CONGRUENCES OF THE ELLIPTIC TYPE*

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Introduction.

Let L be a line of a rectilinear congruence; X, Y, Z, its direction-cosines; and x, y, z, the coördinates of the point in which L cuts a surface of reference S. These six quantities are functions of two parameters, say u and v, which we assume to be real. As usual we put

$$(1)\qquad \begin{gathered} E = \Sigma\left(\frac{\partial X}{\partial u}\right)^2, \qquad F = \Sigma\frac{\partial X}{\partial u}\frac{\partial X}{\partial v}, \qquad G = \Sigma\left(\frac{\partial X}{\partial v}\right)^2, \\ e = \Sigma\frac{\partial x}{\partial u}\frac{\partial X}{\partial u}, \quad f = \Sigma\frac{\partial x}{\partial v}\frac{\partial X}{\partial u}, \quad f' = \Sigma\frac{\partial x}{\partial u}\frac{\partial X}{\partial v}, \quad g = \Sigma\frac{\partial x}{\partial v}\frac{\partial X}{\partial v}. \end{gathered}$$

Of all the ruled surfaces, formed by lines of the congruence, which pass through L two at most are developable. They are defined by the equation

$$(2)\qquad (Ef' - Fe)\,du^2 + (Eg + Ff' - Ff - Ge)\,du\,dv + (Fg - Gf)\,dv^2 = 0.$$

In the case of normal congruences, congruences of Guichard, cyclic congruences and congruences of tangents to a real family of curves on a surface, the integrals of equation (2) are real. But there is a large variety of congruences for which the integrals of this equation are imaginary. We say that a congruence is of the *hyperbolic* or *elliptic type* according as the two values of dv/du given by (2) are real or imaginary. This paper deals with congruences of the latter type, and particularly with pairs of ruled surfaces which are real only in this case and which possess properties analogous to those of the developable surfaces of a congruence of the hyperbolic type.

One of these systems may be defined analytically by means of the following theorem of CIFARELLI: †

Given two quadratic differential forms

$$(3)\qquad a_1\,du^2 + 2a_2\,du\,dv + a_3\,dv^2, \qquad b_1\,du^2 + 2b_2\,du\,dv + b_3\,dv^2,$$

of which the first is definite, that is, $a_1 a_3 - a_2^2 > 0$; if one forms the Jacobian

* Presented to the Society September 13, 1909.

† *Le Congruenze,* Annali di Matematica, ser. 3, vol. 2 (1899), p. 148.

of these forms and equates to zero the Jacobian of the resulting quadratic form and of the first of (3), the solutions of the resulting differential equation define a real transformation, say $u' = \phi(u, v)$, $v' = \psi(u, v)$, which changes the forms (3) into two

$$a_1' du'^2 + 2a_2' du'dv' + a_3' dv'^2, \qquad b_1' du'^2 + 2b_2' du'dv' + b_3' dv'^2,$$

which are such that

$$(4) \qquad \frac{a_1'}{b_1'} = \frac{a_3'}{b_3'}, \qquad a_2' = 0.$$

Since the left-hand member of equation (2) is a definite quadratic form, it may be taken as the first of (3). If we take for the second the square of the linear element of the spherical representation, namely

$$(5) \qquad d\sigma^2 = Edu^2 + 2Fdu\,dv + Gdv^2,$$

the parametric ruled surfaces $u' =$ const., $v' =$ const. constitute a system which we shall study in detail. We call them the *characteristic ruled surfaces* of the congruence.

In § 1 the equation of the characteristic ruled surfaces is derived, and therefrom we discover properties of the lines of striction of these surfaces, of their spherical representation, and the fact that their parameters of distribution are equal to one another and to the harmonic mean of the maximum and minimum values of the parameter of distribution of all the surfaces through the line.

The determination of a congruence with an assigned spherical representation of its characteristic ruled surfaces is investigated in § 2, and the results are applied in § 3 to the discussion of congruences whose characteristic surfaces meet the surface of reference in its asymptotic lines. These congruences are of the RIBAUCOUR type. The necessary and sufficient conditions for their existence are found, and an example of such congruences is given in § 4. Incidentally a theorem is derived concerning the case where the characteristic lines on a surface correspond to the asymptotic lines on an associate surface.

Since the quadratic form (2) is definite, there exist an infinity of real transformations of variables such that in terms of the new variables the expression (2) is of the form $\lambda(du'^2 + dv'^2)$. We say that such a parametric system is *isothermic*. Section 5 deals with the determination of congruences with an assigned spherical representation of an isothermic system of ruled surfaces. If ρ denotes the distance from the middle point of a line to one of its two conjugate purely imaginary focal points, the surfaces which are the loci of the points at the distances $i\rho$, $-i\rho$ from the middle point are of interest in this theory. We call them the *pseudofocal surfaces* of the congruence.

When the pseudofocal surfaces correspond with parallelism of tangent planes, as in § 6, the congruences are of the kind studied by LILIENTHAL. He took three functions of $u + iv$, written

$$x_1 + ix_2 = f_1(u + iv), \qquad y_1 + iy_2 = f_2(u + iv), \qquad z_1 + iz_2 = f_3(u + iv),$$

and considered the congruence of lines joining corresponding points of the surfaces which are loci of the points (x_1, y_1, z_1), (x_2, y_2, z_2). We say that two such surfaces are *conjugate-potential.* These *congruences of Lilienthal* are of the RIBAUCOUR type, and the spherical representation of their imaginary developables is similar to that of congruences whose focal surfaces are curves.

Conjugate-potential surfaces are associate to one another. Section 7 deals with congruences which consist of lines joining corresponding points on associate surfaces. Certain congruences of RIBAUCOUR possess this property in an infinity of ways, and of this group are the congruences of LILIENTHAL.

§ 1. *Characteristic Ruled Surfaces.*

Given a congruence of the elliptic type expressed in terms of any parametric system and with an arbitrary surface of reference. If we put

$$(6) \qquad A = \frac{Ef' - Fe}{H}, \qquad B = \frac{Eg + F(f' - f) - Ge}{2H}, \qquad C = \frac{Fg - Gf}{H},$$

where $H = \sqrt{EG - F^2}$, the equation of the developables (2) may be written

$$(7) \qquad Adu^2 + 2Bdudv + Cdv^2 = 0.$$

If we apply the theorem of Cifarelli to the left-hand member of this equation and to the right-hand member of (5), we find that the differential equation of the characteristic surfaces is reducible to

$$(8) \qquad \begin{aligned}[t] &[A(AG - CE) - 2B(AF - BE)]\,du^2 + 2[B(AG + CE) \\ &\quad - 2ACF]\,dudv + [2B(BG - CF) - C(AG - CE)]\,dv^2 = 0.\end{aligned}$$

In order that the characteristic ruled surfaces be parametric, we must have

$$(9) \qquad \begin{aligned} (AG - CE)A - 2B(AF - BE) &= 0, \\ (AG - CE)C - 2B(BG - CF) &= 0, \end{aligned}$$

which equations are reducible to

$$(10) \qquad \frac{A}{E} = \frac{C}{G}, \qquad B = 0,$$

unless

$$(11) \qquad A(BG - CF) - C(AF - BE) = 0.$$

But in the latter case the middle term also of (8) vanishes, so that the characteristic ruled surfaces are indeterminate in this case. Hence equations (10), or in other form

$$(12)\qquad \begin{aligned} (f+f') - F\left(\frac{e}{E}+\frac{g}{G}\right) &= 0, \\ F(f'-f) + Eg - Ge &= 0, \end{aligned}$$

constitute a necessary and sufficient condition that the characteristic surfaces be parametric.

If the middle surface of the congruence is taken as the surface of reference, we have *

$$(13)\qquad (f+f')F - EG\left(\frac{e}{E}+\frac{g}{G}\right) = 0.$$

Combining this equation and the first of (12), we have

$$(14)\qquad f+f' = 0 \qquad \frac{e}{E}+\frac{g}{G} = 0,$$

and the second of equations (12) may be given either of the forms

$$(15)\qquad Eg + Ff' = 0, \qquad Ff + Ge = 0.$$

In order to interpret the second of equations (14), we recall that the abscissa r, measured from the surface of reference, of the point P on a line L where the line of striction of the ruled surface defined by a value of dv/du meets L is given by †

$$(16)\qquad r = -\frac{edu^2 + (f+f')dudv + gdv^2}{Edu^2 + 2Fdudv + Gdv^2}.$$

From this and (14) it follows that the values of r for the two characteristic surfaces through L differ only in sign. Hence we have

THEOREM 1. *The lines of striction of the two characteristic ruled surfaces through a line L of the congruence meet L in points equidistant from the middle point.*

We call these the *characteristic points*.

Before proceeding we return to the consideration of the particular case for which equation (11) holds as well as (9), that is when the characteristic ruled surfaces are indeterminate. These conditions are equivalent to

$$(17)\qquad \frac{AG-CE}{2B} = \frac{AF-BE}{A} = \frac{BG-CF}{C} = \lambda,$$

where λ denotes the factor of proportionality to be determined. The condition that these equations be consistent is

$$\begin{vmatrix} G & 2\lambda & E \\ F-\lambda & E & 0 \\ 0 & -G & F+\lambda \end{vmatrix} = 2\lambda(EG - F^2 + \lambda^2) = 0.$$

* E., p. 401. A reference in this form is to my *Differential Geometry*, Boston, 1909.

† E., p. 395.

Since the quantities are real, we must have $\lambda = 0$, and consequently equations (17) are reducible to

$$\frac{A}{E} = \frac{B}{F} = \frac{C}{G}.$$

Hence the congruences sought are such that their developables are represented on the sphere by the minimal lines of the latter. This is a characteristic property of isotropic congruences.* Hence we have:

THEOREM 2. *The characteristic ruled surfaces of an elliptic congruence are real and determinate, except when the congruence is isotropic.*

Isotropic congruences will be excluded from the subsequent discussion.

Let $\bar{r}$ and R denote the abscissæ, measured from the middle point of a line L, of a limit point P and of the characteristic point C on the same side of the middle point. If ω denotes the angle which the tangent plane at C to the corresponding characteristic surface makes with the tangent plane to the principal surface at P, we have from Hamilton's equation †

$$R = \bar{r}\,(\cos^2\omega - \sin^2\omega).$$

In like manner for the other characteristic ruled surface we have

$$R = -\bar{r}(\cos^2\omega' - \sin^2\omega').$$

From these equations we obtain

$$(18) \qquad \cos 2\omega + \cos 2\omega' = 0.$$

Consequently the curves on the sphere which represent the characteristic ruled surfaces either form an orthogonal system, or they are equally inclined to the curves which bisect the angles between the curves representing the principal ruled surfaces.

In the former case, when these surfaces are parametric, we should have $F = 0$ and also $f + f' = 0$, from the first of (12). Moreover, from the second of (12) we should have $Eg = Ge$; that is, the congruence is isotropic. Hence $F \neq 0$, and we have

THEOREM 3. *The curves on the sphere which represent the characteristic ruled surfaces are equally inclined to the curves which bisect the angles between the images of the principal surfaces.*

The developables of a congruence of the hyperbolic type possess the same property.

By means of the preceding results we obtain the quadratic equation which the abscissæ of the characteristic points satisfy, when the parameters and surface of reference are any whatsoever. If R_1 and R_2 denote these abscisssæ

* E. p. 412.

† E, p. 397.

and r_1, r_2, the abscissæ of the limit points, we have from Hamilton's equation

$$R_1 = r_1 \cos^2 \omega + r_2 \sin^2 \omega,$$

$$R_2 = r_1 \cos^2 \omega' + r_2 \sin^2 \omega',$$

and from (18)

$$\cos^2 \omega' = \sin^2 \omega, \qquad \sin^2 \omega' = \cos^2 \omega.$$

From these equations we deduce

$$R_1 + R_2 = r_1 + r_2,$$

$$R_1 R_2 = (r_1 + r_2)^2 \frac{\cos^2 \theta_0}{4} + r_1 r_2 \sin^2 \theta_0,$$

where θ_0 denotes the angle between the spherical images of the characteristic surfaces. If the differential equation of the latter surfaces be written in the form

$$L du^2 + 2M du dv + N dv^2 = 0,$$

it is readily shown that

$$\cos^2\theta_0, \sin^2\theta_0 = \frac{(EN - 2FM + GL)^2, \qquad 4H^2(M^2 - NL)}{E^2N^2 - 4EFMN + 2EG(2M^2 - NL) - 4FGLM + 4F^2LN + G^2L^2}.$$

Since moreover

$$r_1 + r_2 = \frac{(f + f')F - gE - eG}{EG - F^2}, \qquad r_1 r_2 = \frac{4eg - (f + f')^2}{EG - F^2},$$

we can readily determine the expressions of the coefficients of the quadratic equation satisfied by the abscissæ of the characteristic points. On account of the involved form of this equation we will not give it here.

We shall obtain another property of the characteristic ruled surfaces by recalling that the parameter of distribution of the ruled surface determined by a value of dv/du is given by*

$$p = \frac{A du^2 + 2B du dv + C dv^2}{E du^2 + 2F du dv + G dv^2}. \tag{19}$$

The first of equations (10) expresses the fact that the parameters of distribution of the two characteristic surfaces through a line are equal.

The differential equation (8) can be written

$$\frac{AG + CE - 2BF}{2(AC - B^2)} = \frac{E du^2 + 2F du dv + G dv^2}{A du^2 + 2B du dv + C dv^2}. \tag{20}$$

If now p_1 and p_2 denote the maximum and minimum values of p for all the ruled surfaces through a line, and if p_c denotes the value for the characteristic

* E, p. 424.

surfaces, equation (20) may be written, in consequence of (19),

$$\frac{1}{2}\left(\frac{1}{p_1}+\frac{1}{p_2}\right)=\frac{1}{p_c}.$$

Hence we have

THEOREM 4. *The parameters of distribution of the two characteristic ruled surfaces through a line have the same value, namely the harmonic mean of the maximum and minimum values of the parameter for all the ruled surfaces through the line.*

The first part of this theorem is a consequence of the following theorem which can be readily proved:

THEOREM 5. *If the lines of striction of two ruled surfaces through a line L meet the latter at points equidistant from its middle point, the parameters of distribution of the two surfaces have the same value; and conversely.*

§ 2. *Spherical Representation of Characteristic Ruled Surfaces.*

It can be shown that the first derivatives of the coefficients of the surface of reference of a congruence are expressible in the form *

$$(21)\qquad \begin{aligned}\frac{\partial x}{\partial u}&=\frac{eG-f'F}{H^2}\frac{\partial X}{\partial u}+\frac{f'E-eF}{H^2}\frac{\partial X}{\partial v}+\gamma X,\\ \frac{\partial x}{\partial v}&=\frac{fG-gF}{H^2}\frac{\partial X}{\partial u}+\frac{gE-fF}{H^2}\frac{\partial X}{\partial v}+\gamma_1 X,\end{aligned}$$

and similar equations in y and z, where γ and γ_1 are functions which must satisfy the conditions

$$(22)\qquad \begin{aligned}&\frac{\partial e}{\partial v}-\frac{\partial f}{\partial u}-\left\{\begin{smallmatrix}12\\1\end{smallmatrix}\right\}'e+\left\{\begin{smallmatrix}11\\1\end{smallmatrix}\right\}'f-\left\{\begin{smallmatrix}12\\2\end{smallmatrix}\right\}'f'+\left\{\begin{smallmatrix}11\\2\end{smallmatrix}\right\}'g+F\gamma-E\gamma_1=0,\\ &\frac{\partial f'}{\partial v}-\frac{\partial g}{\partial u}-\left\{\begin{smallmatrix}22\\1\end{smallmatrix}\right\}'e+\left\{\begin{smallmatrix}12\\1\end{smallmatrix}\right\}'f-\left\{\begin{smallmatrix}22\\2\end{smallmatrix}\right\}'f'+\left\{\begin{smallmatrix}12\\2\end{smallmatrix}\right\}'g+G\gamma-F\gamma_1=0,\\ &\frac{\partial \gamma}{\partial v}-\frac{\partial \gamma_1}{\partial u}+f-f'=0,\end{aligned}$$

the Christoffel symbols being $\left\{\begin{smallmatrix}rs\\t\end{smallmatrix}\right\}'$ are formed with respect to the quadratic form (5).

We consider now a congruence of the elliptic type referred to its characteristic surfaces. We assume that the middle surface is the surface of reference and that R is the abscissa of the characteristic point of the surface $u=$ const. From (14), (15) and (16) we have

$$(23)\qquad R=\frac{e}{E}=\frac{-g}{G}=-\frac{Ff}{EG}=\frac{Ff'}{EG}.$$

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* E, pp. 406, 407.

The abscissæ of the limit and focal points are given by the respective equations *

$$(24)\qquad \bar{r}^2 = \frac{(f+f')^2 - 4eg}{4H^2}, \qquad \rho^2 = \frac{ff' - eg}{H^2}.$$

From equations (23) and (24) we obtain

$$(25)\qquad R = \bar{r}\sin\theta_0, \qquad R = i\rho\cos\theta_0,$$

where θ_0 denotes the angle between the central planes of the two characteristic surfaces.

In consequence of (23) equations (21) are reducible to

$$(26)\qquad \frac{\partial x}{\partial u} = \frac{ER}{F}\frac{\partial X}{\partial v} + \gamma X, \qquad \frac{\partial x}{\partial v} = -\frac{GR}{F}\frac{\partial X}{\partial u} + \gamma_1 X.$$

From the first two of (22) we find that

$$(27)\qquad \begin{aligned} \gamma &= -\frac{\partial}{\partial v}\left(\frac{ER}{F}\right) - \frac{R}{F}\left(E\left\{ {22 \atop 2} \right\}' + G\left\{ {11 \atop 2} \right\}'\right), \\ \gamma_1 &= \frac{\partial}{\partial u}\left(\frac{GR}{F}\right) + \frac{R}{F}\left(E\left\{ {22 \atop 1} \right\}' + G\left\{ {11 \atop 1} \right\}'\right), \end{aligned}$$

and the last of (22) may be written

$$(28)\qquad \begin{aligned} &\frac{\partial^2}{\partial u^2}\left(\frac{GR}{F}\right) + \frac{\partial^2}{\partial v^2}\left(\frac{ER}{F}\right) + \frac{\partial}{\partial u}\left[\frac{R}{F}\left(E\left\{ {22 \atop 1} \right\}' + G\left\{ {11 \atop 1} \right\}'\right)\right] \\ &\qquad + \frac{\partial}{\partial v}\left[\frac{R}{F}\left(E\left\{ {22 \atop 2} \right\}' + G\left\{ {11 \atop 2} \right\}'\right)\right] + 2\frac{EG}{F}R = 0. \end{aligned}$$

Since equations (22) are sufficient conditions upon the functions e, f, f', g γ and γ_1 that equations (21) define the surface of reference of a congruence with a given spherical representation of linear element (5),† we have

THEOREM 6. *Given any system of curves on the sphere and let E, F, G denote the fundamental coefficients; each solution of equation* (28) *determines a congruence whose characteristic ruled surfaces are represented on the sphere by the given system of curves.*

From (26) we obtain by differentiation with respect to u and v the following

$$(29)\qquad \begin{aligned} \frac{\partial^2 x}{\partial u^2} &= \left(\frac{\partial}{\partial u}\log\frac{ER}{F} + \left\{ {12 \atop 2} \right\}'\right)\frac{\partial x}{\partial u} - \left(\frac{E}{G}\left\{ {12 \atop 1} \right\}' + \frac{F\gamma}{GR}\right)\frac{\partial x}{\partial v} + A_{11}X, \\ \frac{\partial^2 x}{\partial u\partial v} &= -\frac{G}{E}\left\{ {11 \atop 2} \right\}'\frac{\partial x}{\partial u} - \frac{E}{G}\left\{ {22 \atop 1} \right\}'\frac{\partial x}{\partial v} + A_{12}X, \\ \frac{\partial^2 x}{\partial v^2} &= \left(\frac{F\gamma_1}{ER} - \frac{G}{E}\left\{ {12 \atop 2} \right\}'\right)\frac{\partial x}{\partial u} + \left(\frac{\partial}{\partial v}\log\frac{GR}{F} + \left\{ {12 \atop 1} \right\}'\right)\frac{\partial x}{\partial v} + A_{22}X, \end{aligned}$$

* E., pp. 396, 399.

† E., p. 407.

where, for the sake of brevity, we have put

$$A_{11} = \gamma\gamma_1 \frac{F}{GR} - \gamma\left(\left\{\begin{smallmatrix}12\\2\end{smallmatrix}\right\}' + \frac{\partial}{\partial u}\log\frac{ER}{F}\right) + \gamma_1 \frac{E}{G}\left\{\begin{smallmatrix}12\\1\end{smallmatrix}\right\}' - ER + \frac{\partial \gamma}{\partial u},$$

$$(30)\qquad \begin{aligned} A_{12} &= \frac{E}{G}\left\{\begin{smallmatrix}22\\1\end{smallmatrix}\right\}'\gamma_1 + \frac{G}{E}\left\{\begin{smallmatrix}11\\2\end{smallmatrix}\right\}'\gamma + \frac{\partial \gamma}{\partial v} - \frac{GE}{F}R, \\ &= \frac{E}{G}\left\{\begin{smallmatrix}22\\1\end{smallmatrix}\right\}'\gamma_1 + \frac{G}{E}\left\{\begin{smallmatrix}11\\2\end{smallmatrix}\right\}'\gamma + \frac{\partial \gamma_1}{\partial u} + \frac{GE}{F}R, \end{aligned}$$

$$A_{22} = -\gamma\gamma_1 \frac{F}{ER} - \gamma_1\left(\left\{\begin{smallmatrix}12\\1\end{smallmatrix}\right\}' + \frac{\partial}{\partial v}\log\frac{GR}{F}\right) + \gamma \frac{G}{E}\left\{\begin{smallmatrix}12\\2\end{smallmatrix}\right\}' + GR + \frac{\partial \gamma_1}{\partial v}.$$

§ 3. *Congruences whose Characteristic Ruled Surfaces meet their Middle Surfaces in Asymptotic Lines.*

We consider in particular the case where the characteristic surfaces cut the middle surface in its asymptotic lines. From (29) and (30) it is seen that a necessary and sufficient condition for this is

$$A_{11} = A_{22} = 0.$$

In this case we have

$$A_{11}G + A_{22}E = G\frac{\partial \gamma}{\partial u} + E\frac{\partial \gamma_1}{\partial v} - \gamma G\frac{\partial}{\partial u}\log\frac{ER}{F} - \gamma_1 E\frac{\partial}{\partial v}\log\frac{GR}{F} = 0.$$

When the values of γ and γ_1 from (27) are substituted in this equation, the result is reducible to

$$(31)\qquad \frac{\partial}{\partial u}\left[\frac{G}{E}\left\{\begin{smallmatrix}11\\2\end{smallmatrix}\right\}' + \left\{\begin{smallmatrix}22\\2\end{smallmatrix}\right\}' + \frac{\partial}{\partial v}\log\sqrt{\frac{E}{G}}\right] = \frac{\partial}{\partial v}\left[\frac{E}{G}\left\{\begin{smallmatrix}22\\1\end{smallmatrix}\right\}' + \left\{\begin{smallmatrix}11\\1\end{smallmatrix}\right\}' + \frac{\partial}{\partial u}\log\sqrt{\frac{G}{E}}\right].$$

But this is a necessary and sufficient condition that the parametric curves on the sphere be the spherical representation of the characteristic conjugate system on a unique surface Σ.* The coördinates $\bar{x}$, $\bar{y}$, $\bar{z}$ of Σ are given by the quadratures

$$(32)\qquad \frac{\partial \bar{x}}{\partial u} = -\frac{\lambda}{G}\left(G\frac{\partial X}{\partial u} - F\frac{\partial X}{\partial v}\right), \qquad \frac{\partial \bar{x}}{\partial v} = \frac{\lambda}{E}\left(F\frac{\partial X}{\partial u} - E\frac{\partial X}{\partial v}\right),$$

where λ is given by

$$(33)\qquad \begin{aligned} \frac{\partial \log\lambda}{\partial u} &= \frac{\partial}{\partial u}\log\sqrt{\frac{E}{G}} - \left\{\begin{smallmatrix}22\\1\end{smallmatrix}\right\}'\frac{E}{G} - \left\{\begin{smallmatrix}11\\1\end{smallmatrix}\right\}', \\ \frac{\partial \log\lambda}{\partial v} &= \frac{\partial}{\partial v}\log\sqrt{\frac{G}{E}} - \left\{\begin{smallmatrix}11\\2\end{smallmatrix}\right\}'\frac{G}{E} - \left\{\begin{smallmatrix}22\\2\end{smallmatrix}\right\}'. \end{aligned}$$

* EISENHART, *Three particular systems of lines on a surface*, Transactions of the American Mathematical Society, vol. 5 (1904), p. 434.

From (26) and (32) we have

$$(34)\qquad \Sigma\frac{\partial x}{\partial u}\frac{\partial \bar{x}}{\partial u}=0,\qquad \Sigma\frac{\partial x}{\partial u}\frac{\partial \bar{x}}{\partial v}+\Sigma\frac{\partial x}{\partial v}\frac{\partial \bar{x}}{\partial u}=0,\qquad \Sigma\frac{\partial x}{\partial v}\frac{\partial \bar{x}}{\partial v}=0.$$

Hence S and Σ correspond with orthogonality of linear elements and consequently the congruence under consideration is of the Ribaucour type,* and Σ is the director-surface.

Referring to (29), we have that another necessary condition that the asymptotic lines on S be parametric is

$$\frac{\partial}{\partial v}\left(\frac{\partial}{\partial u}\log\frac{ER}{F}+\left\{{12 \atop 2}\right\}'\right)=\frac{\partial}{\partial u}\left(\frac{\partial}{\partial v}\log\frac{GR}{F}+\left\{{12 \atop 1}\right\}'\right).$$

This equation and (31) are equivalent to the two

$$(35)\qquad \frac{\partial^2}{\partial u\,\partial v}\log\frac{G}{E}+\frac{\partial}{\partial u}\left\{{12 \atop 1}\right\}'-\frac{\partial}{\partial v}\left\{{12 \atop 2}\right\}'=0,\qquad \frac{\partial}{\partial u}\frac{G}{E}\left\{{11 \atop 2}\right\}'=\frac{\partial}{\partial v}\frac{E}{G}\left\{{22 \atop 1}\right\}'.$$

We shall show that these conditions are sufficient.

When equations (35) are satisfied, equation (31) is true and there exists a surface Σ, defined by (32). Since the parametric curves on Σ are the characteristic lines, we have †

$$(36)\qquad \frac{\bar{D}}{E}=\frac{\bar{D}''}{G},\qquad \bar{D}'=0,$$

and consequently the second of equations (35) is reducible to ‡

$$(37)\qquad \frac{\partial}{\partial u}\left\{{12 \atop 1}\right\}_1=\frac{\partial}{\partial v}\left\{{12 \atop 2}\right\}_1,$$

where the Christoffel symbols are formed with respect to the linear element of Σ. This is the condition that there exist a surface S corresponding to Σ with orthogonality of linear elements and such that its asymptotic lines are parametric.§ If we take S for the middle surface of a congruence of Ribaucour whose director-surface is Σ, we have a congruence of the kind sought.

Since the developables of a congruence of Ribaucour correspond to the asymptotic lines on the director surface, the conditions (10) that the characteristic ruled surfaces be parametric are equivalent to equations (36). Hence we have

THEOREM 7. *The characteristic ruled surfaces of a congruence of Ribaucour cut the middle surface in the curves which correspond to the characteristic conjugate system on the director-surface.*

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*E., p. 420.

† Loc. cit., p. 435.

‡ E., p. 201.

§ E., p. 380.

In order that these curves on the middle surface may be asymptotic lines, the condition (37) must be satisfied, and consequently we have

THEOREM 8. *Let Σ be a surface of positive curvature whose characteristic lines form a conjugate system with equal point invariants and let S be the unique surface corresponding to Σ with orthogonality of linear elements in such a way that its asymptotic lines correspond to the characteristic lines on Σ; then S is the middle surface and Σ the director-surface of a congruence of Ribaucour whose characteristic ruled surfaces cut S in its asymptotic lines; and these are the only congruences of this sort.*

§ 4. *The Case $E = G$.*

We shall establish the existence of such congruences, by showing that there exist upon the unit sphere systems of curves which are such that when they are parametric the conditions $E = G$ and (35) are satisfied. The latter conditions become in this case

$$(38) \qquad \frac{\partial}{\partial u}\left\{ {12 \atop 1} \right\}' = \frac{\partial}{\partial v}\left\{ {12 \atop 2} \right\}', \qquad \frac{\partial}{\partial u}\left\{ {11 \atop 2} \right\}' = \frac{\partial}{\partial v}\left\{ {22 \atop 1} \right\}'.$$

If we put

$$(39) \qquad E = G = \lambda, \qquad F = \lambda \cos \omega,$$

the first of these conditions may be written

$$\frac{\partial}{\partial u}\left(\frac{\dfrac{\partial \log \lambda}{\partial v} - \cos \omega \dfrac{\partial \log \lambda}{\partial u}}{\sin \omega} \right) = \frac{\partial}{\partial v}\left(\frac{\dfrac{\partial \log \lambda}{\partial u} - \cos \omega \dfrac{\partial \log \lambda}{\partial v}}{\sin \omega} \right),$$

and the second is reducible by means of the first to

$$\frac{\partial}{\partial u}\left(\frac{1}{\sin \omega} \frac{\partial \omega}{\partial u} \right) = \frac{\partial}{\partial v}\left(\frac{1}{\sin \omega} \frac{\partial \omega}{\partial v} \right),$$

or in other form

$$\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) \log \tan \frac{\omega}{2} = 0 .$$

The general solution of this equation may be given the form

$$(40) \qquad \tan^2 \frac{\omega}{2} = \frac{\psi(u - v)}{\phi(u + v)},$$

where ϕ and ψ are arbitrary functions.

From the first of equations (38) it follows that there is a surface S_0 whose asymptotic lines have the given representation on the sphere. Since $E = G$, the equation of the lines of curvature on S_0 is $du^2 - dv^2 = 0$, so that if

$$u_1 = u + v, \qquad v_1 = u - v,$$

the curves $u_1 =$ const., $v_1 =$ const. are lines of curvature. From (40) it follows that

$$(40') \qquad \cos\omega = \frac{\phi(u+v) - \psi(u-v)}{\phi(u+v) + \psi(u-v)} = \frac{U(u_1) - V(v_1)}{U(u_1) + V(v_1)},$$

and consequently the linear element of the spherical representation is

$$d\sigma^2 = \frac{\lambda}{U+V}(U du_1^2 + V dv_1^2).$$

Moreover, the linear element of S_0 is of the form *

$$ds_0^2 = \mu\left(\frac{du_1^2}{U} + \frac{dv_1^2}{V}\right),$$

where μ is a determinate function. Hence S_0 is an isothermic surface whose lines of curvature are represented on the sphere by an isothermal system. Conversely, it can be shown that when a surface of this kind is referred to its asymptotic lines conditions (38) and (40) are satisfied. †

To this group of surfaces belong the quadrics, cyclides of Dupin, surfaces of revolution, minimal surfaces, certain surfaces with plane lines of curvature in both systems, ‡ and a group of systems recently discussed by A. E. Young. §

Suppose that S_0 is such a surface referred to its asymptotic lines, and that S_1 is the unique surface whose characteristic lines have the same spherical representation as S_0. In consequence of (39) these lines on S_1 form an isothermal-conjugate system, and since the second of equations (38) is satisfied, this conjugate system has equal point invariants.|| Hence there exists a surface S, corresponding to S_1 with orthogonality of linear elements, whose asymptotic lines correspond to the characteristic lines on S_1. From Theorem 7 it follows that the congruence of Ribaucour for which S is the middle surface and S_1 the director surface is such that the characteristic ruled surfaces cut S in its asymptotic lines, and thus we have a congruence of the kind sought.

It is evident that S_0 is an associate surface of S_1 and consequently determines a surface S' corresponding to S_1 with orthogonality of linear elements. When S_0 is referred to its asymptotic lines, the parametric curves on S' form a conjugate system. In view of Theorem 7 we have that the characteristic ruled surfaces of the congruence of Ribaucour, whose director-surface is S_1 and middle surface S_1', cut S' in a conjugate system.

In general, when the characteristic ruled surfaces of a congruence of Ribaucour

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* E., p. 192.

† Cf. A. E. Young, *On a certain class of isothermic surfaces*, Transactions of the American Mathematical Society, vol. 10 (1909), pp. 79–94.

‡ Eisenhart, *Isothermal-conjugate systems of lines on surfaces*, American Journal of Mathematics, vol. 25 (1902), pp. 239–248.

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|| E., p. 380.

cut the middle surface in a conjugate system, the spherical representation of these surfaces is that of the asymptotic lines on that associate surface of the director-surface which is determined by the middle surface. Hence we have

THEOREM 9. *A necessary and sufficient condition that the characteristic ruled surfaces of a congruence of Ribaucour cut the middle surface in a conjugate system is that the spherical representation of these surfaces satisfy the conditions*

$$(41)\quad \begin{aligned} &\frac{\partial}{\partial u}\left\{ {12 \atop 1} \right\}' = \frac{\partial}{\partial v}\left\{ {12 \atop 2} \right\}', \\ &\frac{\partial^2 \log \frac{E}{G}}{\partial u \partial v} + \frac{\partial}{\partial u}\left(\frac{G}{E}\left\{ {11 \atop 2} \right\}' \right) - \frac{\partial}{\partial v}\left(\frac{E}{G}\left\{ {22 \atop 1} \right\}' \right) = 0. \end{aligned}$$

The knowledge of one such system of curves on the sphere leads to the determination of another. This results from

THEOREM 10. *If the characteristic lines on a surface correspond to the asymptotic lines on an associate surface, the characteristic lines on the latter correspond to the asymptotic lines on the former.*

For, let S_1 and S_0 be two associate surfaces with the characteristic lines and asymptotic lines respectively parametric; then

$$(42)\quad \frac{D_1}{E_1} = \frac{D_1''}{G_1}, \qquad D_1' = 0, \qquad D_0 = D_0'' = 0, \qquad D_0' = -\mu D_1'' = \sigma D_1,$$

where μ and σ are two functions such that the coördinates of the two surfaces are in the relations

$$\frac{\partial x_0}{\partial u} = -\mu \frac{\partial x_1}{\partial v}, \qquad \frac{\partial \bar{x}_0}{\partial v} = \sigma \frac{\partial x_1}{\partial u},$$

and similarly for the y's and z's.* From these we have

$$(43)\qquad E_0 = \mu^2 G_1, \qquad G_0 = \sigma^2 E_1,$$

the functions E_1, G_1 in (42) and (43) being the coefficients of the linear element of S_1, and E_0, G_0 the corresponding functions for S_0. The differential equation of the characteristic lines on S_0 is reducible to†

$$E_0 du^2 + G_0 dv^2 = 0,$$

which by means of (42) and (43) is equivalent to

$$D_1 du^2 + D_1'' dv^2 = 0;$$

consequently the theorem is proved.

---

* E., pp. 378, 380.

† E., p. 131.

Since asymptotic lines and characteristic lines are real only on surfaces of negative and positive curvature respectively, it follows that if a system satisfying (41) is real, the similar system obtained by means of Theorem 10 is imaginary, and *vice-versa.*

§ 5. *Isothermic Systems of Ruled Surfaces. The Pseudofocal Surfaces.*

Since the quadratic form

$$\Phi \equiv A du^2 + 2 B du dv + C dv^2$$

is definite for a congruence of the elliptic type, there exist double systems of ruled surfaces which are such that, when a system of this kind is parametric, the coefficients of the form Φ satisfy the conditions

$$A = C, \qquad B = 0. \tag{44}$$

The determination of these double systems of ruled surfaces is the same analytical problem as that of isothermic orthogonal systems of curves on a surface.* For this reason we say that ruled surfaces satisfying the conditions (44) form an *isothermic system.*

On the assumption that such a system is parametric, we have from (6)

$$Ef' + Gf - F(e+g) = 0, \qquad Eg - Ge + F(f' - f) = 0. \tag{45}$$

Furthermore, if we take the middle surface of the congruence for the surface of reference, we have the condition (13). Combining the latter and (45), we find that

$$\frac{e}{F} = -\frac{f}{E} = \frac{f'}{G} = -\frac{g}{F} = \sigma, \tag{46}$$

where σ denotes the factor of proportionality.

When the values (46) are substituted in the first two of equations (22), we obtain

$$\gamma = -\frac{\partial \sigma}{\partial v} - \sigma S, \qquad \gamma_1 = \frac{\partial \sigma}{\partial u} + \sigma T, \tag{47}$$

where we have put

$$S = \left\{ {11 \atop 2} \right\}' + \left\{ {22 \atop 2} \right\}', \qquad T = \left\{ {22 \atop 1} \right\}' + \left\{ {11 \atop 1} \right\}'. \tag{48}$$

And the third of equations (22) is reducible to

$$\frac{\partial^2 \sigma}{\partial u^2} + \frac{\partial^2 \sigma}{\partial v^2} + T\frac{\partial \sigma}{\partial u} + S\frac{\partial \sigma}{\partial v} + \left(\frac{\partial T}{\partial u} + \frac{\partial S}{\partial v} + E + G\right)\sigma = 0. \tag{49}$$

* Cf., E., p. 93.

Moreover, from (21) we derive the equations of the middle surface in the form

$$(50)\qquad \begin{aligned} \frac{\partial x}{\partial u} &= \sigma\frac{\partial X}{\partial v} - \left(\frac{\partial \sigma}{\partial v} + \sigma S\right)X, \\ \frac{\partial x}{\partial v} &= -\sigma\frac{\partial X}{\partial u} + \left(\frac{\partial \sigma}{\partial u} + \sigma T\right)X. \end{aligned}$$

Conversely, when the parametric lines on the sphere are any whatever, and σ is any solution of equation (49), equations (50) define the coördinates of the middle surface of a congruence referred to an isothermic system of ruled surfaces.*

We shall consider now the two surfaces S_1 and S_2, whose coördinates are given by

$$(51)\qquad \begin{aligned} x_1 &= x - \sigma X, & y_1 &= y - \sigma Y, & z_1 &= z - \sigma Z, \\ x_2 &= x + \sigma X, & y_2 &= y + \sigma Y, & z_2 &= z + \sigma Z. \end{aligned}$$

In the first place we remark that S_1 and S_2, thus defined, are the same surfaces for all isothermic systems. In fact, if the values (46) be substituted in the second of (24), we find the following relation between σ and the abscissa of an imaginary focal point

$$(52)\qquad \sigma = i\rho.$$

Because of this result we call S_1 and S_2 the *pseudofocal surfaces* of the congruence.

§ 6. *Conjugate-Potential Surfaces. Congruences of Lilienthal.*

Of particular interest is the case when the tangent planes at corresponding points of S_1 and S_2 are parallel. In order to investigate this case, we differentiate equations (51), with the result

$$(53)\qquad \begin{aligned} \frac{\partial x_1}{\partial u} &= -\sigma\left(\frac{\partial X}{\partial u} - \frac{\partial X}{\partial v}\right) - \left(\frac{\partial \sigma}{\partial u} + \frac{\partial \sigma}{\partial v} + \sigma S\right)X, \\ \frac{\partial x_1}{\partial v} &= -\sigma\left(\frac{\partial X}{\partial u} + \frac{\partial X}{\partial v}\right) + \left(\frac{\partial \sigma}{\partial u} - \frac{\partial \sigma}{\partial v} + \sigma T\right)X, \end{aligned}$$

$$(54)\qquad \begin{aligned} \frac{\partial x_2}{\partial u} &= \sigma\left(\frac{\partial X}{\partial u} + \frac{\partial X}{\partial v}\right) + \left(\frac{\partial \sigma}{\partial u} - \frac{\partial \sigma}{\partial v} - \sigma S\right)X, \\ \frac{\partial x_2}{\partial v} &= -\sigma\left(\frac{\partial X}{\partial u} - \frac{\partial X}{\partial v}\right) + \left(\frac{\partial \sigma}{\partial u} + \frac{\partial \sigma}{\partial v} + \sigma T\right)X. \end{aligned}$$

If X_1, Y_1, Z_1; X_2, Y_2, Z_2 denote the direction-cosines of the normals to S_1

* Cf. SANNIA, *Nuova esposizione della geometria infinitesimale delle congruenze rettilinee*, Annali di matematica, ser. 3, vol. 15 (1908), pp. 39, 40.

and S_2, we find from (53) and (54)

$$
(55)\qquad
\begin{aligned}
X_1 &= \frac{\sigma^2}{\sqrt{E_1G_1 - F_1^2}}\left\{2XH + \left(Y\frac{\partial Z}{\partial u} - Z\frac{\partial Y}{\partial u}\right)\left(2\,\frac{\partial \log\sigma}{\partial u} + S + T\right)\right. \\
&\qquad\left. + \left(Y\frac{\partial Z}{\partial v} - Z\frac{\partial Y}{\partial v}\right)\left(2\,\frac{\partial \log\sigma}{\partial v} + S - T\right)\right\}, \\
X_2 &= \frac{\sigma^2}{\sqrt{E_2G_2 - F_2^2}}\left\{2XH + \left(Y\frac{\partial Z}{\partial u} - Z\frac{\partial Y}{\partial u}\right)\left(-2\,\frac{\partial \log\sigma}{\partial u} + S - T\right)\right. \\
&\qquad\left. - \left(Y\frac{\partial Z}{\partial v} - Z\frac{\partial Y}{\partial v}\right)\left(2\,\frac{\partial \log\sigma}{\partial v} + S + T\right)\right\},
\end{aligned}
$$

where E_1, F_1, G_1; E_2, F_2, G_2 are the first fundamental coefficients of S_1 and S_2. From these expressions it follows that the necessary and sufficient condition that the tangent planes to S_1 and S_2 be parallel is that

$$
(56)\qquad E_1G_1 - F_1^2 = E_2G_2 - F_2^2
$$

and

$$
(57)\qquad \frac{\partial \log\sigma}{\partial u} = -\tfrac{1}{2}T, \qquad \frac{\partial \log\sigma}{\partial v} = -\tfrac{1}{2}S.
$$

When the values from (57) are substituted in (53) and (54), it is found that

$$
(58)\qquad
\begin{aligned}
\frac{\partial x_1}{\partial u} &= \frac{\partial x_2}{\partial v} = -\sigma\left[\left(\frac{\partial X}{\partial u} - \frac{\partial X}{\partial v}\right) + \tfrac{1}{2}(S - T)X\right], \\
\frac{\partial x_1}{\partial v} &= -\frac{\partial x_2}{\partial u} = -\sigma\left[\left(\frac{\partial X}{\partial u} + \frac{\partial X}{\partial v}\right) - \tfrac{1}{2}(S + T)X\right].
\end{aligned}
$$

From these equations it follows that $x_1 + ix_2$, $y_1 + iy_2$, $z_1 + iz_2$ are analytic functions of $u + iv$. Hence we say that S_1 and S_2 are *conjugate-potential surfaces*, and the congruence consists of the joins of corresponding points of the pair.

Conversely, given any three analytic functions, consider the congruence of linesjoining corresponding points of the surfaces S_1 and S_2 so determined. The direction-cosines of the lines are of the form

$$
X = \frac{x_2 - x_1}{2\sigma}, \qquad Y = \frac{y_2 - y_1}{2\sigma}, \qquad Z = \frac{z_2 - z_1}{2\sigma},
$$

where 2σ denotes the distance between the points on S_1 and S_2. From these we obtain

$$
(59)\qquad
\begin{aligned}
\frac{\partial X}{\partial u} &= -\frac{1}{2\sigma}\left(\frac{\partial x_1}{\partial v} + \frac{\partial x_1}{\partial u} + 2\,\frac{\partial \log\sigma}{\partial u}X\right), \\
\frac{\partial X}{\partial v} &= \frac{1}{2\sigma}\left(\frac{\partial x_1}{\partial u} - \frac{\partial x_1}{\partial v} - 2\,\frac{\partial \log\sigma}{\partial v}X\right).
\end{aligned}
$$

If x, y, z denote the coördinates of the mid-point of the join of corresponding points of S_1 and S_2, equations (59) may be given the form

$$(60)\quad \frac{\partial X}{\partial u} = -\frac{1}{\sigma}\left(\frac{\partial x}{\partial v} + 2\frac{\partial \log \sigma}{\partial u} X\right), \qquad \frac{\partial X}{\partial v} = \frac{1}{\sigma}\left(\frac{\partial x}{\partial u} - 2\frac{\partial \log \sigma}{\partial v} X\right).$$

From these equations follow equations (46). Hence the surface S is the middle surface of the congruence, and the parametric ruled surfaces form an isothermic system. Congruences of this kind were considered by Lilienthal,* and so we shall refer to them as *congruences of Lilienthal.*

The preceding results may be stated thus:

THEOREM 11. *A necessary and sufficient condition that the pseudofocal surfaces of a congruence correspond with parallelism of tangent planes is that the congruence be of the Lilienthal type; in this case the pseudofocal surfaces are conjugate-potential.*

From (57) it follows that a necessary condition that the parametric lines on the sphere represent an isothermic system of ruled surfaces of a congruence of Lilienthal is

$$(61)\qquad \frac{\partial}{\partial v} T = \frac{\partial}{\partial u} S.$$

Furthermore, the function σ given by (57) must satisfy equation (49). This gives the further condition

$$(62)\qquad 2\left[\frac{\partial T}{\partial u} + \frac{\partial S}{\partial v} + 2(E+G)\right] = T^2 + S^2.$$

Conversely, from the general theory it follows that when conditions (61) and (62) are satisfied, there exists a congruence of Lilienthal which can be found by quadratures.

In consequence of the identity †

$$\frac{\partial}{\partial v}\left(\left\{ {11 \atop 1} \right\}' + \left\{ {12 \atop 2} \right\}'\right) = \frac{\partial}{\partial u}\left(\left\{ {22 \atop 2} \right\}' + \left\{ {12 \atop 1} \right\}'\right),$$

equation (61) is equivalent to

$$\frac{\partial}{\partial v}\left(\left\{ {12 \atop 2} \right\}' - \left\{ {22 \atop 1} \right\}'\right) = \frac{\partial}{\partial u}\left(\left\{ {12 \atop 1} \right\}' - \left\{ {11 \atop 2} \right\}'\right).$$

But this is the condition ‡ that the parametric lines on the sphere represent a real isothermal-conjugate system on a surface Σ whose coördinates ξ, η, ζ are given by the equations

$$(63)\quad \frac{\partial \xi}{\partial u} = \frac{t}{H^2}\left(-G\frac{\partial X}{\partial u} + F\frac{\partial X}{\partial v}\right), \qquad \frac{\partial \xi}{\partial v} = \frac{t}{H^2}\left(F\frac{\partial X}{\partial u} - E\frac{\partial X}{\partial v}\right),$$

* *Untersuchungen zur allgemeinen Theorie der krummen Oberflächen und geradlinigen Strahlensysteme* (Bonn, 1886), p. 80.

† E., p. 153.

‡ E., p. 202.

where t is given by

$$\frac{\partial \log t}{\partial u} = \left\{ {12 \atop 2} \right\}' - \left\{ {22 \atop 1} \right\}', \qquad \frac{\partial \log t}{\partial v} = \left\{ {12 \atop 1} \right\}' - \left\{ {11 \atop 2} \right\}'.$$

From (50) and (63) we find that

$$dx\,d\xi + dy\,d\eta + dz\,d\zeta = 0.$$

Hence:

THEOREM 12. *A congruence of Lilienthal is a congruence of Ribaucour.*

If we put

$$u = \bar{u} + \bar{v}, \qquad v = i(\bar{v} - \bar{u}),$$

the ruled surfaces $\bar{u} = \text{const.}$, $\bar{v} = \text{const.}$ are the developables. If $\bar{E}$, $\bar{F}$, $\bar{G}$ denote the fundamental coefficients of the sphere in this case, we find that the conditions (61) and (62) may be given the form

$$(64) \qquad \frac{\partial}{\partial \bar{u}} \left\{ {\overline{12} \atop 1} \right\}' = \frac{\partial}{\partial \bar{v}} \left\{ {\overline{12} \atop 2} \right\}' = \left\{ {\overline{12} \atop 1} \right\}' \left\{ {\overline{12} \atop 2} \right\}' - \bar{F},$$

where the symbols $\left\{ {\overline{rs} \atop t} \right\}$ are formed with respect to $\bar{E}d\bar{u}^2 + 2\bar{F}d\bar{u}\,d\bar{v} + \bar{G}d\bar{v}^2$.

When a real system of lines on the sphere satisfies (64), these lines represent the developables of a congruence whose focal surfaces are curves;* moreover, the semi-focal distance is given by quadratures.

Since the tangent planes to two conjugate-potential surfaces S_1 and S_2 at corresponding points are parallel, it is readily shown that the second fundamental coefficients of these surfaces satisfy the relations

$$D_1 = -D_1'' = D_2', \qquad D_1' = -D_2 = D_2'',$$

and consequently

$$D_1 D_2'' + D_1'' D_2 - 2D_1' D_2' = 0.$$

Hence we have

THEOREM 13. *Conjugate-potential surfaces are associate to each other.*

§ 7. *Congruences which consist of the Lines joining Corresponding Points on Associate Surfaces.*

We have seen that the abscissæ of the focal points are $i\sigma$, $-i\sigma$; hence, the above result follows also from the following known theorem †:

THEOREM 14. *In order that two surfaces S_1 and S_2 corresponding with parallelism of tangent planes be associate surfaces, it is necessary and sufficient that for the congruence formed by the joins of corresponding points M_1 and M_2 of these surfaces, the focal points and the points M_1 and M_2 form a harmonic range.*

* E., p. 412.

† E., p. 425.

We inquire whether a pair of associate surfaces is connected with every congruence after the manner described in the preceding theorem.

We assume that a given congruence possesses this property and that the two associate surfaces meet a line of the congruence in points whose distances from the middle point of the line are denoted by t_1 and t_2. By Theorem 14 we have $t_1 t_2 = \rho^2$, and by (52)

$$t_1 t_2 = -\sigma^2. \tag{65}$$

We must express the condition that the surfaces Σ_1 and Σ_2, defined by

$$\xi_1 = x + t_1 X, \qquad \eta_1 = y + t_1 Y, \qquad \zeta_1 = z + t_1 Z,$$
$$\xi_2 = x + t_2 X, \qquad \eta_2 = y + t_2 Y, \qquad \zeta_2 = z + t_2 Z,$$

correspond with parallelism of tangent planes.

By means of (50) we have

$$\begin{aligned} \frac{\partial \xi_1}{\partial u} &= t_1 \frac{\partial X}{\partial u} + \sigma \frac{\partial X}{\partial v} + \left(\frac{\partial t_1}{\partial u} - \frac{\partial \sigma}{\partial v} - \sigma S\right) X, \\ \frac{\partial \xi_1}{\partial v} &= -\sigma \frac{\partial X}{\partial u} + t_1 \frac{\partial X}{\partial v} + \left(\frac{\partial t_1}{\partial v} + \frac{\partial \sigma}{\partial u} + \sigma T\right) X, \end{aligned} \tag{66}$$

and similar expressions in η_1 and ζ_1. From these we find

$$\begin{aligned} \sqrt{E_1 G_1 - F_1^2}\, X_1 = (t_1^2 + \sigma^2) H X &+ \left[t_1 \left(\frac{\partial t_1}{\partial v} + \frac{\partial \sigma}{\partial u} + \sigma T\right) \right. \\ &+ \left. \sigma \left(\frac{\partial t_1}{\partial u} - \frac{\partial \sigma}{\partial v} - \sigma S\right)\right] \left(Z \frac{\partial Y}{\partial u} - Y \frac{\partial Z}{\partial u}\right) \\ &+ \left[\sigma \left(\frac{\partial t_1}{\partial v} + \frac{\partial \sigma}{\partial u} + \sigma T\right) - t_1 \left(\frac{\partial t_1}{\partial u} - \frac{\partial \sigma}{\partial v} - \sigma S\right)\right] \left(Z \frac{\partial Y}{\partial v} - Y \frac{\partial Z}{\partial v}\right), \end{aligned}$$

where E_1, F_1, G_1 are the first fundamental coefficients of Σ_1 and X_1, Y_1, Z_1 are the direction courses of the normal to Σ_1.

The expressions for X_2, Y_2, Z_2 are similar to the above. The necessary and sufficient conditions that these respective quantities be equal are reducible to

$$t_1 \sqrt{E_2 G_2 - F_2^2} = - t_2 \sqrt{E_1 G_1 - F_1^2}, \tag{67}$$

$$\frac{\partial \theta}{\partial v} + 4\sigma \frac{\partial \sigma}{\partial u} + 2\sigma^2 T + S\theta = 0, \qquad \frac{\partial \theta}{\partial u} - 4\sigma \frac{\partial \sigma}{\partial v} - 2\sigma^2 S + T\theta = 0 \tag{68}$$

where, for the sake of brevity, we have put

$$\theta = t_1 \sigma - \frac{\sigma^3}{t_1}. \tag{69}$$

By means of (49) the condition of integrability of (68) is reducible to

$$
\begin{aligned}
\frac{1}{4}\left(\frac{\partial S}{\partial u}-\frac{\partial T}{\partial v}\right)\theta+\left(\frac{\partial \sigma}{\partial u}\right)^2+\left(\frac{\partial \sigma}{\partial v}\right)^2+\sigma\left(S\frac{\partial \sigma}{\partial v}+T\frac{\partial \sigma}{\partial u}\right)\\
-\frac{\sigma^2}{2}\left[\frac{\partial T}{\partial u}+\frac{\partial S}{\partial v}+2(E+G)-(T^2+S^2)\right]=0.
\end{aligned}
\tag{70}
$$

Hence in order that two associate surfaces Σ_1 and Σ_2 exist it is necessary that the function θ given by (70) satisfy equations (68). Moreover, it is readily shown that these conditions are sufficient.

When in particular $\theta = 0$, equations (68) reduce to (57) and then (70) is satisfied. This is the case of congruences of Lilienthal.

It may be shown that equation (61) is equivalent to the first of equations (64) and consequently is the condition that the congruence be of the Ribaucour type. From this and (70) it results that for the congruences of Ribaucour, for which σ is a solution of (49) and

$$
\begin{aligned}
\left(\frac{\partial \sigma}{\partial u}\right)^2+\left(\frac{\partial \sigma}{\partial v}\right)^2+\sigma\left(S\frac{\partial \sigma}{\partial v}+T\frac{\partial \sigma}{\partial u}\right)\\
-\frac{\sigma^2}{2}\left[\frac{\partial T}{\partial u}+\frac{\partial S}{\partial v}+2(E+G)-(T^2+S^2)\right]=0,
\end{aligned}
\tag{71}
$$

the function θ involves a parameter, and consequently there is an infinity of pairs of associate surfaces such as Σ_1 and Σ_2. Moreover, their determination requires quadratures only.

Since congruences of Lilienthal are of the Ribaucour type, the above result is applicable to these congruences; for equation (71) is satisfied by the function σ given by (57). Now equations (68) reduce to

$$
\frac{\partial \theta}{\partial v}-2\frac{\partial \log \sigma}{\partial v}\cdot\theta=0, \qquad \frac{\partial \theta}{\partial u}-2\frac{\partial \log \sigma}{\partial u}\theta=0,
$$

of which the general solution is

$$
\theta = c\sigma^2,
$$

c being an arbitrary constant. From this result, (69), and (65), we have

$$
t_1 = a\sigma, \qquad t_2 = -\frac{\sigma}{a},
$$

where a is an arbitrary constant. Hence we have

THEOREM 15. *With a congruence of Lilienthal there are associated an infinity of pairs of associate surfaces; two of these are conjugate-potential surfaces which cut a line of the congruence at points distant* σ *and* $-\sigma$ *from the middle point; and corresponding points of any other pair are at distances* $a\sigma$, $-\sigma/a$, *where* a *is a constant.*

PRINCETON, *September*, 1909.