AN APPLICATION OF SYMBOLIC METHODS TO THE TREATMENT OF MEAN CURVATURES IN HYPERSPACE*

BY

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This paper is an application of MASCHKE's symbolic method for discussing invariants of quadratic differential forms, as developed in his article, *A Symbolic Treatment of the Theory of Invariants of Quadratic Differential Quantics of n Variables.*† Extensive use is also made of results and methods contained in two later publications, *Differential Parameters of the First Order*,‡ and *The Kronecker-Gaussian Curvature of Hyperspace.* § Some familiarity with these three articles is implied.

Part I of the present paper is devoted to the study of the curvatures of an n-space R_n in an euclidean $(n+1)$-space S_{n+1}. In §§ 1–3 the equations and some of the properties of the lines of curvature of R_n in S_{n+1} are developed. In particular, equation (28) gives the n curvatures of the n lines of curvature through a given point of R_n. The coefficients $K_1, \cdots, K_n$ of this equation are the so-called curvatures of R_n in S_{n+1}, involving the coefficients a_{ik} and α_{ik} of the two fundamental forms of R_n. With the help of his symbolic method,|| MASCHKE has expressed K_n, when n is even, and K_n^2, when n is odd, as rational integral functions of the coefficients a_{ik} of the first fundamental form and their derivatives.

In §§ 4–6 similar expressions are derived for all the curvatures $K_{2\nu}$ of even index. It does not seem possible to obtain rational results for the curvatures $K_{2\nu+1}$ of odd index. In § 7, however, it is shown that, with the exception of K_1, these curvatures are expressible irrationally in terms of the first fundamental quantities and their derivatives.

The symbolic expressions for $K_{2\nu}$ and K_n^2 show at once that they are differential invariants of the first fundamental quadratic form for R_n, and they have meaning as invariants of any quadratic form in n variables. Part II of this paper considers a space R_λ defined in a space $R_n (n > \lambda)$, which is not neces-

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† These Transactions, vol. 4 (1903), pp. 445–469. This paper is referred to hereafter as M. I.

‡ *Ibid.*, vol. 7 (1906), pp. 69–80; referred to as D. P.

§ *Ibid.*, vol. 7 (1906), pp. 81–93; referred to as K.-G. C.

|| In K.-G. C.

sarily euclidean. The invariants $K_{2\nu}$ and K_λ^2 for R_λ are calculated in terms of the coefficients a_{ik} belonging to the length element of R_n and of the functions $U^{\lambda+1}, \cdots, U^n$ which determine the space R_λ in R_n.

PART I.

Curvatures of an n-space in an $(n+1)$-space.

§ 1. *Parametric Representation for an n-space in an $(n+1)$-space.*

Let $z', z^2, \cdots, z^{n+1}$ be the coördinates* of an euclidean space S_{n+1} of $n+1$ dimensions, i. e., a space whose arc-element is of the form

$$(1) \qquad ds^2 = \sum_{i=1}^{n+1} [dz^i]^2.$$

We define in S_{n+1} any hypersurface, or space R_n, of n dimensions, by expressing each z as a funcion of n independent variables $x_1, \cdots, x_n$:

$$(2) \qquad \begin{aligned} z' &= z'(x_1, \cdots, x_n), \\ &\cdots\cdots\cdots \\ z^{n+1} &= z^{n+1}(x_1, \cdots, x_n). \end{aligned}$$

The arc-element of R_n is given by the equation

$$(3) \qquad ds^2 = \sum_{i,k}^{1,\ldots,n} a_{ik}\, dx_i\, dx_k,$$

where

$$(4) \qquad a_{ik} = \sum_{j=1}^{n+1} \frac{\partial z^j}{\partial x_i} \frac{\partial z^j}{\partial x_k} = \sum_{j=1}^{n+1} z_i^j z_k^j = f_i f_k,$$

differentiation with respect to x_i being indicated here, as in the following pages, by the lower index i.

A space of λ dimensions, $\lambda < n$, would be obtained by using in (2) only λ independent variables $x_1, \cdots, x_\lambda$. In particular, a curve in S_{n+1} is obtained by expressing each z as a function of one new variable x.

If in (2) one puts $x_2 = \cdots = x_n = 0$, the resulting curve is called the x_1-axis of a curvilinear system of coördinates on R_n. By letting $x_2, \cdots, x_n$ represent arbitrary constants, one gets the complete system of x_1-curves; and similarly for the other cases.

$$(5) \qquad \begin{aligned} &x_1\text{-curves if } x_1 \text{ varies and } x_2, \cdots, x_n \text{ are constants,} \\ &\cdots\cdots\cdots\cdots \\ &x_n\text{-curves if } x_n \text{ varies and } x_1, \cdots, x_{n-1} \text{ are constants.} \end{aligned}$$

Equations (3) and (5) give the elements of the new axes,

$$(6) \qquad ds_1^2 = a_{11}\, dx_1^2, \cdots, ds_n^2 = a_{nn}\, dx_n^2,$$

* It is assumed that no confusion will arise from writing the upper index, as MASCHKE does, without parentheses. Exponents are only occasionally used, and will be easily recognized.

where ds_k is the element of the x_k-axis. Represent the direction cosines of the x_k-axis, in the old system, by $\cos(z', x_k), \cdots, \cos(z^{n+1}, x_k)$. Then

$$\cos(z^j, x_k) = \frac{dz^j}{ds_k} = \frac{z_k^j\, dx_k}{\sqrt{a_{kk}}\, dx_k} = \frac{z_k^j}{\sqrt{a_{kk}}} \quad (j=1, \cdots, n+1; k=1, \cdots, n). \tag{7}$$

Let ω_{ik} be the angle between the x_i-axis and the x_k-axis. Then

$$\cos\omega_{ik} = \sum_{j=1}^{n+1} \cos(z^j x_i)\cos(z^j x_k) = \sum_{j=1}^{n+1} \frac{z_i^j z_k^j}{\sqrt{a_{ii}}\sqrt{a_{kk}}} = \frac{a_{ik}}{\sqrt{a_{ii}}\sqrt{a_{kk}}} \quad (i, k=1, \cdots, n), \tag{8}$$

so that *necessary and sufficient conditions for mutual orthogonality of the axes of the new system are*

$$a_{ik} = 0 \qquad (i, k = 1, \cdots, n;\ i \neq k). \tag{9}$$

§ 2. *General Curves on R_n.*

A general curve on R_n may be defined by means of $n-1$ equations,

$$U^2(x_1, \cdots, x_n) = \text{const.}, \cdots, U^n(x_1, \cdots, x_n) = \text{const.} \tag{10}$$

The differential equations of this curve, which we call the U-curve, are

$$\sum_{i=1}^{n} U_i^2 dx_i = 0, \ \cdots, \ \sum_{i=1}^{n} U_i^n dx_i = 0. \tag{11}$$

Its direction is defined by the ratios of $dx_1, \cdots, dx_n$ in (11). In order to solve for these differentials, let p be any function of $x_1, \cdots, x_n$ which satisfies the condition *

$$D = (pU^2 \cdots U^n) = (pU) \neq 0.$$

If A^r denotes the cofactor of p_r in D, equations (11) are identically satisfied by

$$dx_1 = \rho A', \cdots, dx_n = \rho A^n, \tag{12}$$

where ρ is an arbitrary parameter.

The direction cosines $\xi', \cdots, \xi^{n+1}$ of the U-curve are found as follows. From (12),

$$\sum_{i=1}^{n} p_i dx_i = \rho \sum_{i=1}^{n} p_i A^i = \rho(pU). \tag{13}$$

Then

$$\xi^k = \frac{dz^k}{ds} = \frac{1}{ds}\sum_{i=1}^{n} z_i^k dx_i = \frac{\rho}{ds}(z^k U),$$

where ds is arc-element of the U-curve. Now

$$\sum_{k=1}^{n} [\xi^k]^2 = 1 = \sum_{k=1}^{n+1} \left[\frac{\rho}{ds}\right]^2 (z^k U)^2 = \left[\frac{\rho}{ds}\right]^2 (fU)^2.$$

* See M. I., § 2, for an explanation of this invariantive notation.

Hence

$$\frac{\rho}{ds} = \frac{1}{\sqrt{(fU)^2}}.$$

Then the direction cosines of the U-curve on R_n, referred to the original system of axes, are

$$\xi' = \frac{(z'U)}{\sqrt{(fU)^2}}, \quad \cdots, \quad \xi^{n+1} = \frac{(z^{n+1}U)}{\sqrt{(fU)^2}}. \tag{14}$$

If there is given also a V-curve on R_n by equations similar to (10), its direction cosines may be written

$$\eta' = \frac{(z'V)}{\sqrt{(fV)^2}}, \quad \cdots, \quad \eta^{n+1} = \frac{(z^{n+1}V)}{\sqrt{(fV)^2}}. \tag{15}$$

If ω is the angle between the two curves, we have from (14) and (15)

$$\cos\omega = \sum_{i=1}^{n+1} \xi^i \eta^i = \sum_{i=1}^{n+1} \frac{(z^iU)(z^iV)}{\sqrt{(fU)^2}\sqrt{(fV)^2}} = \frac{(fU)(fV)}{\sqrt{(fU)^2}\sqrt{(fV)^2}}. \tag{16}$$

Thus *a necessary and sufficient condition for orthogonality of the two curves is*

$$(fU)(fV) = 0. \tag{17}$$

Equation (17) also defines the orthogonal trajectories of a system of U-curves on R_n. An illustration is found in the case of curves on an ordinary surface.

§ 3. *Lines of Curvature on R.*

A line L drawn on R_n such that the normals to R_n along L (with respect to the enclosing space S_{n+1}) generate a developable surface is called* a line of curvature of R_n in S_{n+1}.

At a point P of R_n there is a unique normal to R_n in S_{n+1}. Let the direction cosines of this normal be $\zeta', \cdots, \zeta^{n+1}$. Choose P as origin of the system of x-axes on R_n. Then, since the normal to R_n at P is orthogonal to every direction on R_n at P, we have from (7)

$$\sum_{i=1}^{n+1} \zeta^i z^i_k = 0 \qquad (k = 1, \cdots, n). \tag{18}$$

The coefficients a_{ik} of the first fundamental form of R_n, given in (3) are the first fundamental quantities. The second fundamental quantities are defined by the equations

$$\alpha_{ik} = \sum_{j=1}^{n+1} \zeta^j z^j_{ik} \qquad (i, k = 1, \cdots, n).$$

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* Cf. BIANCHI, *Lezioni di Geometria Differenziale*, vol. I, p. 125.

By differentiating (18), one obtains

$$(19)\qquad \alpha_{ik}=\sum_{j=1}^{n+1}\zeta^j z^j_{ik}=-\sum_{j=1}^{n+1}\zeta^j_i z^j_k \qquad (i,\ k=1,\ \cdots,\ n).$$

The letters g and γ are to be used in this paper as symbols of the second fundamental quantities

$$(20)\qquad \alpha_{ik}=g_i g_k=\gamma_i\gamma_k=\alpha_{ki}.$$

Let C be the curve of S_{n+1} which is the envelope of the normals along L. Let $M(z', \cdots, z^{n+1})$ be any point of L, and $\overline{M}(\bar{z}', \cdots, \bar{z}^{n+1})$ be the point where the normal at M meets C. Denote by r the distance $M\overline{M}$, which is positive or negative according to the direction of $\overline{M}$ from M. Then

$$(21)\qquad \bar{z}'=z'-r\zeta',\ \cdots,\ \bar{z}^{n+1}=z^{n+1}-r\zeta^{n+1}.$$

Take derivatives of equations (21) with respect to the arc s of L. Then, since C is envelope of the normals along L,

$$(22)\qquad \begin{aligned}\frac{d\bar{z}'}{ds}&=\frac{dz'}{ds}-r\frac{d\zeta'}{ds}-\zeta'\frac{dr}{ds}=q\zeta',\\ &\cdots\cdots\cdots\\ \frac{d\bar{z}^{n+1}}{ds}&=\frac{dz^{n+1}}{ds}-r\frac{dz^{n+1}}{ds}-\zeta^{n+1}\frac{dr}{ds}=q\zeta^{n+1},\end{aligned}$$

where q is a factor of proportionality to be determined. Multiply equations (22) by $\zeta', \cdots, \zeta^{n+1}$ in order and add. We get

$$(23)\qquad q\sum_{i=1}^{n+1}[\zeta^i]^2=\sum_{i=1}^{n+1}\zeta^i\frac{dz^i}{ds}-r\sum_{i=1}^{n+1}\zeta^i\frac{d\zeta^i}{ds}-\frac{dr}{ds}\sum_{i=1}^{n+1}[\zeta^i]^2.$$

Now

$$\sum_{i=1}^{n+1}[\zeta^i]^2=1,\qquad \sum_{i=1}^{n+1}\zeta^i\frac{d\zeta^i}{ds}=0,\qquad \sum_{i=1}^{n+1}\zeta^i\frac{dz^i}{ds}=0,$$

since $\zeta', \cdots, \zeta^{n+1}$ are direction cosines of the normal and $dz'/ds, \cdots, dz^{n+1}/ds$ are direction cosines of L. Substituting these results in (23), one obtains

$$q=-\frac{dr}{ds}.$$

Then equations (22) give

$$(24)\qquad \frac{dz'}{ds}=r\frac{d\zeta'}{ds},\ \cdots,\ \frac{dz^{n+1}}{ds}=r\frac{d\zeta^{n+1}}{ds},$$

or

$$(25)\qquad \frac{dz'}{d\zeta'}=\frac{dz^2}{d\zeta^2}=\cdots=\frac{dz^{n+1}}{d\zeta^{n+1}}=r.$$

This result is expressed in curvilinear coördinates as follows:

$$z_1' dx_1 + \cdots + z_n' dx_n = r(\zeta_1' dx_1 + \cdots + \zeta_n' dx_n),$$
$$\cdots\cdots\cdots\cdots$$
$$z_1^{n+1} dx_1 + \cdots + z_n^{n+1} dx_n = r(\zeta_1^{n+1} dx_1 + \cdots + \zeta_n^{n+1} dx_n).$$

Multiply these equations in order by z_k', $\cdots$, z_k^{n+1} and add. We find

$$\sum_{i=1}^{n+1} z_k^i z_1^i dx_1 + \cdots + \sum_{i=1}^{n+1} z_k^i z_n^i dx_n = r\left[\sum_{i=1}^{n+1} z_k^i \zeta_1^i dx_1 + \cdots + \sum_{i=1}^{n+1} z_k^i \zeta_n^i dx_n\right].$$

Then, by (4) and (19),

$$a_{k1} dx_1 + \cdots + a_{kn} dx_n = - r[\alpha_{k1} dx_1 + \cdots + \alpha_{kn} dx_n] \quad (k = 1, \cdots, n). \tag{26}$$

Equations (26) hold for every line of curvature on R_n. Conversely, if equations (26) be true for any curve L on R_n, then there exists a curve C in S_{n+1} whose tangents are normal to R_n along L, so that L is a line of curvature on R_n by definition.

When equations (26) are written in the form

$$\begin{aligned} &(a_{11} + \alpha_{11} r) dx_1 + \cdots + (a_{1n} + \alpha_{1n} r) dx_n = 0, \\ &\cdots\cdots\cdots\cdots \\ &(a_{n1} + \alpha_{n1} r) dx_1 + \cdots + (a_{nn} + \alpha_{nn} r) dx_n = 0, \end{aligned} \tag{27}$$

it is evident that the curvature $(1/r)$ of each line of curvature through a point P on R_n must satisfy the condition

$$\begin{vmatrix} a_{11} + \alpha_{11} r & a_{12} + \alpha_{12} r & \cdots & a_{1n} + \alpha_{1n} r \\ a_{21} + \alpha_{21} r & a_{22} + \alpha_{22} r & \cdots & a_{2n} + \alpha_{2n} r \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} + \alpha_{n1} r & a_{n2} + \alpha_{n2} r & \cdots & a_{nn} + \alpha_{nn} r \end{vmatrix} = 0, \tag{28}$$

since otherwise equations (27) would have no solution except

$$dx_1 = dx_2 = \cdots = dx_n = 0.$$

Hence the reciprocals of the roots of (28) are exactly the curvatures of the n lines of curvature through a point P on R_n.

The coefficients of r in (28) are called the curvatures of R_n in S_{n+1} and are discussed in §§ 4–7. Before proceeding to that discussion, we derive an important property of lines of curvature.

Expressing (26) in symbolic notation, one finds

$$f_k \sum_{i=1}^{n} f_i dx_i = - r g_k \sum_{i=1}^{n} g_i dx \quad (k = 1, \cdots, n) \tag{29}$$

If now a line of curvature be represented as a U-curve (10), one gets from (13) and (29)

$$(30)\qquad f_k(fU) = -rg_k(gU) \qquad (k=1,\cdots,n).$$

A symmetrical expression for r is obtained by multiplying equations (30) in order by the cofactors of $f_1, \cdots, f_n$ in (fU) and adding:

$$(31)\qquad r = -\frac{(fU)^2}{(gU)^2}.$$

If any two lines of curvature through P be given as U and V-curves, and their respective curvatures be denoted by $1/r'$ and $1/r''$, one gets from (30)

$$g_1(gU) = -\frac{1}{r'}f_1(fU),\ \cdots,\ g_n(gU) = -\frac{1}{r'}f_n(fU),$$

$$g_1(gV) = -\frac{1}{r''}f_1(fV),\ \cdots,\ g_n(gV) = -\frac{1}{r''}f_n(fV).$$

Multiply the equations of the first line in order by the cofactors of $f_1, \cdots, f_n$ in (fV) and add. Also multiply the equations of the second line in order by the cofactors of $f_1, \cdots, f_n$ in (fU) and add. Then

$$(gU)(gV) = -\frac{1}{r'}(fU)(fV) = -\frac{1}{r''}(fU)(fV),$$

so that either $r' = r''$ or $(fU)(fV) = 0$. Hence by (17) we have

Theorem I. *Any two distinct lines of curvature through an ordinary point P of R_n are orthogonal to each other.*

If the lines of curvature through P be taken as parameter lines, then, by (9),

$$a_{ik} = 0 \qquad (i \neq k).$$

It follows at once from (26) that also

$$\alpha_{ik} = 0 \qquad (i, k=1, \cdots, n;\ i \neq k).$$

Theorem II. *If the lines of curvature at an ordinary (not umbilic) point of R_n be taken as parameter lines, then*

$$a_{ik} = 0, \qquad \alpha_{ik} = 0 \qquad (i, k=1, \cdots, n;\ i \neq k).$$

§ 4. *Definition of the Curvatures of R_n in S_{n+1}.*

Equation (28) may be written in the form

$$(32)\qquad H_0 + H_1 r + \cdots + H_{n-1}r^{n-1} + H_n r^n = 0,$$

where*

$$H_0 = |a_{ik}| = 1/\beta^2, \qquad H_n = |\alpha_{ik}|,$$

* M. I., (9).

while for $j=1,\cdots,n$, H_j is the sum of all the determinants obtained from $|a_{ik}|$ by replacing in all possible ways j columns of $|a_{ik}|$ by the corresponding columns of $|\alpha_{ik}|$. Dividing (32) by H_0 one obtains

$$1+K_1r+\cdots+K_{n-1}r^{n-1}+K_nr^n=0. \tag{33}$$

The coefficient K_n (the product of all the curvatures) is the Kronecker-Gaussian curvature of hyperspace. It has been shown to be expressible in terms of the first fundamental quantities and their derivatives (cf. K–G. C.). In this paper the coefficients of (33) are called the n curvatures of R_n in S_{n+1}. By definition

$$K_1=\beta^2\sum_{i,k}^{1,\dots,n}\alpha_{ik}A_k^i,\qquad K_2=\beta^2\sum_{i_1i_2,\,k_1k_2}^{1,\dots,n}\begin{vmatrix}\alpha_{i_1k_1}\alpha_{i_1k_2}\\ \alpha_{i_2k_1}\alpha_{i_2k_2}\end{vmatrix}\cdot A_{k_1k_2}^{i_1i_2}=\sum_{i_1i_2k_1k_2}^{1,\dots,n}\Delta_{\substack{i_1i_2\\k_2k_2}}A_{k_1k_2}^{i_1i_2},\ \cdots,$$

$$K_m=\beta^2\sum_{i_1\dots i_mk_1\dots k_m}^{1,\dots,n}\Delta_{\substack{i_1\dots i_m\\k_1\dots k_m}}\cdot A_{k_1\dots k_m}^{i_1\dots i_m}\qquad(m=1,\cdots,n), \tag{34}$$

where A_k^i is the cofactor of a_{ik} in $|a_{ik}|$, $A_{k_1k_2}^{i_1i_2}$ is the *algebraic complement* of

$$\begin{vmatrix}a_{i_1k_1}a_{i_1k_2}\\ a_{i_2k_1}a_{i_2k_2}\end{vmatrix}$$

in $|a_{ik}|$, while $\Delta_{\substack{i_1i_2\\k_1k_2}}$ is the *second minor* of $|\alpha_{ik}|$ indicated for K_2 above; and similarly for the A's and Δ's in K_m. Both sets $i_1,\cdots,i_m$ and $k_1,\cdots,k_m$ are considered as being in ascending numerical order.

§ 5. *Invariant Symbolic Forms of* $K_1,\cdots,K_n$.

If F_k^i be the cofactor of f_k^i in the functional determinant $\{f',\cdots,f^n\}$, Maschke * has shown that

$$A_k^i=\frac{1}{(n-1)!}F'_iF'_k.$$

Thus

$$K_1=\beta^2\sum_{i,k}^{i,\dots,n}\alpha_{ik}A_k^i=\frac{\beta^2}{(n-1)!}\sum_{i,k}^{i,\dots,n}g_ig_kF'_iF'_k$$

$$=\frac{\beta^2}{(n-1)!}\{gf^2\cdots f^n\}^2=\frac{1}{(n-1)!}(gf)^2.$$

This suggests a method for reducing all the curvatures to convenient invariant forms. Let $F_{i_1\dots i_m}^{1\dots m}$ be the algebraic complement of

$$\begin{vmatrix}f'_{i_1}\cdots f'_{i_m}\\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\\ f^m_{i_1}\cdots f^m_{i_m}\end{vmatrix}$$

* M. I., p. 450.

in $\{f' \cdots f^n\}$. Then the product $F^{1\cdots m}_{i_1\ldots i_m}\cdot F^{1\cdots m}_{k_1\ldots k_m}$ may be written

$$\begin{vmatrix} f_1^{m+1} \cdots f_{i_1-1}^{m+1} f_{i_1+1}^{m+1} \cdots f_{i_m-1}^{m+1} f_{i_m+1}^{m+1} \cdots f_{i_n}^{m+1} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ f_1^{n} \cdots f_{i_1-1}^{n} f_{i_1+1}^{n} \cdots f_{i_m-1}^{n} f_{i_m+1}^{n} \cdots f_{i_n}^{n} \end{vmatrix} F^{1\ldots m}_{k_1\ldots k_m}.$$

If the first determinant of this product be expanded, one finds $(n-m)!$ terms of the form

$$(-1)^\mu f_1 \cdots f_{i_1-1} f_{i_1+1} \cdots f_{i_m-1} f_{i_m+1} \cdots f_n \cdot F^{1\cdots m}_{k_1\ldots k_m},$$

where the suppressed upper indices of the first factor are understood to be any permutation of the numbers $m+1, \cdots, n$, while μ represents the number of inversions in the permutation. Since the equivalent symbols $f^{m+1}, \cdots, f^n$ may be interchanged in all possible ways without altering the value of the term, let them be so interchanged for each term as to reduce the first factor to $(-1)^\mu$ times the principal diagonal term of $F^{1\cdots m}_{i_1\ldots i_m}$. This causes an interchange of rows in the second (determinant) factor $F^{1\cdots m}_{k_1\ldots k_m}$ so that it becomes in each case $(-1)^\mu$ times its original form. Hence the above product becomes

$$(n-m)!\, f_1^{m+1} \cdots f_{i_1-1}^{i_1+m-1} f_{i_1+1}^{i_1+m} \cdots f_{i_m-1}^{i_m} f_{i_m+1}^{i_m+1} \cdots f_n^n \cdot F^{1\cdots m}_{k_1\ldots k_m}.$$

Multiplying each f into the corresponding row of the determinant $F^{1\cdots m}_{k_1\ldots k_m}$ (which has a form similar to that given above for $F^{1\cdots m}_{i_1\ldots i_m}$), we have

$$F^{1\ldots m}_{i_1\ldots i_m}\cdot F^{1\ldots m}_{k_1\ldots k_m} = (n-m)!\, A^{i_1\cdots i_m}_{k_1\ldots k_m},$$

or

$$(35) \qquad A^{i_1\cdots i_m}_{k_1\ldots k_m} = \frac{1}{(n-m)!} F^{1\ldots m}_{i_1\ldots i_m}\cdot F^{1\ldots m}_{k_1\ldots k_m}.$$

Also

$$\Delta_{\substack{i_1\ldots i_m\\ k_1\ldots k_m}} = \begin{vmatrix} a_{i_1k_1} \cdots a_{i_1k_m} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_{i_mk_1} \cdots a_{i_mk_m} \end{vmatrix} = \begin{vmatrix} g'_{i_1} g'_{k_1} \cdots g'_{i_1} g'_{k_m} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ g^m_{i_m} g^m_{k_1} \cdots g^m_{i_m} g^m_{k_m} \end{vmatrix} = g'_{i_1} \cdots g^m_{i_m} \begin{vmatrix} g'_{k_1} \cdots g'_{k_m} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ g^m_{k_1} \cdots g^m_{k_m} \end{vmatrix},$$

$$(36) \qquad \Delta_{\substack{i_1\ldots i_m\\ k_1\ldots k_m}} = \frac{1}{m!} \begin{vmatrix} g'_{i_1} \cdots g'_{i_m} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ g^m_{i_1} \cdots g^m_{i_m} \end{vmatrix} \cdot \begin{vmatrix} g'_{k_1} \cdots g'_{k_m} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ g^m_{k_1} \cdots g^m_{k_m} \end{vmatrix}.$$

Substituting (35) and (36) in (34), one finds, by a well-known theorem of determinants,

$$(37) \quad K_m = \frac{1}{m!(n-m)!}(g' \cdots g^m f^{m+1} \cdots f^n)^2 = \frac{1}{m!(n-m)!}(g' \cdots g^m f)^2.$$

In particular,

$$K_1 = \frac{1}{(n-1)!}(gf)^2 = \Delta_1 g,$$

where, by M. I. (22), $\Delta_1 g$ is the first differential parameter of the first quadratic form (3).

Since the other coefficients are corresponding differential parameters (the number of g's being the same as the subscript of K), it would seem fitting to generalize the notation and set *

$$K_m = \frac{1}{m!\,(n-m)!}(g' \cdots g^m f)^2 = \Delta^m g, \tag{38}$$

with the note that $\Delta' g = \Delta_1 g$.

§ 6. *Expression of $K_{2\nu}$ in terms of the first Fundamental Quantities and Derivatives.*

The generalization of the Gauss equation shows that any second order deter minant of the second fundamental quantities is equal to a Riemann quadruple index symbol, which is expressible in terms of the first fundamental quantities and derivatives.† By K.–G. C. (27),

$$\begin{vmatrix} \alpha_{i_1k_1} & \alpha_{i_1k_2} \\ \alpha_{i_2k_1} & \alpha_{i_2k_2} \end{vmatrix} = (i_1 i_2 k_1 k_2) = \frac{1}{(n-1)!} f'_{i_1} f^2_{i_2} \begin{vmatrix} (fa)'_{k_1} & (fa)'_{k_2} \\ (fa)^2_{k_1} & (fa)^2_{k_2} \end{vmatrix}.$$

By an easy induction, any even order determinant of the α's is expressed in terms of the symbols of the a's as follows: ‡

$$\Delta_{\substack{i_1 \dots i_{2\nu} \\ k_1 \dots k_{2\nu}}} = \frac{\epsilon^\nu}{(2\nu)!} \begin{vmatrix} (fa)'_{i_1} & \cdots & (fa)'_{i_{2\nu}} \\ \cdot & \cdot & \cdot \\ (fa)^{2\nu}_{i_1} & \cdots & (fa)^{2\nu}_{i_{2\nu}} \end{vmatrix} \cdot \begin{vmatrix} f'_{k_1} & \cdots & f'_{k_{2\nu}} \\ \cdot & \cdot & \cdot \\ f^{2\nu}_{k_1} & \cdots & f^{2\nu}_{k_{2\nu}} \end{vmatrix}, \tag{39}$$

where $\epsilon = 1/(n-1)!$; the symbol $(fa)^j$ contains f^j and $a^2 \cdots a^n$, while the symbols a in every consecutive pair $(fa^2 \cdots a^n)^{2\lambda-1}$, $(fa^2 \cdots a^n)^{2\lambda}$ are equal when they have the same index, otherwise they are distinct but equivalent symbols of the first fundamental form (3).

Now from (34), (35), and (39),

$$K_{2\nu} = \frac{\beta^2 \epsilon^\nu}{(2\nu)!\,(n-2\nu)!} \sum_{i_1 \dots i_{2\nu} k_1 \dots k_{2\nu}}^{1 \dots n} \begin{vmatrix} (fa)'_{i_1} & \cdots & (fa)'_{i_{2\nu}} \\ \cdot & \cdot & \cdot \\ (fa)^{2\nu}_{i_1} & \cdots & (fa)^{2\nu}_{i_{2\nu}} \end{vmatrix} \cdot \begin{vmatrix} (f'_{k_1} & \cdots & f'_{k_{2\nu}} \\ \cdot & \cdot & \cdot \\ (f^{2\nu}_{k_1} & \cdots & f^{2\nu}_{k_{2\nu}} \end{vmatrix} \times F^{1 \dots 2\nu}_{i_1 \dots i_{2\nu}} F^{1 \dots 2\nu}_{k_1 \dots k_{2\nu}},$$

or

$$K_{2\nu} = \frac{\epsilon^\nu}{(2\nu)!\,(n-2\nu)!}\big((fa)' \cdots (fa)^{2\nu} f\big)(f). \tag{40}$$

* The use of $\Delta_m g$ would conflict with the second differential parameter of ordinary differential geometry, which has an entirely different meaning. Cf. BIANCHI, *Lezioni di Geometria Differenziale*, vol. I, p. 67.

† M. I., (117)–(126).

‡ Cf. K.–G. C., (28).

This gives Maschke's expression * for K_n when n is even:

$$K_n = \frac{1}{n![(n-1)!]^{n/2}}\big((fa)' \cdots (fa)^n\big)(f) \qquad (n \text{ even}).$$

Theorem. *The mean curvatures $K_{2\nu}$, with even subscript, are represented in* (40) *as rational integral functions of the coefficients of the first fundamental form and their derivatives.*

§ 7. *Expression of $K_{2\nu+1}$ in terms of the first Fundamental Quantities and Derivatives, when ν is greater than zero.*

Use is made of the determinant theorem

$$(41)\quad \Delta^2_{\substack{i_1 \cdots i_{2\nu+1}\\ k_1 \cdots k_{2\nu+1}}} = \frac{1}{2}\sum_{j,r}^{1,\ldots,n} \begin{vmatrix} a_{i_jk_s} & a_{i_jk_t} \\ a_{i_rk_s} & a_{i_rk_t}\end{vmatrix} \cdot \begin{vmatrix} D_{i_jk_s} & D_{i_jk_t} \\ D_{i_rk_s} & D_{i_rk_t}\end{vmatrix} \qquad (s, t = 1, \cdots, 2\nu+1;\ s \neq t),$$

where $\nu \neq 0$ and the D's are cofactors of the corresponding a's in $\Delta^{i_1 \cdots i_{2\nu+1}}_{k_1 \cdots k_{2\nu+1}}$ and are therefore all of even order and expressible by (39). The results are

$$D_{i_jk_s} = \frac{\epsilon^\nu}{(2\nu)!}F'_{i_j}(FA)'_{k_s}, \qquad D_{i_jk_t} = \frac{\epsilon^\nu}{(2\nu)!}\Phi'_{i_j}(\Phi B)'_{k_t},$$

$$D_{i_rk_s} = \frac{\epsilon^\nu}{(2\nu)!}F'_{i_r}(FA)'_{k_s}, \qquad D_{i_rk_t} = \frac{\epsilon^\nu}{(2\nu)!}\Phi'_{i_r}(\Phi B)'_{k_t},$$

where F'_{i_j} is the cofactor of f'_{i_j} in $\{f'_{i_1} \cdots f^{2\nu+1}_{i_{2\nu+1}}\}, \cdots, (\Phi B)'_{k_t}$ is the cofactor of $(\phi b)'_{k_t}$ in $\{(\phi b)'_{k_1} \cdots (\phi b)^{2\nu+1}_{k_{2\nu+1}}\}$. Also, by M. I. (120),

$$\begin{vmatrix} a_{i_jk_s} & a_{i_jk_t} \\ a_{i_rk_s} & a_{i_rk_t}\end{vmatrix} = \epsilon (fc)'_{k_s}(\phi c)'_{k_t}\begin{vmatrix} f'_{i_j} & \phi'_{i_j} \\ f'_{i_r} & \phi'_{i_r}\end{vmatrix},$$

Substituting in (41), we find

$$\Delta^2_{\substack{i_1 \cdots i_{2\nu+1}\\ k_1 \cdots k_{2\nu+1}}} = \frac{\epsilon^{2\nu+1}}{[(2\nu)!]^2}(fc)'_{k_s}(FA)'_{k_s}(\phi c)'_{k_t}(\Phi B)'_{k_t} \times \frac{1}{2}\sum_{j,r}^{1\ldots 2\nu+1}\begin{vmatrix} f'_{i_j} & \phi'_{i_j} \\ f'_{i_r} & \phi'_{i_r}\end{vmatrix} \cdot \begin{vmatrix} F'_{i_j} & \Phi'_{i_j} \\ F'_{i_r} & \Phi'_{i_r}\end{vmatrix}.$$

This last sum expands into

$$\frac{1}{2}\sum_{j,r}^{1,\ldots,2\nu+1}[f'_{i_j}F'_{i_j}\phi'_{i_r}\Phi'_{i_r} - f'_{i_j}\Phi'_{i_j}\phi'_{i_r}F'_{i_r} - f'_{i_r}\Phi'_{i_r}\phi'_{i_j}F'_{i_j} + f'_{i_r}F'_{i_r}\phi'_{i_j}\Phi'_{i_j}]$$
$$= \begin{vmatrix} \{f'_{i_1} \cdots f^{2\nu+1}_{i_{2\nu+1}}\} & \{f'_{i_1}\phi^2_{i_2} \cdots \phi^{2\nu+1}_{i_{2\nu+1}}\} \\ \{\phi'_{i_1}f^2_{i_2} \cdots f^{2\nu+1}_{i_{2\nu+1}}\} & \{\phi'_{i_1} \cdots \phi^{2\nu+1}_{i_{2\nu+1}}\}\end{vmatrix},$$

so that

$$\Delta^2_{\substack{i_1 \cdots i_{2\nu+1}\\ k_1 \cdots k_{2\nu+1}}} = \frac{\epsilon^{2\nu+1}}{[(2\nu)!]^2}(fc)'_{k_s}(FA)'_{k_s}(\phi c)'_{k_t}(\Phi B)'_{k_t} \times \begin{vmatrix} \{f'_{i_1} \cdots f^{2\nu+1}_{i_{2\nu+1}}\} & \{f'_{i_1}\phi^2_{i_2} \cdots \phi^{2\nu+1}_{i_{2\nu+1}}\} \\ \{\phi'_{i_1}f^2_{i_2} \cdots f^{2\nu+1}_{i_{2\nu+1}}\} & \{\phi'_{i_1} \cdots \phi^{2\nu+1}_{i_{2\nu+1}}\}\end{vmatrix}.$$

* K.-G. C., (29).

By (41) this equation holds for all values of s and t from 1 to $2\nu+1$ except $s=t$. When $s=t$, the second member vanishes. Sum the equations given by using all values of s and t from 1 to $2\nu+1$ and divide by $(2\nu+1)2\nu$; also multiply by β^4. Then

$$
\tag{42}
\begin{aligned}
\beta^4\Delta^2_{\substack{i_1\cdots i_{2\nu+1}\\ k_1\cdots k_{2\nu+1}}} &= \frac{\varepsilon^{2\nu+1}}{(2\nu+1)(2\nu)[(2\nu)!]^2}\left((fc)'_{k_1}(fa)^2_{k_2}\cdots(fa)^{2\nu+1}_{k_{2\nu+1}}\right)\\
&\times\left((\phi c)'_{k_1}(\phi b)^2_{k_2}\cdots(\phi b)^{2\nu+1}_{k_{2\nu+1}}\right)\begin{vmatrix}\{f'_{i_1}\cdots f^{2\nu+1}_{i_{2\nu+1}}\} & \{f'_{i_1}\phi^2_{i_2}\cdots\phi^{2\nu+1}_{i_{2\nu+1}}\}\\ \{\phi'_{i_1}f^2_{i_2}\cdots f^{2\nu+1}_{i_{2\nu+1}}\} & \{\phi'_{i_1}\cdots\phi^{2\nu+1}_{i_{2\nu+1}}\}\end{vmatrix}.
\end{aligned}
$$

And by (34)

$$
K_{2\nu+1} = \sum_{i_1\cdots i_{2\nu+1}k_1\cdots k_{2\nu+1}}^{1,\ldots,n}\left[\beta^4\Delta^2_{\substack{i_1\cdots i_{2\nu+1}\\ k_1\cdots k_{2\nu+1}}}\right]^{\frac{1}{2}} A_{\substack{i_1\cdots i_{2\nu+1}\\ k_1\cdots k_{2\nu+1}}} \qquad (\nu>0).
$$

Thus by (34) and (42) we have $K_{2\nu+1}(\nu>0)$ expressed in terms of the first fundamental quantities and derivatives (but only in the irrational form of a sum of square roots).

The case of K_1 presents special difficulty:

$$
K_1 = \beta^2\sum_{ik}^{1,\ldots,n}\alpha_{ik}A^i_k.
$$

In K.–G. C. (p. 24), Maschke suggests a method for expressing the α's in terms of the a's when n is odd. His formula (24) should, however, be written,

$$
\tag{43}
\alpha_{11}\Delta^{n-2} = \begin{vmatrix} A_{22} & \cdots & A_{2n}\\ A_{n2} & \cdots & A_{nn}\end{vmatrix}.
$$

If n is odd, the elements of the second member of (43) are of even order, and therefore expressible by (39), and similarly for every α. But Δ itself is of odd order, and is raised to an odd power ($n-2$ instead of $n-1$).* Equation (43) is true also for even values of n, so that the α's are always expressible by (43) in terms of the first fundamental quantities and derivatives (if $n>2$), but in all cases irrationally.

Using (43), the author has calculated irrational values of K_1 when n is greater than two; but the notation is so complicated that the presentation of the results seems impracticable, if not also useless.†

If $2\nu+1=n$, the sum reduces to a single term and formulas (34) and (42)

*Cf. Bôcher, *Introduction to Higher Algebra*, § 11.

† In a recent paper the author has calculated the value of K_1 as well as of the other curvatures of odd subscript, for a space of $n-1$ dimensions defined in R_n by the equation $U(x_1\cdots x_n)=0$. These values involve only the coefficients of the first fundamental form of R_n and their derivatives, together with the function U.

give a rational value for K_n^2,

$$(44)\ K_n^2=\beta^4\Delta^2=\frac{\epsilon^{n+2}}{n(n-1)}\big((fc)'(fa)^2\cdots(fa)^n\big)\big((\phi c)'(\phi b)^2\cdots(\phi b)^n\big)\begin{vmatrix}(f) & (f'\phi)\\ (\phi' f) & (\phi)\end{vmatrix}.$$

By the method used in K.–G. C. (p. 86), this may be reduced to Maschke's form (31): *

$$(45)\ K_n^2=\frac{1}{n[(n-1)!]^{n+2}}\big((fc)'(fa)^2\cdots(fa)^n\big)\big((\phi c)'(\phi b)^2\cdots(\phi b)^n\big)(f'\phi' f)(f^2\phi^2\phi).$$

The rather unsatisfactory results of this section are then as follows:

If n is odd, K_n^2 is expressed by (45) *as a rational function of the first fundamental quantities and their derivatives. Equations* (34) *and* (42) *give irrational expressions for the curvatures of odd index except K_1, for which no expression is here given.*

PART II.

Invariants of R_λ in R_n.

The quantities $K_{2\nu}$ and K_n^2, for n odd, are by their forms (40) and (45) differential invariants of the first fundamental quadratic form (3). When (3) defines the arc-element of a space R_n of n dimensions contained in an euclidean space S_{n+1} of $n+1$ dimensions, these K's have the geometric meaning already assigned to them. It is our object † to find corresponding invariants of a space R_λ of λ dimensions, represented as differential parameters of a general space R_n of higher dimensions containing R_λ.

§ 1. *Definitions and Preliminary Formulas.*

In the general space R_n, of n dimensions, whose coördinates are $x_1, \cdots, x_n$ and whose arc-element is defined by equation (3), let the space R_λ of λ dimensions ($\lambda < n$) be defined by the $n-\lambda$ equations

$$(46)\qquad U^{\lambda+1}(x_1, \cdots, x_n)=\text{const.}, \cdots, U^n(x_1, \cdots, x_n)=\text{const.}$$

If λ other arbitrarily chosen functions of $x_1, \cdots, x_n$, say $u', \cdots, u^\lambda$, such that

$$\Delta=(u'\cdots u^\lambda U^{\lambda+1}\cdots U^n)\neq 0,$$

are adjoined to these, the space R_λ may also be represented in parametric form

$$(47)\qquad x_1=x_1(u', \cdots, u^\lambda), \cdots, x_n=x_n(u', \cdots, u^\lambda),$$

* In Maschke's reduction there are two slight numerical errors which balance each other. His equation (30) differs from (44) above in that he has divided by n^2 instead of by $n(n-1)$; while in his reduction of (30) there are $n-1$ of the terms which become equal, instead of n.

† Cf. K.-G. C., § 5.

by solving the $n-\lambda$ equations (46) with the λ equations

$$u'(x_1, \cdots, x_n) = u', \cdots, u^\lambda(x_1, \cdots, x_n) = u^\lambda. \tag{48}$$

Any n differentials satisfying the $n-\lambda$ equations, found by differentiating (46),

$$\sum_{i=1}^{n} U_i^{\lambda+1} dx_i = 0, \cdots, \quad \sum_{i=1}^{n} U_i^{n} dx_i = 0$$

determine a certain direction in R_λ. In order to find these differentials in terms of $du', \cdots, du^\lambda$, we differentiate also equations (48) and solve the set

$$\begin{aligned} u_1' dx_1 + \cdots + u_n' dx_n &= du', \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad &\cdot \quad \cdot \\ u_1^\lambda dx_1 + \cdots + u_n^\lambda dx_n &= du^\lambda, \\ U_1^{\lambda+1} dx_1 + \cdots + U_n^{\lambda+1} dx_n &= 0, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad &\cdot \quad \cdot \\ U_1^n dx_1 + \cdots + U_n^n dx_n &= 0. \end{aligned}$$

If A^{kr} be the cofactor of u_r^k in Δ, then

$$dx_r = \frac{1}{\Delta} \sum_{k=1}^{\lambda} A^{kr} du^k$$

and therefore,

$$\sum_{r=1}^{n} p_r dx_r = \frac{1}{\Delta} \sum_{k=1}^{\lambda} \{u' \cdots u^{k-1} p u^{k+1} \cdots u^\lambda U\} du^k, \tag{49}$$

where p is any ordinary function of $x_1, \cdots, x_n$.

In order to find the expression for ds in terms of $u', \cdots, u^\lambda$, we introduce for the differential quantic (3) the symbolic form

$$ds^2 = \sum_{i,k}^{1,\ldots,n} a_{ik} dx_i dx_k = \left[\sum_{i=1}^{n} f_i dx_i \right]^2.$$

Then (49) gives for the length element in R_λ

$$\begin{aligned} ds^2 &= \frac{1}{\Delta^2} \left[\sum_{i=1}^{\lambda} \{ u' \cdots u^{i-1} f u^{i+1} \cdots u^\lambda U \} du_i \right]^2 \\ &= \frac{1}{\beta^2 \Delta^2} \left[\sum_{i=1}^{\lambda} (u' \cdots u^{i-1} f u^{i+1} \cdots u^\lambda U) du_i \right]^2. \end{aligned} \tag{50}$$

We may also introduce for ds^2, as given in terms of $u', \cdots, u^\lambda$, the symbolic form

$$ds^2 = \left[\sum_{i=1}^{\lambda} \mathfrak{f}_i du^i \right]^2. \tag{51}$$

By comparing (50) and (51) we find

$$\mathfrak{f}_i = \frac{1}{\Delta} \{ u' \cdots u^{i-1} f u^{i+1} \cdots u^\lambda U \} = \frac{1}{\beta\Delta} (u' \cdots u^{i-1} f u^{i+1} \cdots u^\lambda U). \tag{52}$$

If we use the symbols of form (51), the invariants $K_{2\nu}$ and K_λ^2 (λ odd) of R_λ may be written, by (40) and (45),

$$(53)\quad (2\nu)!(\lambda-2\nu)![(\lambda-1)!]^\nu K_{2\nu} = G_{2\nu} = \big((\mathfrak{f}\mathfrak{a})'\cdots(\mathfrak{f}\mathfrak{a})^{2\nu}\mathfrak{f}^{2\nu+1}\cdots\mathfrak{f}^\lambda\big)(\mathfrak{f}'\cdots\mathfrak{f}^\lambda),$$

$$(54)\quad \begin{aligned}\lambda[(\lambda-1)!]^{\lambda+2}K_\lambda^2 = G_\lambda^2 = &\big((\mathfrak{f}\mathfrak{c})'(\mathfrak{f}\mathfrak{a})^2\cdots(\mathfrak{f}\mathfrak{a})^\lambda\big)\\ &\times\big((\mathfrak{g}\mathfrak{c})'(\mathfrak{g}\mathfrak{b})^2\cdots(\mathfrak{g}\mathfrak{b})^\lambda\big)(\mathfrak{f}'\mathfrak{g}'\mathfrak{f}^3\cdots\mathfrak{f}^\lambda)(\mathfrak{f}^2\mathfrak{g}^2\cdots\mathfrak{g}^\lambda),\end{aligned}$$

where $G_{2\nu}$ and G_λ^2 are introduced merely for convenience. In all invariantive brackets containing the new symbols, of the quadratic form (51), the differentiation is with respect to the λ variables $u', \cdots, u^\lambda$. This is indicated sufficiently by the German type and the number of symbols inside the brackets. β_u is defined by the equation

$$(\mathfrak{f}'\cdots\mathfrak{f}^\lambda) = \beta_u\{\mathfrak{f}'\cdots\mathfrak{f}^\lambda\}.$$

We now proceed to compute the values of the invariantive expressions used in (53) and (54) in terms of the symbols of the first fundamental form (3), of R_n and the functions $U^{\lambda+1}, \cdots, U^n$ which define R_λ in R_n.

By means of (52) and D. P. (3), we obtain

$$\{\mathfrak{f}'\cdots\mathfrak{f}^\lambda\} = \frac{1}{\Delta^\lambda}\{f'\cdots f^\lambda U\}\{u'\cdots u^\lambda U\}^{\lambda-1} = \frac{1}{\Delta}\{f'\cdots f^\lambda U\},$$

so that

$$(55)\qquad \frac{1}{\beta_u}(\mathfrak{f}'\cdots\mathfrak{f}^\lambda) = \frac{1}{\beta\Delta}(f'\cdots f^\lambda U).$$

To calculate the value of β_u, square (55) and simplify the result by placing $(\mathfrak{f}'\cdots\mathfrak{f}^\lambda)^2 = \lambda!$, according to M. I. (17), and $(f'\cdots f^\lambda U)^2 = \lambda!(n-\lambda)!\Delta^{n-\lambda}U$ by (38). This gives

$$(56)\qquad \beta_u = \omega\beta\Delta, \qquad \omega = \sqrt{\frac{1}{(n-\lambda)!\Delta^{n-\lambda}U}}.$$

Then

$$(\mathfrak{f}'\cdots\mathfrak{f}^\lambda) = \omega(f'\cdots f^\lambda U).$$

The other invariantive forms in (53) and (54) are reduced by the same method, and by interchanging equivalent symbols, giving*

$$(57)\quad \begin{aligned}&(\mathfrak{f}'\cdots\mathfrak{f}^\lambda) = \omega(f'\cdots f^\lambda U),\\ &(\mathfrak{f}'\mathfrak{g}'\mathfrak{f}^3\cdots\mathfrak{f}^\lambda) = \omega(f'\phi' f^3\cdots f^\lambda U), \qquad (\mathfrak{f}^2\mathfrak{g}^2\cdots\mathfrak{g}^\lambda) = \omega(f^2\phi^2\cdots\phi^\lambda U)\\ &\big((\mathfrak{f}\mathfrak{a})'\cdots(\mathfrak{f}\mathfrak{a})^{2\nu}\mathfrak{f}^{2\nu+1}\cdots\mathfrak{f}^\lambda\big)\\ &\qquad = \omega\big(\omega(faU)', \omega(faU)^2, \cdots, \omega(faU)^{2\nu}, f^{2\nu+1}\cdots f^\lambda U\big),\end{aligned}$$

* Inside the invariantive brackets, we have followed MASCHKE's custom of omitting commas between symbols, except where ambiguity might occur. Cf. M. I., p. 448.

$$
(57)\quad \begin{aligned} &\big((\mathfrak{fc})'(\mathfrak{fa})^2\cdots(\mathfrak{fa})^\lambda\big)=\omega\big(\omega(fcU)',\,\omega(faU)^2,\,\cdots,\,\omega(faU)^\lambda U\big),\\ &\big((\mathfrak{gc})'(\mathfrak{gb})^2\cdots(\mathfrak{gb})^\lambda\big)=\omega\big(\omega(\phi cU)',\,\omega(\phi bU)^2,\,\cdots,\,\omega(\phi bU)^\lambda U\big).\end{aligned}
$$

§ 2. *Expression for* $K_{2\nu}$.

By (53) and (57),

$$
G_{2\nu}=\omega^2(\omega(faU)',\,\cdots,\,\omega(faU)^{2\nu},f^{2\nu+1}\cdots f^\lambda U)(f'\cdots f^\lambda U).
$$

Applying D. P. (4) to the second member, we get

$$
G_{2\nu}=\omega^{2\nu+2}\big((faU)'\cdots(faU)^{2\nu}f^{2\nu+1}\cdots f^\lambda U\big)(f'\cdots f^\lambda U)+\omega^{2\nu+1}(f'\cdots f^\lambda U)
$$

$$
\times\sum_{k=1}^{2\nu}(faU)^k\big((faU)'\cdots(faU)^{k-1},\,\omega,\,(faU)^{k+1}\cdots(faU)^{2\nu}f^{2\nu+1}\cdots f^\lambda U\big).
$$

It will now be shown that each term of this last sum vanishes. Aside from the factor $\omega^{2\nu+1}$, each odd term of this sum may be written in the form

$$
\begin{aligned} T=(-1)^{\lambda-1}&(f^{k+1}\cdots f^\lambda f'\cdots f^kU)(faU)^k\\ &\times\big((faU)'\cdots(faU)^{k-1},\,\omega,\,(faU)^{k+1}\cdots(faU)^{2\nu}f^{2\nu+1}\cdots f^\lambda U\big).\end{aligned}
$$

Applying D. P. (1) to the first two brackets of the second member, we obtain

$$
T=(-1)^{\lambda-1}\left[\begin{array}{l} \quad(f^kf^{k+2}\cdots f^\lambda f'\cdots f^kU)(f^{k+1}a^2\cdots a^\lambda U)\\ +(a^2f^{k+2}\cdots f^\lambda f'\cdots f^kU)(f^kf^{k+1}a^3\cdots a^\lambda U)\\ +(a^3f^{k+2}\cdots f^\lambda f'\cdots f^kU)(f^ka^2f^{k+1}a^4\cdots a^\lambda U)\\ +\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\ +(a^\lambda f^{k+2}\cdots f^\lambda f'\cdots f^kU)(f^ka^2\cdots a^{\lambda-1}f^{k+1}U)\end{array}\right]
$$

$$
\times\big((faU)'\cdots(faU)^{k-1},\,\omega,\,(faU)^{k+1}\cdots(faU)^{2\nu}f^{2\nu+1}\cdots f^\lambda U\big).
$$

Of these λ terms, the first vanishes because of two identical rows in the first bracket, while the others become equal to each other if we interchange f^{k+1} with $a^2\cdots a^\lambda$ in turn and in each case restore the original order in $(faU)^{k+1}$ by the interchange of two rows.* Thus

$$
\begin{aligned} T=(-1)^{\lambda-1}(1-\lambda)&(f^{k+1}\cdots f^\lambda f'\cdots f^kU)(f^ka^2\cdots a^\lambda U)\\ &\times\big((faU)'\cdots(faU)^{k-1},\,\omega,\,(faU)^{k+1}\cdots\big)=(-1)^{\lambda-1}(1-\lambda)\,T.\end{aligned}
$$

Hence $T=0$ for odd values of k.

If k is even, each term T may be written

$$
\begin{aligned} T=(&f^{k-1}\cdots f^\lambda f'\cdots f^{k-2}U)(faU)^k\\ &\times\big((faU)'\cdots(faU)^{k-1},\,\omega,\,(faU)^{k+1}\cdots(faU)^{2\nu}f^{2\nu+1}\cdots f^\lambda U\big).\end{aligned}
$$

* Cf. K.-G. C., p. 92.

By applying D. P. (1) to the first two brackets, and proceeding as above, one finds $T=0$ also for even values of k.

With the help of these results (58) becomes

$$G_{2\nu}=\omega^{2\nu+2}\big((faU)'\cdots(faU)^{2\nu}f^{2\nu+1}\cdots f^{\lambda}U\big)(f'\cdots f^{\lambda}U).$$

Then, by (53) and (56),

$$(59)\quad K_{2\nu}=\frac{(\lambda-1)!\big((faU)'\cdots(faU)^{2\nu}f^{2\nu+1}\cdots f^{\lambda}U\big)(f'\cdots f^{\lambda}U)}{(2\nu)!(\lambda-2\nu)![(\lambda-1)!(n-\lambda)!\Delta^{n-\lambda}U]^{\nu+1}}.$$

If $2\nu=\lambda$, (59) becomes

$$(60)\qquad K_{\lambda}=\frac{\big((faU)'\cdots(faU)^{\lambda}U\big)(f'\cdots f^{\lambda}U)}{\lambda[(\lambda-1)!(n-\lambda)!\Delta^{n-\lambda}U]^{(\lambda+2)/2}},$$

which agrees with Maschke's form, K.–G. C. (60). The symbols f and a belong to the quadratic form (3), expressing the length element of R_n. Further, $(faU)^i=(f^ia^2\cdots a^{\lambda}U^{\lambda+1}\cdots U^n)$, in which f^i is equal to f^i in $(f'\cdots f^{\lambda}U)$, while the sets of symbols $a^2\cdots a^{\lambda}$ are equal in any two consecutive brackets $(faU)^{2k-1}$, $(faU)^{2k}$ and otherwise distinct.

The result is then that $K_{2\nu}$, for the space R_{λ}, is expressible rationally in terms of the coefficients of the first fundamental form of R_n and their derivatives, together with the functions $U^{\lambda+1},\cdots,U^n$ (which define R_{λ} in R_n) and their derivatives.

§ 3. *Expression for K_{λ}^2 when λ is odd.**

The invariant K_{λ}^2(λ odd) can be expressed in a manner similar to the above. Substituting from (57) into (54), one gets

$$(61)\quad \begin{aligned}G_{\lambda}^2&=\omega^4\big(\omega(fcU)',\,\omega(faU)^2,\,\cdots,\,\omega(faU)^{\lambda}U\big)\\&\quad\times\big(\omega(\phi cU)',\,\omega(\phi bU)^2,\cdots,\omega(\phi bU)^{\lambda}U\big)\big(f'\phi'f^3\cdots f^{\lambda}U\big)\big(f^2\phi^2\cdots\phi^{\lambda}U\big).\end{aligned}$$

By D. P. (4),

$$\begin{aligned}\big(\omega(fcU)',\,\omega(faU)^2,\,\cdots,\,\omega(faU)^{\lambda}U\big)&=\omega^{\lambda}\big((fcU)'(faU)^2\cdots(faU)^{\lambda}U\big)\\&\quad+\omega^{\lambda-1}(fcU)'\big(\omega,\,(faU)^2\cdots(faU)^{\lambda}U\big)\\&\quad+\omega^{\lambda-1}\sum_{i=2}^{\lambda}(faU)^i\big((fcU)'(faU)^2\cdots(faU)^{i-1},\,\omega,\,(faU)^{i+1}\cdots(faU)^{\lambda}U\big)\\&\equiv\omega^{\lambda}\alpha_1+\omega^{\lambda-1}\alpha_2+\omega^{\lambda-1}\alpha_3.\end{aligned}$$

$$\begin{aligned}\big(\omega(\phi cU)',\,\omega(\phi bU)^2,\cdots,\omega(\phi bU)^{\lambda}U\big)&=\omega^{\lambda}\big((\phi cU)'(\phi bU)^2\cdots(\phi bU)^{\lambda}U\big)\\&\quad+\omega^{\lambda-1}(\phi cU)'(\omega,\,(\phi bU)^2\cdots(\phi bU)^{\lambda}U)\\&\quad+\omega^{\lambda-1}\sum_{k=2}^{\lambda}(\phi bU)^k\big((\phi cU)'(\phi bU)^2\cdots(\phi bU)^{k-1},\,\omega,\,(\phi bU)^{k+1}\cdots(\phi bU)^{\lambda}U\big)\\&\equiv\omega^{\lambda}\beta_1+\omega^{\lambda-1}\beta_2+\omega^{\lambda-1}\beta_3.\end{aligned}$$

* See K.–G. C., p. 93.

The notations α_1, α_2, α_3, β_1, β_2, β_3 are used for brevity to represent the expressions whose relative places they occupy. If we also use

$$\gamma = (f'\phi' f^3 \cdots f^\lambda U), \qquad \delta = (f^2\phi^2 \cdots \phi^\lambda U),$$

then

$$G_\lambda^2 = \omega^{2\lambda+2}[\,\omega\alpha_1 + \alpha_2 + \alpha_3\,][\,\omega\beta_1 + \beta_2 + \beta_3\,]\,\gamma\delta. \tag{62}$$

The nine terms of this product (omitting powers of ω) will now be considered in the following order:

1) $\alpha_1\beta_1\gamma\delta$,	4) $\alpha_3\beta_2\gamma\delta$,	7) $\alpha_1\beta_3\gamma\delta$,
2) $\alpha_1\beta_2\gamma\delta$,	5) $\alpha_2\beta_1\gamma\delta$,	8) $\alpha_3\beta_3\gamma\delta$,
3) $\alpha_2\beta_2\gamma\delta$,	6) $\alpha_2\beta_3\gamma\delta$,	9) $\alpha_3\beta_1\gamma\delta$.

For the first we have $\alpha_1\beta_1\gamma\delta = L$, where

$$\begin{aligned} L = {} & \big((fcU)'(faU)^2 \cdots (faU)^\lambda U\big) \\ & \times \big((\phi cU)'(\phi bU)^2 \cdots (\phi bU)^\lambda U\big)(f'\phi' f^3 \cdots f^\lambda U)(f^2\phi^2 \cdots \phi^\lambda U). \end{aligned} \tag{63}$$

The second is shown to vanish as follows:

$$\begin{aligned} 2)\ \alpha_1\beta_2\gamma\delta = {} & (f'\phi' f^3 \cdots f^\lambda U)(\phi' c^2 \cdots c^\lambda U) \\ & \times \big((fc' \qquad\qquad (faU)^\lambda U)(f^2\phi^2 \cdots \phi^\lambda U)\big(\omega, (\phi bU)^2 \cdots (\phi bU)^\lambda U\big) \\ = {} & \begin{bmatrix} \ (\phi'\phi' f' \qquad U)(f'c^2 \cdots c^\lambda U) \\ +(c^2\phi' f^3 \cdots f^\lambda U)(\phi' f' c^3 \cdots e^\lambda U) \\ +(c^3\phi' f^3 \cdots f^\lambda U)(\phi' c^2 f' c^4 \cdots c^\lambda U) \\ + \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ +(c^\lambda\phi' f^3 \cdots f^\lambda U)(\phi' c^2 \cdots c^{\lambda-1} f' U) \end{bmatrix} \big((fcU)'(faU)^2 \cdots (faU)^\lambda U\big) \cdots \qquad \text{[by D. P. (1)]} \\ = {} & (1-\lambda)(f'\phi' f^3 \cdots f^\lambda U)(\phi' c^2 \cdots c^\lambda U)\big((fcU)'(faU)^2 \cdots (faU)^\lambda U\big) \cdots \\ = {} & (1-\lambda)\,\alpha_1\beta_2\gamma\delta. \end{aligned}$$

Hence the second vanishes. The third and fourth are shown to vanish by applying D. P. (1) to exactly the same expressions.

For the fifth term,

$$\begin{aligned} 5)\ \alpha_2\beta_1\gamma\delta = {} & (f'\phi' f^3 \cdots f^\lambda U)(f'c^2 \cdots c^\lambda U)\big((\phi cU)'(\phi bU)^2 \cdots (\phi bU)^\lambda U\big) \\ & \times \big(\omega, (faU)^2 \cdots (faU)^\lambda U\big)(f^2\phi^2 \cdots \phi\ U) \\ = {} & (\phi' f^3 \cdots f^\lambda f' U)(f'c^2 \cdots c^\lambda U)\big((\phi cU)'(\phi bU)^2 \cdots (\phi bU)^\lambda U\big) \cdots. \end{aligned}$$

By applying D. P. (1) to the first two forms and simplifying as for 2), we find

$$\alpha_2\beta_1\gamma\delta = (1-\lambda)\,\alpha_2\beta_1\gamma\delta.$$

Hence 5) vanishes, and the sixth term is shown to vanish by applying D. P. (1) to the same forms.

For the seventh term,

$$7)\quad \alpha_1\beta_3\gamma\delta = \sum_{k=2}^{\lambda}(\phi^k b^2\cdots b^\lambda U)(f^2\phi^2\cdots\phi^\lambda U)$$
$$\times\big((\phi c U)'(\phi b U)^2\cdots(\phi b U)^{k-1}, \omega, \cdots\big)$$
$$\times(f'\phi' f^3\cdots f^\lambda U)\big((fcU)'(faU)^2\cdots(faU)^\lambda U\big).$$

This sum is shown to vanish for all values of k by the method used for (58), and the vanishing of 8) follows by the same method.

For the last term,

$$9)\quad \alpha_3\beta_1\gamma\delta = \sum_{i=2}^{\lambda}(faU)^i(f'\phi' f^3\cdots f^\lambda U)$$
$$\times\big((fcU)'(faU)^2\cdots(faU)^{i-1}, \omega, (faU)^{i+1}\cdots(faU)^\lambda U\big)$$
$$\times(f^2\phi^2\cdots\phi^\lambda U)\big((\phi c U)'(\phi b U)^2\cdots(\phi b U)^\lambda U\big).$$

The terms in which $i > 3$ vanish by the methods used for (58), but the terms T_2 (for $i=2$) and T_3 (for $i=3$) do not vanish and require special treatment.

We have $T_2 = (f^3\cdots f^\lambda f'\phi' U)(f^2 a^2\cdots a^\lambda U)N$, where

$$N = \big((fcU)', \omega, (faU)^3\cdots(faU)^\lambda U\big)(f^2\phi^2\cdots\phi^\lambda U)$$
$$\times\big((\phi c U)'(\phi b U)^2\cdots(\phi b U)^\lambda U)\big),$$

$$T_2 = \begin{bmatrix} (f^2 f^4\cdots f^\lambda f'\phi' U)(f^3 a^2\cdots a^\lambda U) \\ +(a_2 f^4\cdots f^\lambda f'\phi' U)(f^2 f^3 a^3\cdots a^\lambda U) \\ +(a^3 f^4\cdots f^\lambda f'\phi' U)(f^2 a^2 f^3 a^4\cdots a^\lambda U) \\ + \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ +(a^\lambda f^4\cdots f^\lambda f'\phi' U)(f^2 a^2\cdots a^{\lambda-1} f^3 U) \end{bmatrix} N \qquad \text{[by D. P. (1)]}$$

$$= (f^2 f^4\cdots f^\lambda f'\phi' U)(f^3 a^2\cdots a^\lambda U)N + (1-\lambda)T_2.$$

Hence

$$T_2 = \frac{1}{\lambda}(f^2 f^4\cdots f^\lambda f'\phi' U)(f^3 a^2\cdots a^\lambda U)N.$$

If now, we interchange f^2 and f^3 and then restore the regular order of symbols, we get

$$T_2 = -\frac{1}{\lambda}(f'\phi' f^3\cdots f^\lambda U)(faU)^2\big((fcU)', (faU)^2, \omega, (faU)^4\cdots(faU)^\lambda U\big)$$
$$\times(f^3\phi^2\cdots\phi^\lambda U)\big((\phi c U)'(\phi b U)^2\cdots(\phi b U)^\lambda U\big).$$

Next, T_3 may be written as $(f^2\phi^2\cdots\phi^\lambda U)(f^3 a^2\cdots a^\lambda U)\cdot P$, where

$$P = \big((fcU)'(faU)^2,\ \omega,\ (faU)^4 \cdots (faU)^\lambda U\big) \times (f'\phi' f^3 \cdots f^\lambda U)\big((\phi c U)'(\phi b U)^2 \cdots (\phi b U)^\lambda U\big).$$

$$T_3 = \begin{bmatrix} (f^3\phi^2 \cdots \phi^\lambda U)(f^2 a^2 \cdots a^\lambda U) \\ + (a^2\phi^2 \cdots \phi^\lambda U)(f^3 f^2 a^3 \cdots a^\lambda U) \\ + (a^3\phi^2 \cdots \phi^\lambda U)(f^3 a^2 f^2 a^4 \cdots a^\lambda U) \\ + \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ + (a^\lambda \phi^2 \cdots \phi^\lambda U)(f^3 a^2 \cdots a^{\lambda-1} f^2 U) \end{bmatrix} P \qquad \text{[by D. P. (1)]}$$

$$= (f^3\phi^2 \cdots \phi^\lambda U)(f^2 a^2 \cdots a^\lambda U) P + (1-\lambda) T_3,$$

so that $T_3 = -T_2$.

Thus all nine terms in the second member of (62) vanish except the first, whence

$$G_\lambda^2 = \omega^{2\lambda+4} L. \tag{64}$$

Then, by (54) and (56),

$$K_\lambda^2 = \frac{L}{\lambda[(\lambda-1)!(n-\lambda)!\Delta^{n-\lambda}U]^{\lambda+2}}, \tag{65}$$

where L is given by (63), in which the symbols f, ϕ, a, b, c belong to the quadratic form (3); the form $(faU)^k = (f^k a^2 \cdots a^\lambda U^{\lambda+1} \cdots U^n)$; the f's (also ϕ's and c's) with same index are equal; the sets of symbols $a^2, \cdots, a^\lambda$ (also $b^2, \cdots, b^\lambda$) are equal in any two consecutive brackets of which the first has even index, and otherwise distinct.

Hence K_λ^2 (λ *odd*), *for the space* R_λ, *is expressible rationally in terms of the coefficients of the first fundamental form and their derivatives, together with the functions* $U^{\lambda+1}, \cdots, U^n$ (*which define* R_λ *in* R_n) *and their derivatives.*