the order $P_{02}P_{01}P_{50}$. Take P_{41} on the segment $P_{34}P_{51}$. The segment $P_{41}P_{01}$ contains P_{40} on the segment $P_{34}P_{50}$. $P_{02}P_{40}$ meets $P_{41}P_{12}$ in a point P_{42}. Then $P_{32}P_{42}$ is a line in pencil with l_{034} and l_{134}. By Theorem I, the line $P_{32}P_{42}$ must pass through P_{34}, but this line contains the points P_{32} and P_{34}, and hence is the line l_{234}.

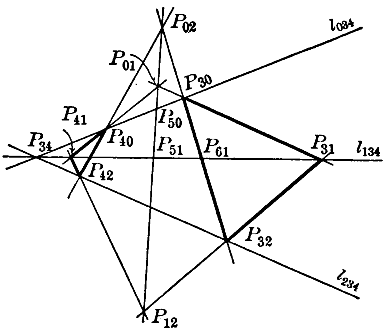

FIG. 4.

COROLLARY. If three lines are in pencil they meet in a point, or else no two of them have a point in common.

DEFINITION 8. A line l_k is *between* the non-intersecting lines l_m and l_n, if every segment P_mP_n (P_m on l_m, P_n on l_n) contains a point P_k of l_k.

DEFINITION 9. If three lines are in pencil and no two of them intersect, we will call the pencil *non-intersecting*.

THEOREM III. If l_{034}, l_{134}, l_{234} are three lines of a *non-intersecting* pencil, Q_{34}, then one of them is always *between* the other two.

LEMMA. If l_k, l_m, l_n are three non-intersecting lines, and if l_m cuts *one* segment joining points of l_k and l_n, then it cuts every such segment, and hence is between l_k and l_n. The proof follows readily from Axiom VIII and is omitted.

Proof of Theorem III. (Fig. 5.) Take P_{30} on l_{034}, P_{31} on l_{134}, and P_{32} on l_{234}. Then not more than one of the lines l_{034}, l_{134}, l_{234} enters the triangle $P_{30}P_{31}P_{32}$, since by hypothesis the pencil is non-intersecting. Then, if *one* does, it cuts the opposite side of the triangle and hence by the lemma it is between the other two lines. We will now prove that the case of all three of the lines l_{034}, l_{134}, l_{234} exterior to the triangle can not occur. Suppose that it did occur. Let the plane be divided by the sides of the triangle into seven regions as in Theorem g, p. 146.

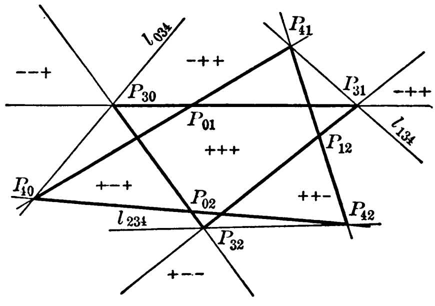

FIG. 5.

Choose P_{40} on l_{034} in $+-+$, and P_{41} on l_{134} in $-++$, and P_{42} on l_{234} in $++-$. Then $P_{40}P_{41}$ meets segment $P_{30}P_{31}$ in a point P_{01}, and $P_{41}P_{42}$ meets segment $P_{31}P_{32}$ in a point P_{12}, and $P_{42}P_{40}$ meets segment $P_{32}P_{30}$ in a point P_{02} and by hypothesis we have (2); but the points P_{12}, P_{02}, P_{01} are not collinear since they are on distinct sides of the triangle. Hence our supposition that no one of the lines enters the triangle is impossible.

DEFINITION 10. If three lines of a pencil are incident at a point, then their common point is the *center* of the pencil.

THEOREM IV. If l_{034}, l_{134}, and l_{234} are three lines in pencil, and such that, if the pencil be non-intersecting,* l_{134} is between l_{034} and l_{234}, and l_{934} ($l_{934} \neq l_{234}$) is a line meeting l_{234} in a point P_{32} (P_{32} not on l_{034} and not on l_{134}), then l_{034}, l_{134}, and l_{934} are not in pencil.

Proof. (A) If the pencil has a center, the theorem follows from Theorem I.

(B) Suppose the pencil has no center. (Fig. 6.)

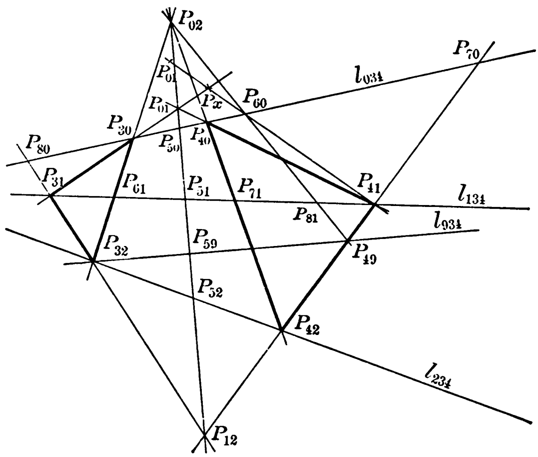

FIG. 6.

Suppose there were a line l_{934} satisfying the conditions of the hypothesis. Then l_{934} can not meet either l_{034} or l_{134}, by Theorem I, and also l_{134} will be between l_{034} and l_{934}. Let P_{80} and P_{70} be any two distinct points on l_{034}. Let P_{12} be any point in the order $P_{80}P_{32}P_{12}$. The segment $P_{80}P_{32}$ meets l_{134} in a point P_{31}. Since neither l_{234} nor l_{934} meets either l_{034} or l_{134}, the line $P_{70}P_{12}$ meets l_{134} in a point P_{41}, l_{234} in a point P_{42}, and l_{934} in a point P_{49}, and these points are in the order $P_{70}P_{41}P_{42}P_{12}$ and $P_{70}P_{41}P_{49}P_{12}$. Take P_{50} on the segment $P_{80}P_{70}$. Then take P_{02} in the order $P_{12}P_{50}P_{02}$ and the line $P_{12}P_{50}$ will meet l_{234} in a point P_{52}, l_{934} in a point P_{59}, and l_{134} in a point P_{51}, and these points are in the orders $P_{12}P_{52}P_{51}P_{50}P_{02}$ and $P_{12}P_{59}P_{51}P_{50}P_{02}$. $P_{02}P_{32}$ contains points P_{30} on l_{034} and P_{61} on l_{134}, in the order $P_{02}P_{30}P_{61}P_{32}$. $P_{02}P_{42}$ meets l_{034} in P_{40} and l_{134} in P_{71}, in the order $P_{02}P_{40}P_{71}P_{42}$. The line $P_{31}P_{30}$ meets $P_{52}P_{50}$ in a point P_{01}, in the orders $P_{31}P_{30}P_{01}$ and $P_{02}P_{01}P_{50}$. $P_{41}P_{40}$ also meets the line $P_{12}P_{02}$ in a point which by Axiom X is the same as P_{01}, from the Desargues configuration (2).

Now the line $P_{02}P_{49}$ meets l_{134} in P_{81}, and l_{034} in P_{60}, and these points are in the order $P_{02}P_{60}P_{81}P_{49}$. The line $P_{41}P_{60}$ meets the segment $P_{02}P_{50}$ in P'_{01}. But P'_{01} can not be P_{01}, since P_{40} and P_{60} are distinct. Also $P_{41}P_{60}$ and $P_{31}P_{30}$, since each enters the triangle $P_{02}P_{30}P_{60}$ at a vertex, meet in a point P_x. But this point is not collinear with P_{12} and P_{02}, since $P_{31}P_{30}$ and $P_{41}P_{40}$ meet $P_{12}P_{02}$ in distinct points, P_{01} and P'_{01}. Then we should have the Desargues configuration

* Note that the proof of the theorem as given would not be valid except for this restriction. The result is proved in general in Theorems VII and VIII.

$$\begin{array}{ccc} & Q_{34} & \\ P_{30} & P_{31} & P_{32} \\ P_{60} & P_{41} & P_{49} \\ \hline P_{12} & P_{02} & P_{x} \end{array}$$

In this configuration, if l_{034}, l_{134}, and l_{934} are in pencil Q_{34}, as we have supposed, then by Axiom X the points P_{12}, P_{02}, P_x must be collinear. But we have proved that they are not collinear, and hence the lines l_{034}, l_{134}, and l_{934} are not in pencil.

THEOREM V. If l_{045}, l_{145}, and l_{245} are three lines in a non-intersecting pencil, and l_{045}, l_{145}, and l_{345} ($l_{245} \neq l_{345}$) are three lines in a non-intersecting pencil, then two of these lines are always between the other two.

Proof. Since no two of the lines meet, it is only necessary to find a single segment joining points of two of them, and containing points of the remaining two, as the lemma to Theorem III shows.

Let the regions into which the plane is divided by the lines l_{045} and l_{145} be denoted by $++$; $-+$; $--$ (cf. Theorem g, p. 146), the first sign referring to division by l_{045} and the second to division by l_{145}, and the notation so used that a point in the $++$ region is on the opposite side of l_{045} from l_{145}, and a point in the $--$ region is on the opposite side of l_{145} from l_{045}. The remaining points not on the lines are in $-+$.

Since neither l_{245} nor l_{345} can meet either l_{045} or l_{145} each must lie wholly in some one of the regions $++$, $-+$, or $--$. We have then the following cases, with the conclusion as given:

Case	l_{245} in	l_{345} in	Conclusion
1	$++$	$++$	See below.
2	$++$	$-+$	l_{045} and l_{345} between other two lines.
3	$++$	$--$	l_{045} and l_{145} " " " "
4	$-+$	$++$	l_{045} and l_{245} " " " "
5	$-+$	$-+$	l_{245} and l_{345} " " " "
6	$-+$	$--$	l_{145} and l_{245} " " " "
7	$--$	$++$	l_{145} and l_{045} " " " "
8	$--$	$-+$	l_{145} and l_{345} " " " "
9	$--$	$--$	See below.

In cases 2 to 8 inclusive, the conclusion is obvious and cases 1 and 9 differ only in notation. The proof is made only for case 9. (Fig. 7.)

Take any points P_{62} and P_{73} on l_{245} and l_{345} respectively, and P_{67} any point in $++$. If P_{62}, P_{73} and P_{67} are collinear, the theorem follows at once. If not, the segment $P_{67}P_{62}$ contains points P_{60} on l_{045} and P_{61} on l_{145}. The segment

$P_{67}P_{73}$ contains points P_{70} on l_{045} and P_{71} on l_{145}. The lines $P_{60}P_{71}$ and $P_{70}P_{61}$ determine a point P_{01}. The points P_{67} and P_{01} determine a line containing points P_{80} on l_{045} and P_{81} on l_{145} in the order $P_{67}P_{80}P_{01}P_{81}$. The segment $P_{61}P_{73}$ meets the line $P_{67}P_{01}$ in a point P_{31}. Either $P_{71}P_{31}$ and $P_{67}P_{62}$ meet in a point P_{63} or they do not. If they do meet in a point, we have the triangles $P_{60}P_{71}P_{63}$ and $P_{70}P_{61}P_{73}$ on the lines l_{045}, l_{145}, and $P_{73}P_{63}$ whose corresponding sides meet

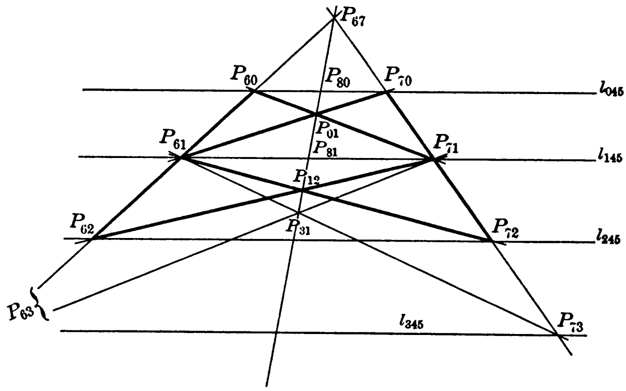

FIG. 7.

in three collinear points, P_{67}, P_{01}, P_{31}. Therefore l_{045}, l_{145}, and$P_{73}P_{63}$ are in pencil, and by Theorem IV, $P_{73}P_{63}$ is l_{345} and hence P_{63} is on l_{345}. Therefore the points P_{60}, P_{61}, P_{62}, and P_{63} are collinear, and are in one or the other of the orders $P_{60}P_{61}P_{62}P_{63}$ or $P_{60}P_{61}P_{63}P_{62}$. In either case we have the theorem. If $P_{71}P_{31}$ and $P_{67}P_{62}$ do not meet, the line $P_{71}P_{62}$ contains a point P_{12} of the line $P_{67}P_{01}$, which is in the order $P_{81}P_{12}P_{31}$; since if the points were in the order $P_{81}P_{34}P_{12}$, then $P_{71}P_{31}$ would cut the segment $P_{61}P_{62}$. Since P_{12} is on the segment $P_{81}P_{31}$, it is within the triangle $P_{71}P_{61}P_{73}$, and hence the line $P_{61}P_{12}$ must cut the segment $P_{71}P_{73}$ in a point P_{72}. The triangles $P_{62}P_{71}P_{60}$ and $P_{72}P_{61}P_{70}$ have their corresponding sides meeting in three collinear points, P_{67}, P_{01} and P_{12}. Therefore $P_{72}P_{62}$ is in pencil with l_{045} and l_{145}, and by Theorem IV, P_{72} is on l_{245}. Hence $P_{70}P_{73}$ contains P_{71} on l_{145} and P_{72} on l_{245}, and hence l_{145} and l_{245} are between l_{045} and l_{345}. This completes the proof.

COROLLARY. A triangle can always be so chosen that it will contain segments of each of the four lines l_{045}, l_{145}, l_{245} l_{345} in its interior, and such that within it no two of the lines meet, and within such a triangle the order in which the four lines are cut by a transversal is invariant.

THEOREM VI. If l_{045}, l_{145}, l_{245} are three lines in pencil, and l_{045}, l_{145}, l_{345} are three lines in pencil, then l_{045}, l_{245}, l_{345} are in pencil, and l_{145}, l_{245}, l_{345} are in pencil.

Proof. If l_{045} and l_{145} meet in a point P_k, then l_{245} and l_{345} pass through P_k, and we have at once the conclusion, by the corollary to Theorem II.

If* l_{045} and l_{145} do not meet (Fig. 8), we will consider the interior of a triangle T chosen as in the corollary to Theorem V.

All of the points chosen below are to be taken in the interior of T. Let the notation l_w, l_x, l_y, l_z be so assigned to the four lines l_{045}, l_{145}, l_{245}, l_{345} that if they are cut by any transversal in T in the points P_w, P_x, P_y, P_z these points

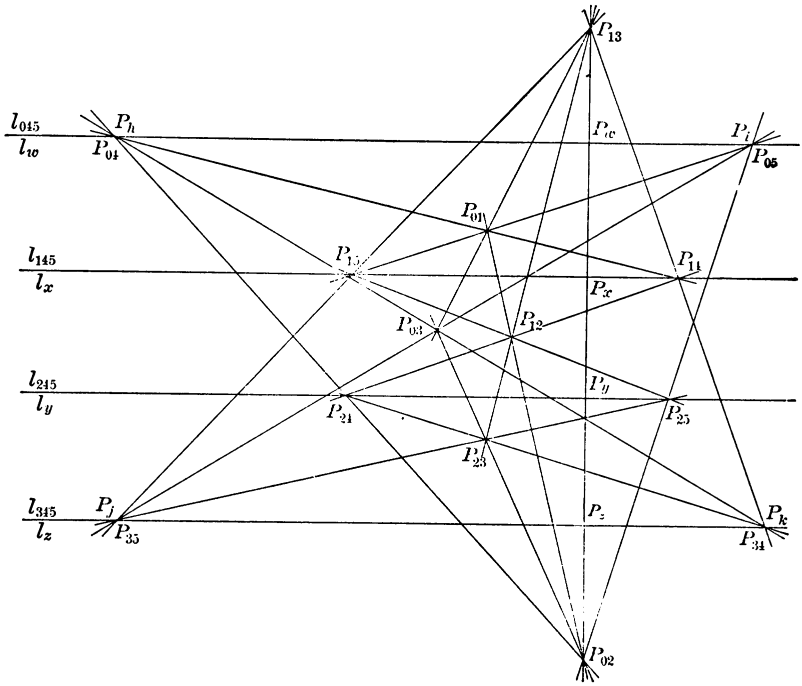

FIG. 8.

will be in the order $P_w P_x P_y P_z$. Choose P_{13} and P_{02} in the order $P_{13} P_w P_x P_y P_z P_{02}$. Then take two points P_j and P_k on l_z in the order $P_j P_z P_k$, and two points P_h and P_i on l_w, in the order $P_h P_w P_i$, and such that P_h and P_j are on the same side of the line $P_{13} P_{02}$. The line $P_{13} P_j$ meets l_{145} in a point P_{15} and l_{345} in a point P_{35}. The line $P_{13} P_k$ meets the line l_{145} in a point P_{14} and l_{345} in a point P_{34}. The line $P_{02} P_h$ meets l_{245} in a point P_{24} and l_{045} in a point P_{04}. The line $P_{02} P_i$ meets l_{245} in a point P_{25} and l_{045} in a point P_{05}. Then the points P_{14} and P_{04} are on opposite sides of the line $P_{15} P_{05}$, hence the lines $P_{14} P_{04}$ and $P_{15} P_{05}$ meet in a point P_{01}. Similarly, $P_{15} P_{25}$ and $P_{24} P_{14}$ meet in a point P_{12}; $P_{04} P_{34}$ and $P_{05} P_{35}$ meet in a point P_{03}; $P_{24} P_{34}$ and $P_{25} P_{35}$ meet in a point P_{23}.

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* The configuration obtained in the proof of this theorem is a plane section of the three-space configuration containing six points, no three of them collinear. The configuration contains fifteen points and twenty lines.

Then the hypothesis l_{045}, l_{145}, l_{245} in pencil, and the hypothesis l_{045}, l_{145}, l_{345} in pencil give respectively the Desargues configurations

$$(A), (B) \qquad \begin{array}{ccc} & Q_{45} & \\ P_{04} & P_{14} & P_{24} \\ P_{05} & P_{15} & P_{25} \\ \hline P_{12} & P_{02} & P_{01} \end{array} \qquad\qquad \begin{array}{ccc} & Q_{45} & \\ P_{04} & P_{14} & P_{34} \\ P_{05} & P_{15} & P_{35} \\ \hline P_{13} & P_{03} & P_{01} \end{array}$$

Then by using the conclusions of (A) and (B), and remembering that any three lines through a point are in pencil, we have

$$(C), (D) \qquad \begin{array}{ccc} & Q_{14} & \\ P_{31} & P_{21} & P_{01} \\ P_{34} & P_{24} & P_{04} \\ \hline P_{02} & P_{03} & P_{23} \end{array} \qquad\qquad \begin{array}{ccc} & Q_{04} & \\ P_{30} & P_{20} & P_{10} \\ P_{34} & P_{24} & P_{14} \\ \hline P_{12} & P_{13} & P_{23} \end{array}$$

In view of (C) and (D) respectively, we have

$$(E), (F) \qquad \begin{array}{ccc} P_{23} & P_{03} & P_{02} \\ \hline P_{04} & P_{24} & P_{34} \\ P_{05} & P_{25} & P_{35} \\ & Q_{45} & \end{array} \qquad\qquad \begin{array}{ccc} P_{23} & P_{13} & P_{12} \\ \hline P_{14} & P_{24} & P_{34} \\ P_{15} & P_{25} & P_{35} \\ & Q_{45} & \end{array}$$

The conclusion of the theorem is expressed by (E) and (F) in the configurational notation.

Corollary. The set of all lines through a point constitutes a pencil which is determined by any pair of the lines.

Theorem VII. If l_{034} and l_{134} are two lines and P_{32} is a point not on either l_{034} or l_{134}, then there is one and only one line l_{234} which passes through the point P_{32} and which is in pencil with l_{034} and l_{134}.

This is a generalization of Theorem IV, in which we have the proof for all cases except when l_{034} and l_{134} do not meet, and l_{234} is between l_{034} and l_{134}. We can now prove this last case and have the more general theorem by the aid of Theorem VI. We will first prove that there can not be two such lines. Suppose l_{234} and l_{534} are two distinct lines through P_{34} each in pencil with l_{034} and l_{134}. Then by Theorem VI, l_{034}, l_{234}, and l_{534} are in pencil, and by Theorem I, l_{034}, l_{134}, l_{534} must meet in P_{32}, but this contradicts the hypothesis that l_{034} and l_{134} do not meet, and hence there can not be two such lines.

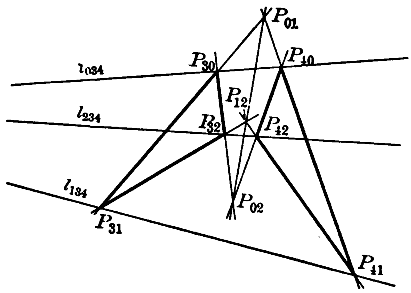

Fig. 9.

One line, and by this proof the only line l_{234}, through P_{32} and in pencil with l_{034} and l_{134} will now be exhibited. (Fig. 9.) Take any point P_{30} on l_{034}, and any point P_{02} not on l_{134}, in the order $P_{30}P_{32}P_{02}$, and such that the segment $P_{32}P_{02}$ contains no point of l_{134}. Then take P_{31} on the line l_{134}, but not on the line $P_{30}P_{32}$, and then any point P_{01} in the order $P_{31}P_{30}P_{01}$. Choose $P_{40} \neq P_{30}$ on l_{034} but not on the line $P_{01}P_{02}$, and any point $P_{41} \neq P_{31}$ on the line l_{134}, and such that these points are in the order $P_{01}P_{40}P_{41}$. Then $P_{31}P_{32}$ and $P_{01}P_{02}$ meet in a point P_{12}, and the segment $P_{40}P_{02}$ and the segment $P_{41}P_{12}$ meet in a point P_{42}. Hence we have the Desargues configuration:

$$\begin{array}{ccc} P_{12} & P_{02} & P_{01} \\ \hline P_{30} & P_{31} & P_{32} \\ P_{40} & P_{41} & P_{42} \\ & Q_{34} & \end{array}$$

whence the line $P_{32}P_{42}$ is the line l_{234} of the theorem.

COROLLARY. Any two lines determine a pencil.

In view of Theorem VII we can now speak of P_xQ_y, where P_x is not the center of Q_y as a definite line, viz., the line of the pencil Q_y which passes through P_x. We will also speak of Q_x as collinear with P_y and P_z if the line P_xP_y belongs to the pencil Q_z; also if the line P_xQ_y is a line of the pencil Q_z, $P_xQ_yQ_z$ will be said to be collinear. Collinearity, when used in this sense, of course, does *not imply order relations*. A line can belong to any number of pencils, viz., the pencils determined by it and each of the lines through some point not on it. But two pencils can not have more than one *line* in common, by Theorem VII. Since a point Q_x always determines uniquely a pencil, viz., the pencil of which it is the center, we may use any point as a pencil, when we mean this pencil. Points will be so used, in speaking of ranges, as in the definition of "in range," points might have been used for the pencils, as indicating the pencils of which they are centers.

SECTION II.

Consequences of Axioms I–XI.

The theorems of Section 2 will have to do with ranges and their properties. It is evident from the definitions of pencils and "in range," that any three points of a line are the centers of three pencils which are in range. The two triangles whose sides are lines of three pencils in range, and the joins of whose vertices are in pencil, will be said to be *perspective triangles*.

Suppose Q_1, Q_2, Q_3 are any three distinct pencils, and T is any triangle not containing in its interior or on any of its sides any segment common to any two of these pencils. Let $P_aP_b''P_c$ be any three points in T such that P_aP_b'' is a line of Q_1 and $P_a P_c$ is a line of Q_2. (Fig. 10.) Then the lines $P_c Q_1$ and $P_b'' Q_2$ meet in a point P_d of T, or do not. If not, let any point of $P_c Q_1$ on the same side of $P_a P_c$ that P_b is, be called P_d; then $P_d Q_2$ will meet $P_a P_b''$ in a point P_b which may be used for P_b''.

Such a figure $(P_a P_b P_c P_d)$ will be called a *quadrilateral* in T on Q_1 and Q_2.

Some one at least of the lines $P_a Q_3$, $P_b Q_3$, $P_c Q_3$, $P_d Q_3$ enters the quadrilateral $P_a P_b P_c P_d$, and by a proper assignment of notation it may be considered $P_a Q_3$. Then $P_a Q_3$ will meet either the segment $P_c P_d$ or the segment $P_b P_d$ in a point P_d'' or P_d' respectively. In the first case, the line $P_d'' Q_2$ meets the segment $P_a P_b$ in a point P_b', and P_d'' and P_b' may be used as a new P_d and P_b, respectively. In the second case, the line $P_d' Q_1$ meets the segment $P_a P_c$ in a point P_c', and P_d' and P_c' may be used as P_d and P_c respectively, and hence the points $P_a P_b P_c P_d$ can be so chosen that they form a quadrilateral on Q_1 and Q_2, and so that $P_a P_d$ is a line of Q_3. Such a figure will be called a quadrilateral on Q_1, Q_2 and Q_3, in T. Any line l_1 of Q_1 and any line l_2 of Q_2, which are within the quadrilateral, meet each other in a point of the quadrilateral.

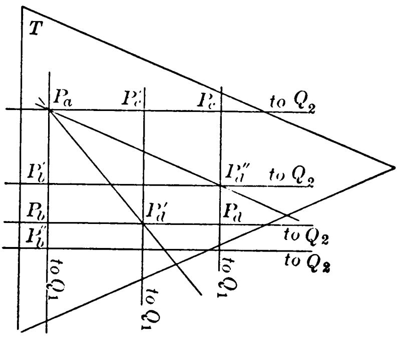

FIG. 10.

The notation for a Desargues configuration will sometimes be written, when the notation is in the form of Definition 7, D. C. (34)–(012), and when in the form of Axiom XI, D. C. (012)–(34), for shortness in writing as this may readily be expanded to the previous form.

THEOREM VIII. If Q_{01} and Q_{02} are any two distinct pencils, and l is any line not belonging to either Q_{01} or Q_{02}, then l is in one and only one pencil Q_{12} which is in range with Q_{01} and Q_{02}. (Fig. 11.)

Proof. Let T be any triangle containing in its interior a segment of l, but no segment of a line common to Q_{01} and Q_{02}. Choose P_{51} and P_{52} on l, in T, and such that $P_{51} Q_{01}$ and $P_{52} Q_{02}$ meet in a point P_{50} of T, and such that $P_{50} Q_{01}$ and $P_{51} Q_{02}$ meet in a

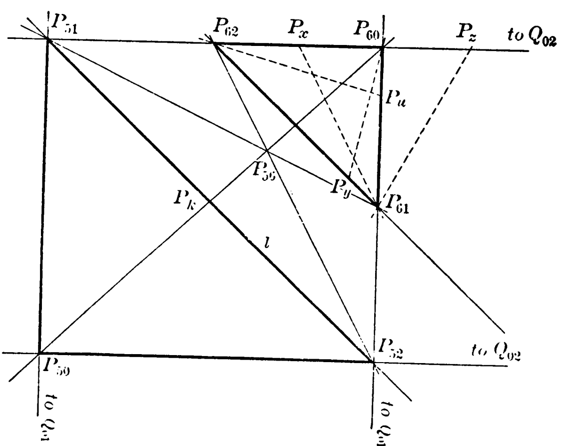

FIG. 11.

point P_{60} of T. Then the segments $P_{51}P_{52}$ and $P_{60}P_{50}$ meet in a point P_k; choose P_{56} in the order $P_kP_{56}P_{60}$. Then $P_{52}P_{56}$ meets the segment $P_{51}P_{60}$ in a point P_{62}, and $P_{51}P_{56}$ meets the segment $P_{52}P_{60}$ in a point P_{61}. Then from the triangles $P_{50}P_{51}P_{52}$ and $P_{60}P_{61}P_{62}$ perspective in P_{56}, $P_{51}P_{52}$ and $P_{61}P_{62}$ determine a pencil Q_{12} which is in range with Q_{01} and Q_{02}.

Suppose there is a pencil Q'_{12}, distinct from Q_{12}, such that l is a line of Q'_{12}, and such that Q_{01}, Q_{02}, and Q'_{12} are in range. Then at least one of the lines $P_{60}Q'_{12}$, $P_{61}Q'_{12}$, $P_{62}Q'_{12}$ must enter the triangle $P_{60}P_{61}P_{62}$, unless $P_{60}Q'_{12}$ is identical with $P_{60}Q_{01}$ or $P_{60}Q_{02}$. But either of these was avoidable by a new choice of P_{52} and P_{51}, since P_{52} and P_{51} might have been any points on the segment $P_{51}P_{52}$.

Case I. Suppose $P_{60}Q'_{12}$ cuts the segment $P_{61}P_{62}$ in P_y. Then choose a point P_z in the order $P_{51}P_{60}P_z$ such that $P_zQ'_{12}$ passes through P_{61}, or a point P'_{61} on the segment $P_{61}P_{60}$. Then since P_{56} was chosen arbitrarily on the segment P_kP_{60}, it may be chosen so that for some point P_z in order $P_{51}P_{60}P_z$, $P_zQ'_{12}$ passes through P_{61}. Then since $Q_{01}Q_{02}Q'_{12}$ are in range, the triangles $P_{51}P_{52}P_{50}$ and $P_{61}P_zP_{60}$ are in perspective. But $P_{61}P_{51}$ and $P_{60}P_{50}$ meet in P_{56}, and $P_{52}P_z$ does not pass through P_{56}. Hence this case is impossible.

Case II. $P_{61}Q'_{12}$ meets the segment $P_{60}P_{62}$ in a point P_x. Then since Q_{01}, Q_{02}, and Q'_{12} are in range, by hypothesis, $P_{60}P_{61}P_x$ and $P_{50}P_{51}P_{52}$ are in perspective. But $P_{60}P_{50}$ and $P_{61}P_{51}$ meet in P_{56} and P_xP_{52} does not pass through P_{56}. Hence this case is impossible.

Case III. $P_{62}Q'_{12}$ meets the segment $P_{61}P_{62}$ in P_u. The proof in this case is identical with that of Case II, except for notation. Hence there can be no such pencil Q'_{12}, and the theorem is proved.

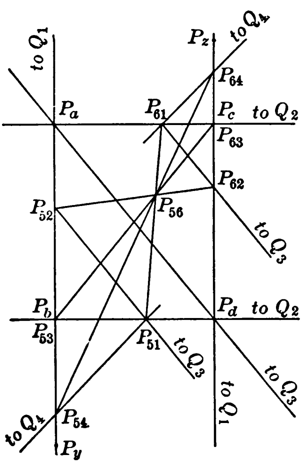

FIG. 12.

THEOREM IX. If Q_1, Q_2, Q_3 and Q_4 are four distinct pencils, and Q_1, Q_2, Q_3 are in range, and Q_1, Q_2, Q_4 are in range, then also Q_1, Q_3, Q_4 are in range. (Fig. 12.)

Proof. Let $P_aP_bP_cP_d$ be a quadrilateral on Q_1, Q_2, Q_3 in a region T not containing any point of a line common to any pair of the pencils Q_1, Q_2, Q_3, Q_4. Let P_y and P_z be any points in T in the orders $P_aP_bP_y$ and $P_dP_cP_z$ respectively. Let P_{51} be any point on the segment P_bP_d such that $P_{51}Q_4$ meets the segment P_aP_y in a point P_{54}. Let P_{61} be any point on the segment P_aP_c, such that $P_{61}Q_4$ meets the segment P_dP_z in a point P_{64}. $P_{51}Q_3$ meets the segment P_aP_b in a point P_{52}, and $P_{61}Q_3$ meets the segment P_cP_d in a point P_{62}. Since P_{61}, P_{51} are on one pair of opposite sides of the quadrilateral, and

P_{62}, P_{52} are on the other pair of opposite sides, $P_{61}P_{51}$ and $P_{62}P_{52}$ meet in a point P_{56}, of the interior of the quadrilateral. If P_b and P_c be called P_{53} and P_{63} respectively, we have the Desargues configuration (123)–(56), whence $P_{53}P_{63}$ also passes through the point P_{56}. But from the hypothesis that Q_1, Q_2 and Q_4 are in range, and the triangles $P_{51}P_{52}P_{54}$ and $P_{61}P_{62}P_{64}$, we have the Desargues configuration (124)–(56), whence $P_{54}P_{64}$ also passes through P_{56}. Hence, from triangles $P_{51}P_{53}P_{54}$ and $P_{61}P_{63}P_{64}$, perspective in P_{56}, we have the Desargues configuration (56)–(134), whence Q_1, Q_3, and Q_4 are in range, and the theorem is proved.

COROLLARY. Any two pencils of a range determine the range.

THEOREM X. If three points P_{01}, P_{02}, P_{12} are in range, they are collinear.

Proof. (See Fig. 13.) Let P_{30} be a point not on any of the lines $P_{01}P_{02}$, $P_{01}P_{12}$, $P_{02}P_{12}$. Take P_{31} on the line $P_{30}P_{01}$ such that the line $P_{31}P_{12}$ will

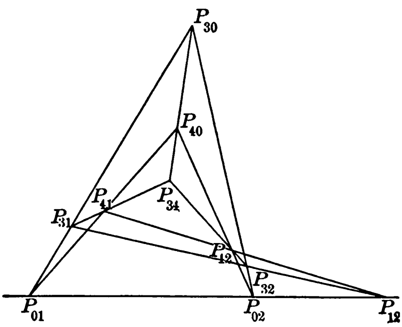

FIG. 13.

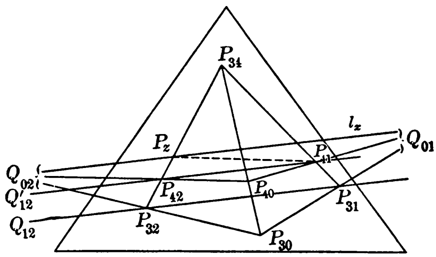

FIG. 14.

meet the segment $P_{30}P_{02}$ in a point P_{32}, and such that no point of any of the lines $P_{01}P_{02}$, $P_{01}P_{12}$, $P_{02}P_{12}$ is within the triangle $P_{30}P_{31}P_{32}$. Let P_{34} be any point within the triangle $P_{30}P_{31}P_{32}$. Take P_{40} in the order $P_{30}P_{40}P_{34}$. Then the line $P_{40}P_{01}$ will meet the segment $P_{31}P_{34}$ in a point P_{41}, and the line $P_{40}P_{02}$ will meet the segment $P_{34}P_{32}$ in a point P_{42}. But the lines $P_{42}P_{41}$ and $P_{32}P_{31}$ determine a pencil Q'_{12}, which, by definition, is in range with P_{01} and P_{02}. Hence by Theorem VIII Q_{12} and Q'_{12} are identical, and we have the theorem.

THEOREM XI. If Q_{01}, Q_{02}, Q_{12} are in range and l_x is a line belonging to Q_{01} and Q_{02}, then l_x belongs also to Q_{12}. (Fig. 14.)

Proof. Let T be a triangle containing in its interior a segment of the line l_x, but no center of any one of the three pencils. Let P_{30} be any point in T not on l_x, and P_{31} a point in T, on line $P_{30}Q_{01}$, and P_{32} a point in T on $Q_{02}P_{30}$, such that $P_{31}P_{32}$ is a line of Q_{12}, and P_{30}, P_{31}, P_{32} are all on the same side of l_x. Take P_{34} on opposite side of l_x from P_{30}, not on $P_{32}Q_{12}$. Suppose now l_x is *not* a line of Q_{12}; let P_z be the point in which $P_{32}P_{34}$ meets l_x. Then P_zQ_{12} will meet the segment $P_{34}P_{31}$ in a point P_{41}, not on l_x. $P_{41}Q_{01}$ meets $P_{30}P_{34}$ in a point P_{40}, not on l_x. $P_{40}Q_{02}$ meets $P_{32}P_{34}$ in a point P_{42}, not on l_x. Hence

$P_{31}P_{32}$ and $P_{41}P_{42}$ determine a pencil Q'_{12} which is in range with Q_{01} and Q_{02}. By Theorem VIII Q'_{12} is Q_{12}. But $P_{41}P_x$ was supposed to be a line of Q_{12}, and that would make P_{41} the center of Q_{12}. But Q_{12} does not have its center in T, and P_{41} is in T, hence l_x must be a line of Q_{12}.

THEOREM XII. If l_x is a line of a pencil Q_{12}, and P_{02} and P_{01} are any two distinct points of l_x not the center of Q_{12}, then Q_{12}, P_{02} and P_{01} are in range.

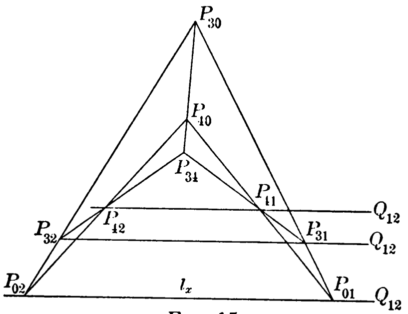

FIG. 15.

Proof. (See Fig. 15.) Let P_{30} be any point not on l_x, and P_{31} and P_{32} two points on the same side of l_x that P_{30} is, and such that $P_{30}P_{31}$ is a line of Q_{01}, $P_{30}P_{32}$ is a line of Q_{02}, and $P_{31}P_{32}$ is a line of Q_{12}. Let P_{34} be any point within the triangle $P_{30}P_{31}P_{32}$. Let P_{40} be any point in the order $P_{30}P_{40}P_{34}$. Then $P_{40}Q_{1}$ meets the segment $P_{31}P_{34}$ in a point P_{41}, and $P_{40}Q_{02}$ meets the segment $P_{34}P_{32}$ in a point P_{42}. Then the line $P_{42}P_{41}$ belongs to the pencil Q_{12}, from the Desargues configuration, (043)–(12), and hence the theorem is true.

THEOREM XIII. If l_x belongs to Q_{01} and Q_{02}, then any other pencil Q_{12} to which l_x belongs is in range with Q_{01} and Q_{02}. Let P_x and P_y be two points of l_x not centers of Q_{01} or Q_{02} or Q_{12}. Then $P_xP_yQ_{01}$, $P_xP_yQ_{02}$, $P_xP_yQ_{12}$ are in range, by Theorem XII. Then by Theorem IX, $P_xQ_{01}Q_{02}$ and $P_xQ_{01}Q_{12}$ and therefore Q_{01}, Q_{02}, Q_{12} are in range.

THEOREM XIV. Any two distinct ranges r_{023} and r_{123} have in common one and only one pencil, Q_{23}. (Fig. 16.)

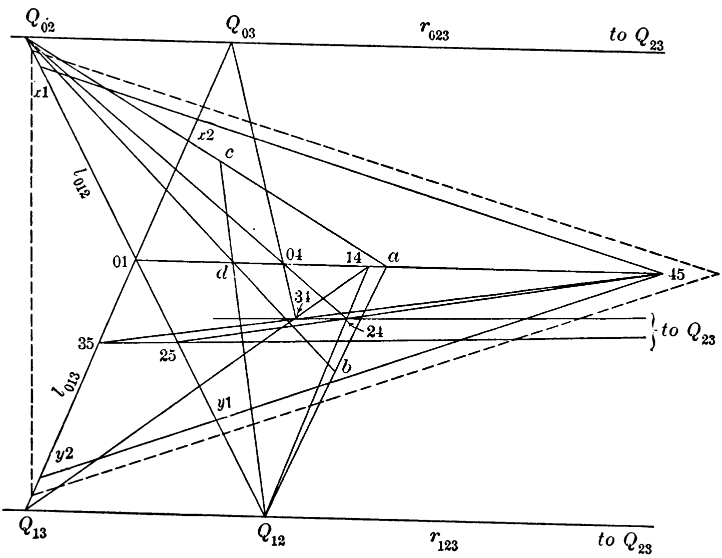

FIG. 16.

Proof. Through any point P_{01}, not a point of either r_{023} or r_{123}, draw lines l_{013} and l_{012}, which with r_{023} and r_{123} determine respectively the pencils Q_{03} and Q_{13}, Q_{02} and Q_{12}. Choose P_{45}, P_{x2} and P_{y1} such that P_{x2} is on the line l_{013} and P_{y1} is on the line l_{012}, and such that $P_{45}P_{x2}$ meets l_{012} in a point P_{x1} and $P_{45}P_{y1}$ meets l_{013} in a point P_{y2} in the orders $P_{x1}P_{01}P_{y1}$ and $P_{x2}P_{01}P_{y2}$, and such that all of these points lie within a triangle containing no points of r_{123} or r_{023}. Take a quadrilateral $P_aP_bP_cP_d$ on Q_{12} or Q_{02}, and P_{01}, and such that P_a and P_d are on the segment $P_{01}P_{45}$, and such that P_cP_{45} and P_bP_{45} are lines within the triangles $P_{45}P_{x1}P_{01}$ and $P_{45}P_{y2}P_{01}$ respectively. Take P_{04} and P_{14} on the segment P_aP_d, and such that $P_{04}P_{03}$ and $P_{14}Q_{13}$ meet at a point P_{34} within the quadrilateral $P_aP_bP_cP_d$; $P_{04}Q_{02}$ and $P_{14}Q_{12}$ meet at a point P_{24} within the quadrilateral. Then $P_{45}P_{24}$ meets l_{012} in a point P_{25} and $P_{45}P_{34}$ meets the line l_{013} in a point P_{35}. Then $P_{25}P_{35}$ and $P_{24}P_{34}$ determine a pencil Q_{23} which is in range with $Q_{02}Q_{03}$ from D. C. (45)–(023), and in range with $Q_{12}Q_{13}$ from D. C. (45)–(123).*

That there is only one such pencil is an immediate consequence of Theorem IX. For if there were two such pencils, then these would determine any range containing them, and hence the ranges r_{023} and r_{123} would not be distinct.

THEOREM XV. If any ten pencils Q_{ij} $(i, j=1, \cdots, 5;\ i \neq j)$ are such that no four of them are in range, and $Q_{ij}\,Q_{ik}\,Q_{jk}$ $(i,\ j,\ k=1,\ 2,\ 3,\ 4,\ 5;$ $i \neq j \neq k$, and $i, j, k \neq 1, 2, 3$ in any order) are in range r_{ijk}, then also Q_{12}, Q_{13} and Q_{23} are in range r_{123}, and conversely.

The theorem is the general Desargues theorem, pencils and ranges replacing points and lines, and forming the configuration $\left|\begin{smallmatrix}10 & 3\\ 3 & 10\end{smallmatrix}\right|$. This configuration, for the purpose of the proof, is enlarged to a configuration $\left|\begin{smallmatrix}21 & 3\\ 5 & 35\end{smallmatrix}\right|$, P_{ij} or Q_{ij}, l_{ijk} or r_{ijk} $(i,\ j,\ k=1, \cdots, 7)$, the new lines or points being within or crossing a region T, say the inside of a triangle, containing no point of the original ranges. The eleven points added form two perspective five-points $P_{1,2,3,4,5-6}$ and $P_{1,2,3,4,5-7}$, perspective from a point P_{67}.

Choose P_{36}, any point in T (Fig. 17), and draw $P_{36}\,Q_{3j}$ $(j=1,\ 2,\ 4)$; in T, choose points P_{16} and P_{26}, on $P_{36}Q_{13}$ and $P_{36}Q_{23}$ respectively, and such that $P_{16}Q_{14}$ and $P_{26}Q_{24}$ meet in a point P_{46} of T, and $P_{26}Q_{25}$ and $P_{16}Q_{15}$ meet in a point P_{56} of T. Take P_{67} in T, but neither collinear with any two points, nor on any line yet chosen, and draw $P_{67}P_{6j}$ $(j=1 \cdots 5)$. Take P_{17} on the segment $P_{67}P_{16}$, and draw $P_{17}Q_{12}$. This will cut the segment $P_{67}P_{26}$ in a point P_{27}. $P_{17}P_{14}$ meets segment $P_{67}P_{46}$ in a point P_{47}, which is also on the line $P_{27}\,Q_{24}$, from the D. C. (67)–(124). [That is, $P_{27}P_{47}$ and $P_{26}P_{46}$ determine a pencil which is in range with Q_{14} and Q_{12}, and since Q_{24} is a pencil of the line $P_{26}P_{46}$, it is this pencil.] Also $P_{17}Q_{15}$ meets the segment $P_{56}P_{67}$ in a point P_{57}, and from

* The configuration used in the proof of this theorem is a degenerate case of the configuration $\left|\begin{smallmatrix}15 & 4\\ 3 & 20\end{smallmatrix}\right|$, of notation P_{ij}, l_{ijk} $(i,\ j,\ k=0, \cdots, 5)$ in which the points P_{01}, P_{05} and P_{15} are coincident.

the D. C. (67)–(125), $P_{26}P_{56}$ and $P_{27}P_{57}$ determine a pencil which is in range with Q_{15} and Q_{12}, and is therefore Q_{25}. From the D. C. (67)–(ijk), ($i=1, 2$; $j, k = 3, 4, 5, j \neq k$) $P_{j6}P_{k6}$ and $P_{j7}P_{k7}$ belong to Q_{jk}. Hence Q_{34}, Q_{35},

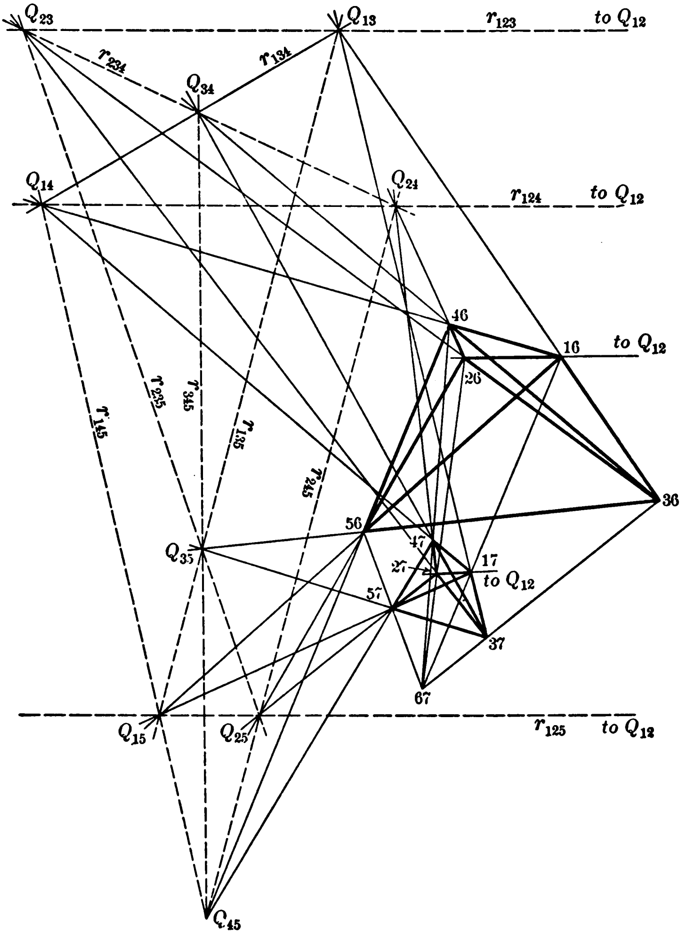

FIG. 17.

Q_{45} are in range, from the D. C. (67)–(345), and we have established the first part of the theorem.

The same configuration serves for the proof of the converse theorem, if the points are chosen in a slightly different order. Choose in T a point P_{36}, then P_{46} and P_{56} such that $P_{46}P_{56}$ is a line of Q_{56}, $P_{56}Q_{25}$ and $P_{46}Q_{24}$ meet in a point P_{26} of T, and $P_{56}Q_{15}$ and $P_{46}Q_{14}$ meet in a point P_{16} of T. Choose P_{67} in T, not on any line or range of the figure so far determined. On the segment $P_{67}P_{36}$ choose P_{37}. Then $P_{37}Q_{35}$ meets the segment $P_{56}P_{67}$ in a point P_{57}. The line $P_{57}Q_{45}$ meets the segment $P_{46}P_{67}$ in a point P_{47}. Then from the D. C. (67)–(345), $P_{37}P_{47}$ is a line of Q_{34}. Then $Q_{14}P_{47}$ meets the segment $P_{16}P_{67}$ in a point P_{17}. From D. C. (67)–(145) and Theorem VIII, $P_{17}P_{57}Q_{15}$ are collinear. We have now hypotheses for the configuration (67)–(134) and (67)–(135), from which, and Theorem VIII, we have the collinearities $P_{16}P_{36}Q_{13}$, $P_{17}P_{37}Q_{13}$. The line $P_{47}Q_{24}$ meets the segment $P_{26}P_{67}$ in a point P_{27}, and from the D. C. (67)–(245) we have $P_{27}P_{57}Q_{25}$ collinear. From the D. C.'s (67)–(234) and (67)–(235) we have the collinearities $P_{26}P_{36}Q_{23}$ and $P_{27}P_{37}Q_{23}$.

But from the D. C.'s (67)–(123), (67)–(124), (67)–(125), the lines $P_{16}P_{26}$ and $P_{17}P_{27}$ determine a pencil Q_{12} which is in range with $Q_{13}Q_{23}$, $Q_{14}Q_{24}$, $Q_{15}Q_{25}$ respectively, and this proves the theorem.

Section III.

Generalization of Order.

In this section it is shown that if some one range of pencils is omitted from the set of all pencils, the remaining subset of pencils has the property, that, with a suitable definition of order, its elements satisfy the Axioms I–XI, a well-known property of the elements of a projective plane.

Before giving the definition of order it will be useful to prove certain lemmas.

Lemma I. If Q_h, Q_i, Q_j, Q_k are any four distinct pencils of a range r_{hijk}, and $T_x \equiv P_{xhi}P_{xhj}P_{xij}$ is a triangle whose sides belong to Q_h, Q_i, Q_j respectively (i. e., the line $P_{xhi}P_{xhj}$ belongs to the pencil Q_h, etc.), all of whose vertices are on the same side of r_{hijk}, then of the three statements

1) $P_{xhi}Q_k$ cuts the segment $P_{xjh}P_{xij}$ in a point P_{xjk};
2) $P_{xhj}Q_k$ cuts the segment $P_{xhi}P_{xij}$ in a point P_{xik};
3) $P_{xij}Q_k$ cuts the segment $P_{xhi}P_{xhj}$ in a point P_{xhk};

one and only one is true.

Proof. Since the vertices of the triangle are all on one side of the range r_{hijk}, no side of the triangle is r_{hijk}. Then the lines $P_{xhi}Q_k$, $P_{xhj}Q_k$, $P_{xij}Q_k$ are all distinct lines of a pencil, whose center, if it exists, is not within the triangle. If the pencil has a center, the lemma follows at once from theorem g of the introduction. If it has no center, it follows from Theorem III. It is evident that by a proper assignment of the notation Q_h, Q_i, Q_j the true statement may be made any desired one of the three.

Lemma II. If T_3 and T_4 are two triangles satisfying the hypothesis of Lemma I, and such that in the triangle T_3, the line $P_{3hj}Q_k$ meets the segment $P_{3hi}P_{3ij}$ in a point P_{3ik}, then also the line $P_{4hj}Q_k$ meets the segment $P_{4hi}P_{4ij}$ in a point P_{4ik}.

Proof. Case I. (Fig. 18.) When no pair of corresponding sides of the triangles T_3 and T_4 lie in the same line, and no vertices are in common. The lines

$P_{3hi}P_{4hi}$, $P_{3hj}P_{4hj}$, $P_{3ij}P_{4ij}$ are in pencil Q_{34}. The line $P_{3ik}Q_{34}$ meets the segment $P_{4hi}P_{4ij}$ in a point P_{4ik}, by theorems on division of a plane by lines and Theorem I, if Q_{34} has a center, and by Theorem IV, if Q_{34} has no center. But

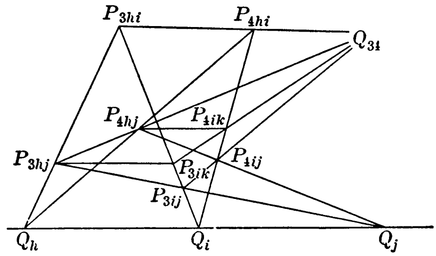

FIG. 18.

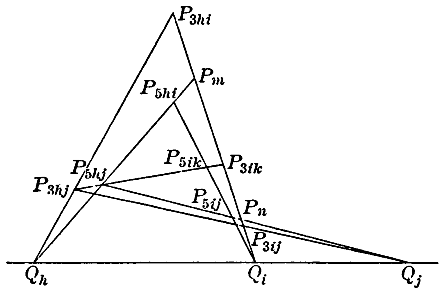

FIG. 19.

$P_{4hj}P_{4ik}$ and $P_{3hj}P_{3ik}$ determine a pencil in range with Q_h and Q_i, and hence Q_k.

Case II. (Fig. 19.) In case some parts of the triangles T_3 and T_4 are coincident, we can make the proof as follows. It is always possible to obtain a triangle T_5, satisfying hypothesis of Lemma II, all of whose parts are distinct from all the parts of T_3 and T_4, and such that $P_{5hj}Q_k$ meets the segment $P_{5hi}P_{5ij}$ in a point P_{5ik}, whence the theorem follows by Case I. Such a triangle T_5 is obtained as follows. Within the triangle T_3 and on the segment $P_{3hj}P_{3ik}$ but on no side of T_4, choose a point P_{5hj}. The line $P_{5hj}Q_h$ will meet the segment $P_{3hi}P_{3ik}$ in a point P_m, and the line $P_{5hj}Q_j$ will meet the segment $P_{3ik}P_{3ij}$ in a point P_n. Choose a point P_{5hi} on the segment $P_{5hj}P_m$ and not on any line of T_4. Then $P_{5hi}Q_i$ will meet the lines $P_{5hj}P_n$ and $P_{5hj}P_{3ik}$ in points P_{5ij} and P_{5ik} respectively, and the triangle $P_{5hj}P_{5hi}P_{5ij}$ is the triangle required.

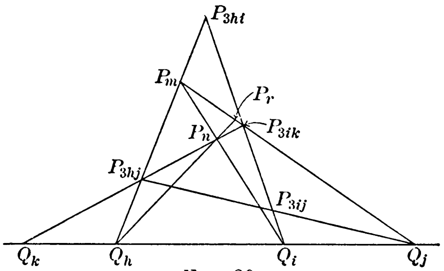

FIG. 20.

LEMMA III. (Fig. 20.) If Q_h, Q_i, Q_j, Q_k are four distinct pencils of a range, r_{hijk}, and T_3 is a triangle satisfying the hypotheses of Lemma II, then in any triangle

$$T_x \equiv \begin{Bmatrix} P_{xhi}P_{xhk}P_{xik} \\ P_{xhj}P_{xhk}P_{xjk} \\ P_{xij}P_{xik}P_{xjk} \end{Bmatrix}$$

all of whose vertices are on the same side of r_{hijk}, and whose sides belong to the pencils

$$\begin{Bmatrix} Q_hQ_iQ_k \\ Q_hQ_jQ_k \\ Q_iQ_jQ_k \end{Bmatrix},$$

respectively, the line and segment

$$\left\{\begin{matrix} P_{xik}Q_j \\ P_{xhj}Q_i \\ P_{xik}Q_h \end{matrix}\right\}, \qquad \left\{\begin{matrix} P_{xhi}P_{xhk} \\ P_{xhk}P_{xjk} \\ P_{xij}P_{xjk} \end{matrix}\right\},$$

meet in a point

$$\left\{\begin{matrix} P_{xhj} \\ P_{xki} \\ P_{xjh} \end{matrix}\right\}.$$

Proof. The line $Q_j P_{3ik}$ can not meet the segment $P_{3hj}P_{3ij}$, hence meets the segment $P_{3hj}P_{3hi}$ in a point P_m, and hence from the triangle $P_{3hi}P_{3hj}P_{3ik}$ and Lemma II we have the first part of the lemma. Similarly, $P_m Q_i$ must meet the segment $P_{3hj}P_{3ik}$ in a point P_n, and hence from the triangle $P_m P_{3hj} P_{3ik}$ and Lemma II we have the second part of the lemma. Also $P_n Q_h$ must meet the segment $P_m P_{3ik}$ in a point P_r, and from the triangle $P_m P_n P_{3ik}$ and Lemma II we have the third part of the lemma.

The range of pencils omitted from the set Q_k will be called r_∞, and pencils not on r_∞ will be denoted by the notation S_k.

Definition of Order. (Fig. 21.) Three pencils have order relations if and only if they are distinct, lie in a range, and no one of them lies on r_∞. Three such pencils S_i, S_j, S, of a range r_{ijk} meeting r_∞ in Q_∞, are in the order $S_i S_j S_k$ if and only if in a triangle $P_{xij}P_{xik}P_{xjk}$, whose sides belong to the pencils S_i, S_j, S_k respectively, and all of whose vertices are on the same side of r_∞ and r_{ijk}, the line $P_{xik}Q_\infty$ meets the segment $P_{xij}P_{xjk}$ in a point $P_{j\infty}$.

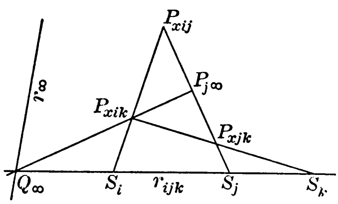

Fig. 21.

It will now be shown that the set of pencils S, subject to the above definition of order, satisfies the Axioms I–XI.

Axiom I is evident, since every point is the center of a pencil.

Axiom II is evident from the symmetry of the definition of order.

Axiom III follows from Lemma I.

Axiom IV is evident from definition.

Axiom V is evident from definition and Lemma III.

Axiom VI is a consequence of Theorem IX.

Axiom VII is obvious.

Axioms IX, X, XI are evident, since all points are centers of pencils, and they hold for the set of *all* pencils.

Axiom VIII. (Fig. 22.) This will be paraphrased as follows. If in the set S_k, S_{145}, S_{245}, and S_{345} are any three pencils not in range, and S_{235} is a pencil in the order $S_{245}S_{345}S_{235}$, and S_{135} is a pencil in the order $S_{145}S_{135}S_{345}$, then there exists a pencil S_{125} which is in the order $S_{145}S_{125}S_{245}$, and which is in range with S_{135} and S_{235}.

Proof. The ranges $S_{135}S_{235}$ and $S_{145}S_{245}$ are distinct, and hence they determine a pencil Q_{125}. Select a triangle T containing in its interior no point of any of the ranges $S_{245}S_{125}$, $S_{245}S_{345}$, $S_{145}S_{345}$, r_∞. All points selected below are to be taken within the triangle T. Call the pencils common to r_∞ and $S_{145}S_{245}$,

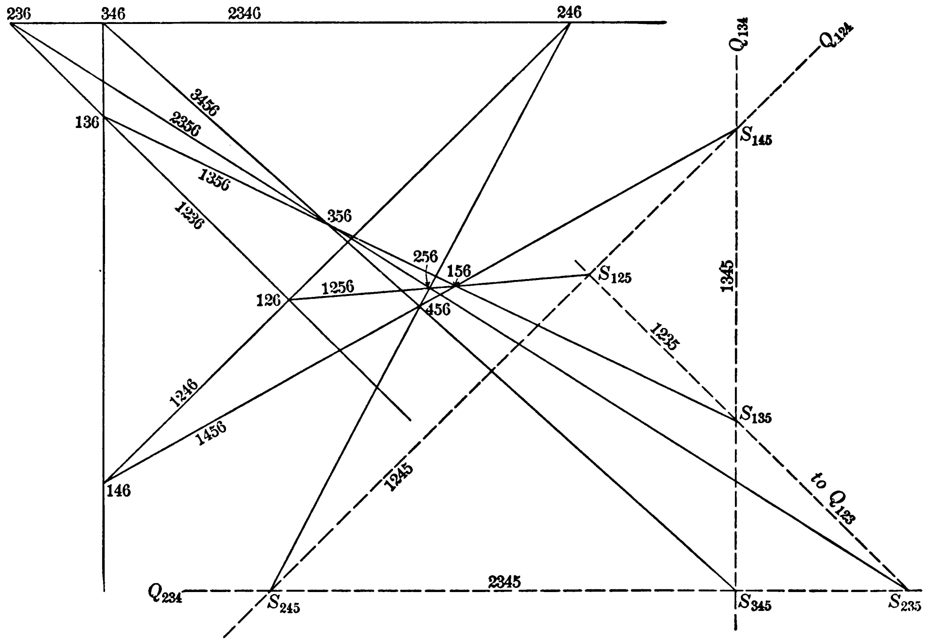

FIG. 22.

$S_{245}S_{345}$, $S_{145}S_{345}$, $S_{135}S_{235}$, respectively Q_{124}, Q_{234}, Q_{134}, Q_{123}. Choose in T a point P_{456}, and then points P_{146}, P_{246}, P_{346} on the lines $P_{456}S_{145}$, $P_{456}S_{245}$, $P_{456}S_{345}$, respectively, and such that the lines $P_{346}P_{246}$, $P_{346}P_{146}$, and $P_{146}P_{246}$ belong to the pencils Q_{234}, Q_{134}, and Q_{124}, respectively. From the order $S_{245}S_{345}S_{235}$ of the hypothesis, $P_{346}S_{235}$ meets the segment $P_{456}P_{246}$ of the triangle $P_{246}P_{346}P_{546}$. Hence it is possible to find in T points $P_{236}P_{356}P_{256}$ in the orders $P_{236}P_{346}P_{246}$, $P_{346}P_{356}P_{456}$, $P_{246}P_{256}P_{456}$, $P_{236}P_{356}P_{256}$. (Equally well might another set of orders have been secured.)

Then $P_{356}S_{135}$ meets $P_{456}S_{145}$ in a point P_{156} of T or does not. If it does, P_{156} is in the order $P_{146}P_{456}P_{156}$, from the order $S_{145}S_{135}S_{345}$ of the hypothesis and the triangle $P_{346}P_{146}P_{456}$. If not, choose in T a point P'_{156} in the order $P_{146}P_{456}P'_{156}$, and from the order $S_{145}S_{135}S_{345}$ of the hypothesis, and the triangle $P_{346}P_{146}P_{456}$, by Lemmas I, III, $P_{456}S_{135}$ meets the segment $P_{346}P_{146}$, and $P'_{156}S_{135}$ will meet the segment $P_{456}P_{356}$ in a point P'_{356}. The line

$P'_{356}S_{235}$ will meet the segment $P_{456}P_{256}$ in P'_{256}, and will or will not meet $P_{346}P_{246}$ in a point P_x of T. Take P'_{236} in the order $P'_{256}P'_{356}P'_{236}$, and in the order $P'_{356}P'_{236}P_x$ if P_x exists, and such that the line $P'_{236}Q_{234}$ meets the segment $P_{346}P_{146}$ in a point P'_{346}, and the segment $P_{146}P_{246}$ in P'_{246}. In this case use the points marked with primes in the further discussion, where points without the primes are used. The line $P_{356}S_{135}$ will meet the segment $P_{346}P_{146}$ in a point P_{136}. The sets of three pencils, $P_{256}P_{356}P_{456}$ and $S_{215}S_{315}S_{415}$ have corresponding ranges belonging to the pencils Q_{345}, Q_{245}, and Q_{235} of the range r_{2345}, and the lines $S_{456}S_{145}$ and $P_{356}S_{135}$ meet in P_{156}. Hence the line $P_{256}S_{125}$ also passes through the point P_{156} (by Theorem XIII), and from the order $P_{456}P_{356}P_{346}$, the point P_{136} is in the orders $P_{156}P_{356}P_{136}$ and $P_{346}P_{136}P_{146}$. Also, the line $P_{236}P_{136}$ meets the line $P_{146}P_{246}$ in a point P_{126} in the orders $P_{236}P_{136}P_{126}$ and $P_{146}P_{126}P_{246}$. From the sets of three pencils $P_{136}P_{236}P_{436}$ and $S_{135}S_{235}S_{435}$ "perspective" in P_{356}, the line $P_{136}P_{236}$ belongs to the pencil Q_{123}. From the sets of three pencils $S_{125}S_{325}S_{425}$ and $P_{126}P_{326}P_{426}$, whose sides belong in pairs to the three pencils Q_{124}, Q_{234}, Q_{123}, respectively, P_{126} is on the line $P_{256}P_{156}$. From the triangle $P_{156}P_{256}P_{456}$, the line $P_{456}Q_{124}$ cuts the segment $P_{156}P_{256}$, since it is not either side line of the triangle, and since it can not meet the segment $P_{146}P_{246}$. Hence if Q_{125} is not on r_∞, it is S_{125} and in the order $S_{145}S_{125}S_{245}$. That it is not on r_∞ may be seen as follows: $S_{245}S_{145}$ meets r_∞ in Q_{124}, of which $P_{146}P_{246}$ is a line. If this line were the same as $P_{126}P_{256}$, P_{156} would be on the line $P_{146}P_{246}$, contrary to the hypothesis. Hence the axiom holds.

The configuration used in the proof may be indicated schematically by $\left|\begin{smallmatrix}20 & 4\\ 3 & 15\end{smallmatrix}\right|$, as may readily be verified from the figure.

SECTION IV.

In terms of the plane set of pencils S, already exhibited in Section III, a new set of elements will now be defined which will be shown to form a three dimensional space, as mentioned in the introduction. Upon the assumption of a plane satisfying the ordinary incidence and order relations, and, furthermore, the parallel axiom, Hilbert* demonstrates that the validity of the Desargues theorem is the necessary and sufficient condition that the plane constitutes a part of a three-dimensional geometry with analogous relations. That the Desargues theorem holds for a plane of such a three space is well known. On the other hand, by the aid of an algebra of segments based upon the Desargues theorem and the parallel axiom, and a resulting analytic geometry, Hilbert exhibited an analytic three-space containing the original plane, while SCHOR† later exhibited a geo-

* *Grundlagen der Geometrie, Festschrift,* § 29.

† SCHOR, D., *Neuer Beweis eines Satzes aus den Grundlagen der Geometrie von Hilbert.* Mathematische Annalen, vol. 58 (1904), pp. 427–433.

metric three-space without the algebra of segments. The method here used is analogous to that of Schor, but makes no use of the parallel axiom; this is possible in view of the use of the ideal elements already introduced on the basis of the Desargues theorem. The ideal elements of the resulting three-space, which were not considered by either Hilbert or Schor, are also developed in terms of the original plane system.

Let three special distinct pencils Q_u, Q_v, Q_w of r_∞ be chosen. Ranges S_k distinct from r_∞ and belonging to Q_u, Q_v, Q_w respectively, will be denoted by u_k, v_k, w_k, respectively. A 3-point will then be defined as follows:

Every triple of ranges u_k, v_k, w_k, not in pencil, is a 3-point. Also since every pencil S_k, with the pencils Q_u, Q_v, Q_w, respectively, determines three ranges u_k, v_k, w_k, respectively, which are in pencil, and conversely every three ranges u_k, v_k, w_k which are in pencil determine a pencil S_k, every pencil S_k will also be called a 3-point, and the notation $T_k \equiv u_k, v_k, w_k$ will be used for 3-points of either class. If T_x is a 3-point, u_x and v_x, u_x and w_x, v_x and w_x, determine three pencils W_x, V_x and U_x, respectively, which are coincident or distinct according as T_k is or is not a pencil.

DEFINITION. Three 3-points T_x, T_y, T_z are in the order $T_x T_y T_z$ if and only if the pencils U_x, U_y, U_z are in the order $U_x U_y U_z$ as pencils.

In view of the Lemma to Theorem III, the order $U_x U_y U_z$ implies also the orders $V_x V_y V_z$ and $W_x W_y W_z$, and conversely. Also all three, or any one, of the three 3-points having order relations may be pencils, but if two are pencils the third is a pencil.

If three 3-points T_x, T_y, T_z are collinear, U_x, U_y, U_z are in range, V_x, V_y, V_z are in range, W_x, W_y, W_z are in range, and these three ranges are in pencil Q_{xyz}.

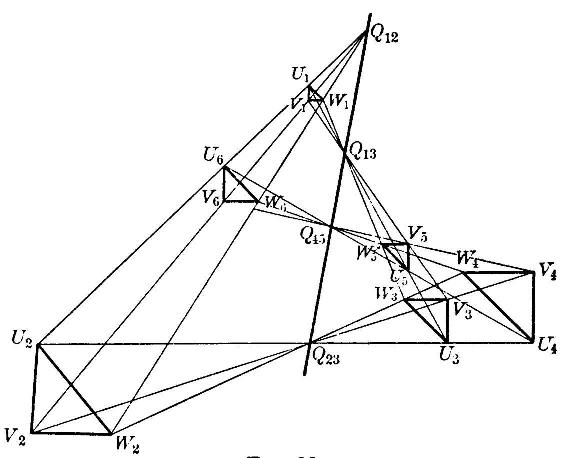

FIG. 23.

It will now be shown that the set of elements T_k satisfies the Axioms I–XI, except IX, and in place of IX possesses a property IX′, characteristic of three-dimensionality.

The validity of Axioms I–VII inclusive follows immediately from our definitions and their validity with respect to the elements S.

AXIOM VIII. (Fig. 23.) *General Case.* If T_1, T_2, T_3 are any three non-collinear 3-points, forming a triangle, and T_4 is a 3-point in the order $T_2 T_3 T_4$, and T_5 is a 3-point in the order $T_3 T_5 T_1$, then there exists a 3-point T_6 in the order $T_1 T_6 T_2$, and on the line $T_4 T_5$.

Proof. Let Q_{12} be the pencil determined by the ranges U_1U_2, V_1V_2, W_1W_2; Q_{45} the pencil determined by the ranges U_4U_5, V_4V_5, W_4W_5; Q_{23} the pencil determined by U_2U_3, V_2V_3, W_2W_3; and Q_{13} the pencil determined by the ranges U_1U_3, V_1V_3, W_1W_3.

Then we have two Desargues configurations.

$$
\begin{array}{ccc}
 & (A) & \\
 & Q_w & \\
V_1 & V_2 & V_3 \\
U_1 & U_2 & U_3 \\
\hline
Q_{23} & Q_{13} & Q_{12}
\end{array}
\qquad\qquad
\begin{array}{ccc}
 & (B) & \\
 & Q_w & \\
U_3 & U_4 & U_5 \\
V_3 & V_4 & V_5 \\
\hline
Q_{45} & Q_{13} & Q_{23}
\end{array}
$$

Hence from (A) and (B) together we have Q_{23}, Q_{13}, Q_{12}, Q_{45} in range, and this gives us the hypotheses of the Desargues configuration (C), (D), (E).

$$
\begin{array}{ccc}
 & (C) & \\
 & Q_{23} & \\
Q_{45} & W_4 & U_4 \\
Q_{12} & W_2 & U_2 \\
\hline
Q_v & U_6 & W_6
\end{array}
\qquad\qquad
\begin{array}{ccc}
 & (D) & \\
 & Q_{23} & \\
Q_{45} & W_4 & V_4 \\
Q_{12} & W_2 & V_2 \\
\hline
Q_u & V_6 & W_6
\end{array}
\qquad\qquad
\begin{array}{ccc}
 & (E) & \\
 & Q_{23} & \\
Q_{45} & U_4 & V_4 \\
Q_{12} & U_2 & V_2 \\
\hline
Q_w & V_6 & U_6
\end{array}
$$

From (C), (D), and (E), we know that the set of ranges U_6W_6, U_6V_6, V_6W_6 is a 3-point. It still remains to show that this 3-point is in the order $T_1T_6T_2$. But the figure formed by the pencils Q_u, Q_v, Q_w, U_1, U_2, U_3, U_4, U_5, U_6 is precisely the figure of Axiom VIII, Section III, whence U_6 is in the order $U_1U_6U_2$, and hence T_6 is in the order $T_1T_6T_2$. This also shows that no one of the three pencils U_6, V_6, W_6 could be on the range r_∞.

It is obvious that one or more of the 3-points T_1, T_2, T_3, T_4, T_5, T_6 could be pencils, but the above general proof is easily modified to fit all the special cases, and the details of these proofs are omitted.

Axiom IX is evidently *not* satisfied.

Axiom X. The hypothesis of Axiom X gives at once

$$
\begin{array}{ccc}
X_{12} & X_{02} & X_{01} \\
\hline
X_{40} & X_{41} & X_{42} \\
X_{30} & X_{31} & X_{32} \\
 & X^*_{34} &
\end{array}
\qquad (X = U,\ V,\ W),
$$

the asterisk indicating that U^*_{34}, V^*_{34}, W^*_{34} may or may not be a 3-point, but simply that U^*_{34}, V^*_{34}, W^*_{34} exist as pencils, whether or not on r_∞.

We have then at once the configurations,

$$
\begin{array}{ccc}
 & Q_z & \\
X_{m4} & X_{m3} & X_{mn} \\
Y_{m4} & Y_{m3} & Y_{mn} \\
\hline
Q_{mn3} & Q_{mn4} & Q_{m34}
\end{array}
\qquad\qquad
\begin{array}{ccc}
 & Q_z & \\
X_{n4} & X_{n3} & X_{nm} \\
Y_{n4} & Y_{n3} & Y_{nm} \\
\hline
Q_{nm3} & Q_{nm4} & Q_{n34}
\end{array}
$$

for $z, X, Y = u, V, W; v, W, U; w, U, V$; and $m, n = 1, 2; 0, 1; 0, 2$.

From these configurations we have at once the collinearities $Q_{124}Q_{134}Q_{123}Q_{234}$, $Q_{013}Q_{014}Q_{034}Q_{134}$ and $Q_{023}Q_{024}Q_{034}Q_{234}$. These give the hypotheses for the configurations

$$\begin{array}{ccc} & Q_{jmn} & \\ X_{jm} & Y_{jm} & Q_{34m} \\ X_{jn} & Y_{jn} & Q_{34n} \\ \hline Y^*_{34} & X^*_{34} & Q_z \end{array}$$

for $j = 3, 4$; $m, n = 0, 1;\ 0, 2;\ 1, 2$; and $X, Y, z = U, V, w;\ V, W, u;\ W, U, v$.

From this it follows that U^*_{34}, V^*_{34}, W^*_{34} is a 3-point if these pencils do not lie on r_∞, and if it is not a 3-point they do lie on r_∞.

Now let $T_{31}, T_{41};\ T_{30}, T_{40};\ T_{32}, T_{42}$ be any pairs whatever of 3-points on the same 3-lines, such that T_{01}, T_{02}, T_{12} exist, and it must be shown that T_{01}, T_{02}, T_{12} are collinear.

We have at once the Desargues configurations

$$\begin{array}{ccc} & X^*_{34} & \\ X_{31} & X_{30} & X_{32} \\ X_{41} & X_{40} & X_{42} \\ \hline X_{02} & X_{12} & X_{01} \end{array} \qquad (X = U, V, W),$$

and then the configurations

$$\begin{array}{ccc} & Q_z & \\ X_{ik} & X_{il} & X_{ij} \\ Y_{ik} & Y_{il} & Y_{ij} \\ \hline Q_{ijl} & Q_{ijk} & Q_{ikl} \end{array}$$

where $i, j, k, l = 0, 1, 2, 3, 4$ but all different and $z, X, Y = u, V, W;\ v, W, U;\ w, U, V$. These give the collinearities $Q_{ijk}, Q_{ijl}, Q_{ikl}, Q_{jkl}$ $(i, j, k, l = 0, 1, 2, 3, 4$ but all different).

These give the hypotheses for the configurations

$$\begin{array}{ccc} & Q_i & \\ Q_{i4j} & X_{i4} & Y_{i4} \\ Q_{i3j} & X_{i3} & Y_{i3} \\ \hline Q_z & Y_{ij} & X_{ij} \end{array} \left(\begin{array}{l} i, j = 0, 1;\ 0, 2;\ 1, 2 \\ z, X, Y = \begin{cases} u, V, W \\ v, W, U \\ w, U, V \end{cases} \end{array}\right), \qquad \begin{array}{ccc} Q_u & Q_v & Q_w \\ \hline U_{ij} & V_{ij} & W_{ij} \\ U_{ik} & V_{ik} & W_{ik} \\ & Q_{012} & \end{array} \left(\begin{array}{c} i, j, k = 0, 1, 2, \\ i \neq j \neq k \neq i \end{array}\right),$$

whence T_{01}, T_{02}, and T_{12} are collinear, and we have the conclusion of the theorem.

Axiom XI. The hypothesis of three 3-points in a range implies the existence of a pair of triangles the joins of whose corresponding vertices are in a pencil.

But, conversely, by Axiom X, this implies that the three 3-points are in a 3-line. But any pair of triangles with sides through these three collinear points are in pencil, by definition. Hence Axiom XI holds.

Definition. A tetrahedron is a set of four planes, determined by four non-coplanar points taken in triples. The existence of such points follows from the fact that Axiom IX does not hold.

Definition. A 3-space is the set of all points collinear with two distinct points of a tetrahedron.

Theorem. (Fig. 24.) The set of 3-points T constitutes a 3-space.

Proof. Let $T_3 \equiv U_3, V_3, W_3$ (distinct) be any 3-point. Then, since U_3, V_3, W_3 as pencils of the set S, are themselves 3-points, we have a tetrahedron, T_3, U_3, V_3, W_3. Let $T_4 \equiv U_4, V_4, W_4$ be any other 3-point. Then U_3U_4, V_3V_4, W_3W_4, belong to a pencil Q_{34}. If Q_{34} is not a pencil of the range r_∞, it is a 3-point of the plane determined by U_3, V_3, W_3, and T_4 is collinear with T_3 and Q_{34}, and hence is a point of the 3-space $T_3U_3V_3W_3$.

If Q_{34} is on r_∞, choose a new pencil V'_3 in the order $V_3V'_3U_3$. Then V'_3Q_u meets U_3Q_v in W'_3, and we have a 3-point T'_3. T'_3 is in the plane determined by $V_3U_3T_3$, and T'_3 and T_4 belong to a pencil $Q'_{34} \not\equiv Q_{34}$, hence not on r_∞. Hence T_4 is collinear with two distinct points of the tetrahedron, and the points T constitute a 3-space.

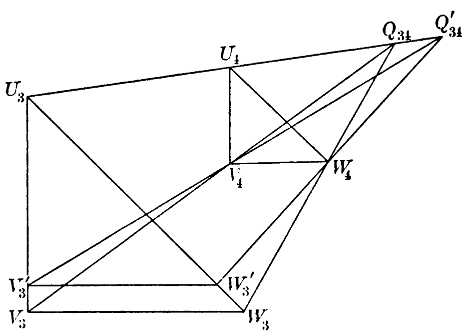

Fig. 24.

Ideal elements of the 3-space T. We have now a three-dimensional set of points satisfying order relations, and the ideal 3-space elements might be introduced in it directly, by the methods of Bonola, Schur, or Veblen.* It is interesting, from the point of view of the present paper, to do this in terms of the original plane elements.

Let $T_3 \equiv U_3, V_3, W_3$ and $T_4 \equiv U_4, V_4, W_4$ be any two 3-points. The ranges U_3U_4, V_3V_4, W_3W_4 meet the range r_∞ in three pencils $Q_{u'}, Q_{v'}, Q_{w'}$, which are in general distinct. The set of three pencils $Q_{u'}, Q_{v'}, Q_{w'}$, whether or not distinct, will be defined as an *ideal element* of the 3-space T, and will be said to lie on the 3-line (U_3U_4, V_3V_4, W_3W_4). If these ideal 3-points are adjoined to the set T, giving a set T', the points of T' will not satisfy the order relations as defined, since the set Q did not, but will, evidently, have the following three properties:

Any two planes determine one and only one line.

* Cf. the previous citations.

Any two lines of a plane determine a point.

The general Desargues theorem.

In short, the set of elements T' has all the intersectional properties of a projective 3-space, except those which are dependent on Pascal's theorem. These must be excepted, since Pascal's theorem is independent of the Desargues theorem (cf. Hilbert).

SECTION V.

Extension to n-Space.

The method of Section IV may now be extended to cover the definition of n-space in terms of the plane set of elements already considered. We choose on the line r_∞ of the set of elements Q, n pencils, $Q_1, Q_2, \cdots, Q_n$. Any set of n ranges r_i $(i = 1, 2, \cdots, n)$, one belonging to each of the pencils $Q_1, \cdots, Q_n$, respectively, and each distinct from the range r_∞, will be defined as an n-point, and denoted by the notation N_x. These n ranges, taken in pairs, determine $n(n-1)/2$ pencils (not necessarily distinct), $S_{x,ij}$ $(i, j = 1, \cdots, n;\ i \neq j)$. Let two such n-points be N_x and N_y. The pencils $S_{x,ij}$ and $S_{y,ij}$ $(i, j = 1, \cdots, n;\ i \neq j)$ determine a range $r_{xy,ij}$ such that $r_{xy,ij}$, $r_{xy,ik}$, $r_{xy,jk}$ $(i, j, k = 1, \cdots, n;\ i \neq j \neq k \neq i)$ are in pencil.

The set of $\frac{1}{2}n(n-1)$ ranges $r_{xy,ij}$ will be called an n-line. An n-point N_z will be said to be on the n-line n_{xy}, if and only if, the pencil $S_{z,ij}$ belongs to the range $r_{xy,ij}$ $(i, j = 1, \cdots, n;\ i \neq j)$. Its *order* will be said to be the same as the order of the pencils $S_{x,ij}$, $S_{y,ij}$, $S_{z,ij}$. That this definition of order is unique is easily seen as follows. Each i, j, k may be thought of as fixing a 3-space geometry in which the order for ij, ik, jk is unique, and by a sequential process the result holds for all of the ranges, and hence we have the general result for the n-space. Hence we may write the order $N_x N_y N_z$ as implying the orders $S_{x,ij} S_{y,ij} S_{z,ij}$ for each triple of values for i, j, k, and conversely. The n-point N_x will be said to be the same as the pencil S_x if $S_{x,ij}$ is S_x for every ij, since S_x will in this case completely determine the n-point.

It can now be shown that this n-point geometry satisfies all the axioms except that of closure. Axioms I–VII inclusive follow immediately from the definitions. The validity of Axiom VIII may be shown as follows. The axiom states that N_1, N_2, N_3 are the vertices of a triangle, and if N_4 and N_5 are in the orders $N_2 N_3 N_4$ and $N_1 N_5 N_3$ respectively, then there exists an n-point N_6 in the order $N_1 N_6 N_2$ and on the line $N_4 N_5$. Consider any triple of the pencils $Q_1, \cdots, Q_n$, of r_∞, say Q_i, Q_j, Q_k. Then we have the existence of the three ranges $r_{6,i}$, $r_{6,j}$, $r_{6,k}$, of the n-point, N_6, from the corresponding theorem of the 3-space theory. Similarly, if i, j, k be allowed to take each set of values (distinct and of the set $1 \cdots n$), we obtain all of the ranges of the set forming the n-point N_6, and each of the pencils $S_{6,ij}$ will be in the order required from the corresponding 3-space theorem. Hence Axiom VIII is valid.

That Axiom IX does not hold is evident, since each pencil S_x is an n-point, but there are other n-points which do not lie in the plane of the pencils Q.

Axioms X and XI may be easily shown to hold by considering the pencils $Q_1, \cdots, Q_n$ of the range r_∞ in sets of triples, and then applying the theorem already proved for each of the resulting 3-spaces, thus obtaining the theorem for n-space.

That the space thus obtained is actually an n-space may be shown as follows: By the same method of proof as in Section IV, if for $Q_1, \cdots, Q_n$, we have an n-space, then by increasing the number of pencils by adding Q_{n+1} to the set, we introduce new elements into the geometry. But these may all be seen to be collinear with two points of $n + 1$ n-spaces, and hence constitute an $(n + 1)$-space. But the theorem is true for 3-space, and hence by induction, for any value of n.

The ideal elements which must be adjoined to this n-space to make it projective are the pencils (n in number, but not necessarily distinct) belonging to the range r_∞, in which any other n-line (i. e., its component ranges) meet this range r_∞. If we call each such set an ideal n-point, and adjoin all such ideal points to the set already obtained, we have a set possessing the properties that any two n-points determine an n-line, any two n-lines determine an n-point, and the general Desargues Theorem. The first two of these statements are evident, and the last may readily be seen by considering the parts of the figure separately, as in the proof of Axiom VIII.

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