NATURAL SYSTEMS OF TRAJECTORIES GENERATING FAMILIES OF LAMÉ*

BY

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As in the author's previous papers in these Transactions,† the term *natural* is employed to designate those quadruply-infinite systems of space curves which can be identified with the totality of extremals of a variation problem of the form

$$(1) \qquad \int F(x, y, z)\, ds = \text{minimum} \qquad (ds^2 = dx^2 + dy^2 + dz^2).$$

Such a system has the characteristic property that if ∞^2 of the curves are orthogonal to one surface they are necessarily orthogonal to ∞^1 surfaces. The equation of these ∞^1 surfaces may be written in the form

$$(2) \qquad f(x, y, z) = \text{constant}$$

where f satisfies the Hamilton-Jacobi equation

$$(3) \qquad f_x^2 + f_y^2 + f_z^2 = F^2$$

associated with (1).

A set of ∞^1 surfaces of this sort will be termed a set of wave fronts or, more simply, a *wave set*. The natural system being given, every surface determines one of these wave sets. The phraseology employed is of course suggested by the optical interpretation of (1), in which $F(x, y, z)$ represents the index of refraction in a non-homogeneous isotropic medium: the curves of the natural system are then the possible paths of light; and a disturbance starting from any surface will be propagated by means of a set of wave fronts.

The question arises whether wave sets may be families of Lamé, that is, simply infinite families of surfaces which can form part of a triply-orthogonal system. Ordinarily this is not the case. The condition that (2) shall represent a Lamé family is that f satisfy the Darboux-Cayley equation of third order ‡

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† *Natural families of trajectories: conservative fields of force*, vol. 10 (1909), pp. 201–219; *The theorem of Thomson and Tait and natural families of trajectories*, vol. 11 (1910), pp. 121–140.

‡ DARBOUX, *Leçons sur les systèmes orthogonaux*, vol. 1 (1898), p. 23.

$$
(4) \qquad \begin{vmatrix}
H_{xx} & H_{yy} & H_{zz} & H_{yz} & H_{xz} & H_{xy} \\
f_{xx} & f_{yy} & f_{zz} & f_{yz} & f_{xz} & f_{xy} \\
1 & 1 & 1 & 0 & 0 & 0 \\
2f_x & 0 & 0 & 0 & f_z & f_y \\
0 & 2f_y & 0 & f_z & 0 & f_x \\
0 & 0 & 2f_z & f_y & f_x & 0
\end{vmatrix} = 0,
$$

where

$$
(5) \qquad H \equiv (f_x^2 + f_y^2 + f_z^2)^{-\frac{1}{2}}.
$$

The problem of the present paper is to find those natural systems for which all the associated wave sets are families of Lamé. The solution together with optical and kinetic formulations follow:

THEOREM I. *The only natural systems whose wave sets are of the Lamé type are those composed of the* ∞^4 *circles orthogonal to a fixed sphere. The corresponding variation problem* (1) *is of the form*

$$
\int \frac{ds}{a_0(x^2+y^2+z^2)+a_1x+a_2y+a_3z+a_4} = \textit{minimum}.
$$

THEOREM II. *The only isotropic media for which every disturbance spreads out by means of a Lamé family of surfaces are those in which the index of refraction is of the form*

$$
[\,a_0(x^2+y^2+z^2)+a_1x+a_2y+a_3z+a_4\,]^{-1};
$$

its value at any point varies inversely as the power of that point with respect to a fixed sphere.

THEOREM III. *The only conservative fields of force such that, when the energy constant is taken equal to zero, every set of surfaces of equal action is of the Lamé type, are those in which the potential at any point varies inversely as the square of the power of the point with respect to a fixed sphere.*

The analytic formulation of our problem is as follows: Determine the function F in such a way that every function f which satisfies the Hamilton-Jacobi equation (3) shall also satisfy the Darboux-Cayley equation (4).

It will be convenient to write the former equation in the form

$$
(6) \qquad f_x^2 + f_y^2 + f_z^2 = 2W,
$$

so that

$$
(7) \qquad 2W \equiv F^2.
$$

By differentiation, we have

$$
(8)\qquad
\begin{aligned}
f_x f_{xx} + f_y f_{yx} + f_z f_{zx} &= W_x, \\
f_x f_{xy} + f_y f_{yy} + f_z f_{zy} &= W_y, \\
f_x f_{xz} + f_y f_{yz} + f_z f_{zz} &= W_z.
\end{aligned}
$$

Hence of the six partial derivatives of the second order which appear in (4) we may eliminate say f_{xx}, f_{yy}, f_{zz}. The final result may be put into the form

$$
(9)\qquad
\begin{vmatrix}
A & B & C & H_{yz} & H_{xz} & H_{xy} \\
W_x & W_y & W_z & f_{yz} & f_{xz} & f_{xy} \\
f_x & f_y & f_z & 0 & 0 & 0 \\
2W & 0 & 0 & 0 & f_z & f_y \\
0 & 2W & 0 & f_z & 0 & f_x \\
0 & 0 & 2W & f_y & f_x & 0
\end{vmatrix} = 0,
$$

where

$$
(10)\qquad
\begin{aligned}
A &\equiv f_x H_{xx} + f_y H_{xy} + f_z H_{xz}, \qquad B \equiv f_x H_{yx} + f_y H_{yy} + f_z H_{yz}, \\
C &\equiv f_x H_{zx} + f_y H_{zy} + f_z H_{zz},
\end{aligned}
$$

$$
(11)\qquad H \equiv (2W)^{-\frac{1}{2}} \equiv F^{-1}.
$$

Equation (9) may be expanded in the form

$$
(12)\qquad \alpha f_{yz} + \beta f_{xz} + \gamma f_{xy} + \delta = 0,
$$

where the coefficients involve only derivatives of first order. Since, in virtue of (8), the quantities f_{yz}, f_{xz}, f_{xy} are numerically arbitrary, it follows that the above coefficients must vanish individually.

It will be sufficient to calculate γ, the minor of f_{xy} in (9). The resulting condition is found to be

$$
(13)\quad f_z\{H_{xy}(f_x^2 - f_y^2) + (H_{yy} - H_{xx}) f_x f_y\} - (f_x^2 + f_y^2)(f_x H_{yz} - f_y H_{xz}) = 0.
$$

This is to hold as a consequence of (6); hence the left hand member of (13), a homogeneous cubic in f_x, f_y, f_z, must be divisible by the non-homogeneous quadratic $f_x^2 + f_y^2 + f_z^2 - 2W$. It follows that (13) must be an identity. Equating the coefficients to zero and adjoining other conditions obtained by cyclic permutation of the letters x, y, z, we find

$$
(14)\qquad H_{xx} = H_{yy} = H_{zz}, \qquad H_{yz} = H_{zx} = H_{xy} = 0.
$$

It follows that the function H must be of the form

$$
(15)\qquad H = a_0(x^2 + y^2 + z^2) + a_1 x + a_2 y + a_3 z + a_4.
$$

It is easily verified that, for this form of H, equation (9) holds identically and is thus certainly satisfied by all the solutions of (6).

The result of our analytic discussion is that the only cases in which all the solutions of (3) are also solutions of (4) are defined by

$$(16)\qquad F = [a_0(x^2+y^2+z^2) + a_1x + a_2y + a_3z + a_4]^{-1}.$$

The natural system associated with such a function is easily shown to consist of the ∞^4 circles orthogonal to the fixed sphere S

$$(17)\qquad a_0(x^2+y^2+z^2) + a_1x + a_2y + a_3z + a_4 = 0.$$

The Lamé families corresponding to the form (15) are well known.* The ∞^1 surfaces of each of these families have for orthogonal trajectories circles orthogonal to S. These circles cut any two of the surfaces and S in four points whose anharmonic ratio is constant.

The ten-parameter group of conformal transformations of space converts natural systems into natural systems and Lamé families into Lamé families; and, therefore, solutions of our problem into solutions. With respect to real transformations there are three types according as the radius of the sphere S is zero, real, or imaginary. The function H may be reduced to one of the forms

$$(18)\qquad x^2+y^2+z^2, \qquad x^2+y^2+z^2-k^2, \qquad x^2+y^2+z^2+k^2,$$

In the optical interpretation the index of refraction is F or $1/H$. It is thus of one of the forms

$$(19)\qquad \frac{1}{r^2}, \qquad \frac{1}{r^2-k^2}, \qquad \frac{1}{r^2+k^2},$$

where r is the distance from the origin and k is a real constant. These three media furnish representations of the parabolic, elliptic, and hyperbolic geometries respectively.

In the kinetic interpretation, the work function W, according to (11), is proportional to H^{-2}, and is thus

$$(20)\qquad \frac{1}{r^4}, \qquad \frac{1}{(r^2-k^2)^2}, \qquad \frac{1}{(r^2+k^2)^2}.$$

The force is hence central and varies as

$$(21)\qquad \frac{1}{r^5}, \qquad \frac{r}{(r^2-k^2)^3}, \qquad \frac{r}{(r^2+k^2)^3}.$$

It is to be remembered that the constant of energy must here be taken equal to zero. The dynamical trajectories are then the systems of circles described above. For other values of the energy constant, the trajectories are represented by elliptic functions and the systems do not give rise to Lamé families.

* They were first studied by RIBAUCOUR. Cf. DARBOUX, *loc. cit.*, pp. 53, 56.

It is easily seen that the natural systems obtained are the only ones composed exclusively of circles. This follows from the fact that any family of surfaces whose orthogonal trajectories are circles is necessarily a Lamé family. We thus have

THEOREM IV. *The only natural systems of circles are those formed by the circles orthogonal to a fixed sphere.*

The media (19) are hence the only simply-refracting media in which all light rays are circular.

In conclusion we observe that, since natural systems are characterized by the property of Thomson and Tait,* and Lamé families are characterized by a certain property of Ribaucour,† our first theorem may be restated in the following purely geometric form:

THEOREM V. *If a system of ∞^4 curves in space is to be such that, for an arbitrarily selected surface Σ: 1) there are ∞^2 curves of the system orthogonal to Σ and these form a normal congruence; 2) the osculating circles of these ∞^2 curves for the points of Σ also form a normal congruence; then the system must be composed of the ∞^4 circles orthogonal to a fixed sphere.*‡

Theorem IV is seen to be included in this result as a special case.

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*These Transactions, vol. 11 (1910), p. 121.

† DARBOUX, *loc. cit.*, p. 78.

‡ It is sufficient, in fact, to require merely that the osculating circles shall form a normal congruence. That the system is then natural follows from Theorem II, these Transactions, vol. 11 (1910), p. 130.