# ON THE ORDER OF LINEAR HOMOGENEOUS GROUPS*

## (FOURTH PAPER)

BY

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1. In the writer's theorems † on finite groups of linear homogeneous substitutions of determinant unity, a group of special nature called *a self-conjugate* (or *invariant*) *subgroup* $H$ plays an important rôle. There is a lack of completeness to these theorems due to the fact it has not been proved that $H$ is actually less than the transitive (irreducible) group $G$ in which, under certain conditions, it is contained; i. e., the groups $G$ and $H$ may be identical so far as the theorems are concerned. Account had to be taken of this fact in constructing the collineation-groups for the plane and space.‡

The relation between $G$ and $H$ is as follows. Let the number of variables be $n$, and let $V$ be any substitution of $G$. The sum of the multipliers of $V$ (weight, characteristic) we shall indicate by $(V)$, which, therefore, represents the sum of $n$ roots of unity:

$$(V) = \sum_{i=1}^{n} \alpha_i \beta_i .$$

Each of these roots we write as the product of one $\alpha_i$, whose index is prime to a given prime number $p$, and one $\beta_i$, whose index is a power of $p$. By $(V)_p$ we indicate the quantity obtained by replacing in $(V)$ every root $\beta_i$ by unity:

$$(V)_p = \sum_{i=1}^{n} \alpha_i .$$

Then, under certain conditions, as mentioned above, there is in $G$ a self-conjugate subgroup $H$ whose substitutions $T$ have the property

$$(VT)_p \equiv (V)_p \pmod{p};$$

from which, in particular,

$$(T)_p \equiv n \pmod{p}.$$

* Presented to the Society (San Francisco), September 24, 1910.

† The theorems chiefly concerned are as follows: Theorems 10 and 11 of the article *On the Order of Linear Homogeneous Groups* (*Second Paper*), these Transactions, vol. 5 (1904), pp. 315, 317; Theorem 14 of the article *On the Order of Linear Homogeneous Groups* (*Supplement*), *ibid.*, vol. 7 (1906), p. 523. These two articles will be referred to by $LGII$ and $LGS$ respectively.

‡ Mathematische Annalen, vol. 60 (1905), p. 214, § 16; vol. 63 (1907), p. 563, § 15.

2. The writer has recently been able to prove that $H$ (or more strictly, a modified form of it) is actually less than $G$ when this group is primitive, so that in all such cases $G$ cannot be a simple group. Furthermore, it follows that

$$n \equiv 0 \pmod{p}.$$

Theorem 14 of $LGS$ may be modified to read:

THEOREM 14′. *If for $n > 1$ a primitive group $G$ in $n$ variables has an abelian subgroup $K$ of order $p^a \geqq p^n$, then $G$ will have an invariant subgroup $H$ which contains at least $p^{a-n+1}$ substitutions of $K$, but which does not contain $K$ entirely. If $T$ be any substitution of $H$, and $V$ any substitution of $G$ whose weight $(V)$ is zero,* then*

$$(VT)_p \equiv 0 \quad (mod\, p).$$

*Moreover,*

$$(T)_p \equiv 0, \qquad and \qquad n \equiv 0 \quad (mod\, p).$$

Let the substitutions of $K$ be represented by $S_0 =$ identity, $S_1, \cdots, S_{p^a-1}$.

In proving Theorem 14, $LGS$, it was first proved that for at least $p^{a-n+1}$ substitutions of $K$,† say

(1) $$S_0,\ S_1,\ \cdots,\ S_i,\ \cdots,$$

we have equations of the form

$$(VS_i) - (V) + (1-\theta)X_i = 0,$$

$V$ being any substitution of $G$, $\theta$ a root of the equation

$$\theta^{p^a} - 1 = 0,$$

and $X_i$ a sum of the weights

(2) $$(VS_j) \qquad (j = 0, 1, \cdots, p^a - 1),$$

multiplied by integral functions of $\theta$, the numerical coefficients entering being integers or fractions whose denominators are prime to $p$.

Now let $V$ be any substitution of $G$ such that $(V)$ is zero. Then

(3) $$(VS_i) + (1-\theta)X_i = 0$$

for all substitutions $S_i$ of the series (1).

Consider all the weights (2).

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* BURNSIDE has proved the following theorem: "In any irreducible group of linear substitutions of finite order, other than a cyclical group in a single variable, at least one of its characteristics is zero," Proceedings of the London Mathematical Society, ser. 2, vol. 1 (1903), p. 115.

† Attention is here called to an omission in $LGS$, page 524. First line below the matrix reads, "Now, to this matrix may be added $p^{a-m+1}$ rows $\cdots$." This should read: "Now, to this matrix may be added $p^{a-m+1} - 1$ rows $\cdots$."

*First*, they may all vanish, whatever be the substitution $V$ chosen, so long as $(V)$ vanishes. By the arguments of § 7, *LGII*, all the substitutions of $K$ and all further substitutions $T$ of $G$ for which $(VT)$ vanishes, form a group $H$, self-conjugate in $G$. If $H$ had any weight $(W)$ which vanished, then every $(WT)$ would vanish, and therefore every $(T)$. But this is impossible, since $(S_0) = n$. Accordingly, every weight of $H$ is non-vanishing, and therefore $H$ is intransitive, by Burnside's Theorem. It follows by Theorem 8, *LGII*, and by Burnside's Theorem that $H$ is composed of similarity-substitutions.

*Second*, the weights (2) do not all vanish. By (3), some of them are divisible by $1 - \theta$, the quotient being expressible as a linear function of a finite number of roots of unity, no numerical coefficient entering having a denominator which is divisible by $p$. Assume that all the weights (2) are divisible by $(1 - \theta)^k$, whenever $V$ represents a substitution of $G$ whose weight is divisible by $(1 - \theta)^k$, $k$ ranging through the values $0, 1, 2, \cdots, m$; but that the weights (2) are not all divisible by $(1 - \theta)^{m+1}$ whenever $(V)$ is divisible by $(1 - \theta)^{m+1}$. In general, we should not expect $m$ to be greater than zero. Now, all the substitutions $T$ of $G$ for which

$$(VT) = (1 - \theta)^{m+1} X,$$

whenever

$$(V) = (1 - \theta)^{m+1} Y,$$

form an invariant subgroup $H$. To $H$ belongs the series (1), but not the entire group $K$.

Since $G$ is assumed to be primitive, $H$, if not composed of similarity-substitutions, must contain a substitution $W$ whose weight vanishes (Theorem 8, *LGII*, and Burnside's Theorem). Then

$$(WT) = (1 - \theta)^{m+1} Z$$

for every substitution $T$ of $H$; i. e., every

$$(WT)_p \equiv 0 \pmod{p},$$

and therefore every

$$(T)_p \equiv 0 \pmod{p}.$$

The Theorems 10 and 11, *LGII*, may be modified in like manner to read that the self-conjugate subgroup $H$ is less than $G$, the latter being primitive. In addition,

$$(T)_p \equiv 0 \pmod{p}$$

for every substitution of $H$.

3. One of the most important problems in the theory of linear homogeneous groups is the determination of the maximum order; i. e., the fixing of a superior limit to $\lambda$ in Jordan's Theorem, the number of variables being given. The

limit known * can now be greatly reduced in special cases. Let $G$ be a primitive group, $n$ the number of variables, and $p$ a prime. Then, using Theorem 14′ in conjunction with Theorem 9, $LGII$, we can prove the

COROLLARY. *If $p$ and $n$ are prime to each other, the highest power of $p$ which divides the order of $G$ must divide $n!\ p^{n-1}$.*

If $n = p$, an invariant subgroup $H$ (assumed to contain and to be greater than the group of similarity-substitutions of order $p$) must be of order $p^k$ and cannot be abelian, $G$ being assumed primitive. Writing $H$ in monomial form we readily find that, if $k > 3$, it possesses one, and only one, invariant of degree $p$ which can be factored into $p$ linear factors. In such a case $G$ cannot be primitive. Hence, $H$ is of order $p^3$ (or $p^2$ when considered as a collineation-group), being generated by the substitutions

$$A:\ x_1' = x_1,\ x_2' = \theta x_2,\ \cdots,\ x_p' = \theta^{p-1} a_p;\ \frac{\theta^p - 1}{\theta - 1} = 0;$$

$$B:\ x_1' = x_2,\ x_2' = x_3,\ \cdots,\ x_p' = x_1.$$

The order of $G$ is a factor of $(p^2 - 1)p \cdot p^2$ when considered as a collineation-group. The corollary above is therefore true also when $n = p > 2$.

THEOREM 17. *Let $G$ be a primitive collineation-group in $n$ variables, $n$ being a prime $> 1$. Then the order of $G$ is a factor of*

$$n!\ (2 \cdot 3 \cdot \cdots \cdot p \cdot \cdots)^{n-1}, \tag{4}$$

*$2, 3, \cdots, p, \cdots$ being the different primes not greater than the greater of the numbers $4n - 3$, $(n-2)(2n+1)$.† The only exception is the octahedral group of order 24 in two variables.*

It will be noticed that a transitive group in a prime number of variables $n$ is either primitive or of monomial type. In either case, the corresponding Jordan factor $\lambda$ must divide the number (4) if $n > 2$.

* See $LGII$, pp. 310, 320–321, and *On Imprimitive Linear Homogeneous Groups*, these Transactions, vol. 6 (1905), p. 232, for an upper limit for all transitive groups. SCHUR has given an upper limit when the weights belong to a given field: Sitzungsberichte der Kgl. Preussischen Akademie der Wissenschaften, 1905, p. 77 ff.

† $LGS$, p. 528.