ON THE ORDER OF LINEAR HOMOGENEOUS GROUPS*

(FOURTH PAPER)

BY

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1. In the writer's theorems † on finite groups of linear homogeneous substitutions of determinant unity, a group of special nature called *a self-conjugate* (or *invariant*) *subgroup* H plays an important rôle. There is a lack of completeness to these theorems due to the fact it has not been proved that H is actually less than the transitive (irreducible) group G in which, under certain conditions, it is contained; i. e., the groups G and H may be identical so far as the theorems are concerned. Account had to be taken of this fact in constructing the collineation-groups for the plane and space.‡

The relation between G and H is as follows. Let the number of variables be n, and let V be any substitution of G. The sum of the multipliers of V (weight, characteristic) we shall indicate by (V), which, therefore, represents the sum of n roots of unity:

$$(V) = \sum_{i=1}^{n} \alpha_i \beta_i .$$

Each of these roots we write as the product of one α_i, whose index is prime to a given prime number p, and one β_i, whose index is a power of p. By $(V)_p$ we indicate the quantity obtained by replacing in (V) every root β_i by unity:

$$(V)_p = \sum_{i=1}^{n} \alpha_i .$$

Then, under certain conditions, as mentioned above, there is in G a self-conjugate subgroup H whose substitutions T have the property

$$(VT)_p \equiv (V)_p \pmod{p};$$

from which, in particular,

$$(T)_p \equiv n \pmod{p}.$$

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† The theorems chiefly concerned are as follows: Theorems 10 and 11 of the article *On the Order of Linear Homogeneous Groups* (*Second Paper*), these Transactions, vol. 5 (1904), pp. 315, 317; Theorem 14 of the article *On the Order of Linear Homogeneous Groups* (*Supplement*), *ibid.*, vol. 7 (1906), p. 523. These two articles will be referred to by $LGII$ and LGS respectively.

‡ Mathematische Annalen, vol. 60 (1905), p. 214, § 16; vol. 63 (1907), p. 563, § 15.

2. The writer has recently been able to prove that H (or more strictly, a modified form of it) is actually less than G when this group is primitive, so that in all such cases G cannot be a simple group. Furthermore, it follows that

$$n \equiv 0 \pmod{p}.$$

Theorem 14 of LGS may be modified to read:

THEOREM 14′. *If for $n > 1$ a primitive group G in n variables has an abelian subgroup K of order $p^a \geqq p^n$, then G will have an invariant subgroup H which contains at least p^{a-n+1} substitutions of K, but which does not contain K entirely. If T be any substitution of H, and V any substitution of G whose weight (V) is zero,* then*

$$(VT)_p \equiv 0 \quad (mod\, p).$$

Moreover,

$$(T)_p \equiv 0, \qquad and \qquad n \equiv 0 \quad (mod\, p).$$

Let the substitutions of K be represented by $S_0 =$ identity, $S_1, \cdots, S_{p^a-1}$.

In proving Theorem 14, LGS, it was first proved that for at least p^{a-n+1} substitutions of K,† say

(1) $$S_0,\ S_1,\ \cdots,\ S_i,\ \cdots,$$

we have equations of the form

$$(VS_i) - (V) + (1-\theta)X_i = 0,$$

V being any substitution of G, θ a root of the equation

$$\theta^{p^a} - 1 = 0,$$

and X_i a sum of the weights

(2) $$(VS_j) \qquad (j = 0, 1, \cdots, p^a - 1),$$

multiplied by integral functions of θ, the numerical coefficients entering being integers or fractions whose denominators are prime to p.

Now let V be any substitution of G such that (V) is zero. Then

(3) $$(VS_i) + (1-\theta)X_i = 0$$

for all substitutions S_i of the series (1).

Consider all the weights (2).

* BURNSIDE has proved the following theorem: "In any irreducible group of linear substitutions of finite order, other than a cyclical group in a single variable, at least one of its characteristics is zero," Proceedings of the London Mathematical Society, ser. 2, vol. 1 (1903), p. 115.

† Attention is here called to an omission in LGS, page 524. First line below the matrix reads, "Now, to this matrix may be added p^{a-m+1} rows $\cdots$." This should read: "Now, to this matrix may be added $p^{a-m+1}-1$ rows $\cdots$."

First, they may all vanish, whatever be the substitution V chosen, so long as (V) vanishes. By the arguments of § 7, *LGII*, all the substitutions of K and all further substitutions T of G for which (VT) vanishes, form a group H, self-conjugate in G. If H had any weight (W) which vanished, then every (WT) would vanish, and therefore every (T). But this is impossible, since $(S_0) = n$. Accordingly, every weight of H is non-vanishing, and therefore H is intransitive, by Burnside's Theorem. It follows by Theorem 8, *LGII*, and by Burnside's Theorem that H is composed of similarity-substitutions.

Second, the weights (2) do not all vanish. By (3), some of them are divisible by $1-\theta$, the quotient being expressible as a linear function of a finite number of roots of unity, no numerical coefficient entering having a denominator which is divisible by p. Assume that all the weights (2) are divisible by $(1-\theta)^k$, whenever V represents a substitution of G whose weight is divisible by $(1-\theta)^k$, k ranging through the values $0, 1, 2, \cdots, m$; but that the weights (2) are not all divisible by $(1-\theta)^{m+1}$ whenever (V) is divisible by $(1-\theta)^{m+1}$. In general, we should not expect m to be greater than zero. Now, all the substitutions T of G for which

$$(VT) = (1-\theta)^{m+1}X,$$

whenever

$$(V) = (1-\theta)^{m+1}Y,$$

form an invariant subgroup H. To H belongs the series (1), but not the entire group K.

Since G is assumed to be primitive, H, if not composed of similarity-substitutions, must contain a substitution W whose weight vanishes (Theorem 8, *LGII*, and Burnside's Theorem). Then

$$(WT) = (1-\theta)^{m+1}Z$$

for every substitution T of H; i. e., every

$$(WT)_p \equiv 0 \pmod{p},$$

and therefore every

$$(T)_p \equiv 0 \pmod{p}.$$

The Theorems 10 and 11, *LGII*, may be modified in like manner to read that the self-conjugate subgroup H is less than G, the latter being primitive. In addition,

$$(T)_p \equiv 0 \pmod{p}$$

for every substitution of H.

3. One of the most important problems in the theory of linear homogeneous groups is the determination of the maximum order; i. e., the fixing of a superior limit to λ in Jordan's Theorem, the number of variables being given. The

limit known * can now be greatly reduced in special cases. Let G be a primitive group, n the number of variables, and p a prime. Then, using Theorem 14′ in conjunction with Theorem 9, *LGII*, we can prove the

COROLLARY. *If p and n are prime to each other, the highest power of p which divides the order of G must divide $n!\ p^{n-1}$.*

If $n = p$, an invariant subgroup H (assumed to contain and to be greater than the group of similarity-substitutions of order p) must be of order p^k and cannot be abelian, G being assumed primitive. Writing H in monomial form we readily find that, if $k > 3$, it possesses one, and only one, invariant of degree p which can be factored into p linear factors. In such a case G cannot be primitive. Hence, H is of order p^3 (or p^2 when considered as a collineation-group), being generated by the substitutions

$$A:\ x_1' = x_1,\ x_2' = \theta x_2,\ \cdots,\ x_p' = \theta^{p-1} a_p;\ \frac{\theta^p - 1}{\theta - 1} = 0;$$

$$B:\ x_1' = x_2,\ x_2' = x_3,\ \cdots,\ x_p' = x_1.$$

The order of G is a factor of $(p^2 - 1)p \cdot p^2$ when considered as a collineation-group. The corollary above is therefore true also when $n = p > 2$.

THEOREM 17. *Let G be a primitive collineation-group in n variables, n being a prime > 1. Then the order of G is a factor of*

$$n!\ (2 \cdot 3 \cdot \cdots \cdot p \cdot \cdots)^{n-1}, \tag{4}$$

$2, 3, \cdots, p, \cdots$ *being the different primes not greater than the greater of the numbers $4n - 3$, $(n-2)(2n+1)$.*† *The only exception is the octahedral group of order 24 in two variables.*

It will be noticed that a transitive group in a prime number of variables n is either primitive or of monomial type. In either case, the corresponding Jordan factor λ must divide the number (4) if $n > 2$.

* See *LGII*, pp. 310, 320–321, and *On Imprimitive Linear Homogeneous Groups*, these Transactions, vol. 6 (1905), p. 232, for an upper limit for all transitive groups. SCHUR has given an upper limit when the weights belong to a given field: Sitzungsberichte der Kgl. Preussischen Akademie der Wissenschaften, 1905, p. 77 ff.

† *LGS*, p. 528.