ON SEMI-DISCRIMINANTS OF TERNARY FORMS*

BY

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§ 1. *Introduction.*

It is well known that the number of independent conditional relations which must exist among the coefficients of a ternary form of order m in order that it should be factorable into linear factors, distinct term for term, is $\frac{1}{2}m(m-1)$. Several writers,† among them BRILL, and GORDAN, have published methods for the determination of such sets of relations. Their results are, as a rule, expressed in the form of a covariant, the identical vanishing of which gives necessary and sufficient conditions for the factorability.

These methods are somewhat indirect, and from certain standpoints are unsatisfactory for the additional reason that the set of conditions given by the identical vanishing of such a covariant is always redundant.

Our aim in this paper has been to develop a direct method of attacking this problem. Our method leads to a set of conditional relations containing the exact minimum number $\frac{1}{2}m(m-1)$; that is, it leads to a set of $\frac{1}{2}m(m-1)$ independent seminvariants of the form, whose simultaneous vanishing gives necessary and sufficient conditions for the factorability. We shall call these seminvariants *semi-discriminants* ‡ of the form. They are all of the same degree $2m-1$; and are readily formed for any order m as simultaneous invariants of a certain set of *binary* quantics related to the original ternary form.

If a polynomial, f_{3m}, of order m, and homogeneous in three variables (x_1, x_2, x_3) is factorable into linear factors, its terms in (x_1, x_2) must furnish the (x_1, x_2) terms of those factors. Call these terms collectively a_{0x}^m, and the terms linear in x_3 collectively $x_3 a_{1x}^{m-1}$. Then if the factors of the former were known, and were distinct, say

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† THAER, "*Ueber die Zerlegbarkeit einer ebenen Linie dritter Ordnung in gerade Linien,*" Mathematische Annalen, vol. 14 (1879); GORDAN, "*Das Zerfallen der Curven in gerade Linien,*" Mathematische Annalen, vol. 45 (1894); BRILL, "*Ueber die Zerfällung der Ternärformen in lineare Factoren,*" Mathematische Annalen, vol. 50 (1898).

‡ It seems desirable to introduce a new term for the members of such a minimum set of functions of the form's coefficients. Since the name discriminant is already in common use, and since discriminants are *invariants,* it seems natural to adopt here the term *semi-discriminant.*

$$a_{0x}^m \equiv a_{00}\prod_{i=1}^{m}(r_2^{(i)}x_1 - r_1^{(i)}x_2) \div \prod_{i=1}^{m}(r_2^{(i)}),$$

the second would give by rational means the terms in x_3 required to complete the several factors. For we could find rationally the numerators of the partial fractions in the decomposition of a_{1x}^{m-1}/a_{0x}^m, viz.:

$$\frac{a_{1x}^{m-1}}{a_{0x}^m} \equiv \frac{\prod_{i=1}^{m} r_2^{(i)}}{a_{00}} \sum_{i=1}^{m} \frac{\alpha_i}{r_2^{(i)}x_1 - r_1^{(i)}x_2},$$

and the factors of the complete form will be, of course,

$$r_2^{(i)}x_1 - r_1^{(i)}x_2 + \alpha_i x_3 \qquad (i=1, 2, \cdots, m).$$

Further, the coefficients of all other terms in f_{3m} are rational integral functions of the $r^{(i)}$ on the one hand, and of the α_i on the other — symmetrical in the sets $(r_2^{(i)}, -r_1^{(i)}, \alpha_i)$. We shall show in general that all these coefficients in the case of any linearly factorable form are *rationally* expressible in terms of those occurring in a_{0x}^m, a_{1x}^{m-1}. Hence will follow the important theorem (§ 5):

Theorem 1. *If a ternary form f_{3m} is decomposable into linear factors, all its coefficients, after certain $2m$, are expressible rationally in terms of those $2m$ coefficients. That is, in the space whose coördinates are all the coefficients of ternary forms of order m, the forms composed of linear factors fill a rational spread of $2m$ dimensions.*

We shall thus obtain the explicit form of the general ternary quantic which is factorable into linear factors (§ 5). Moreover, in case f_{3m} is not factorable a similar development will give the theorem (§ 3),

Theorem 2. *Every ternary form f_{3m}, for which the discriminant D of a_{0x}^m does not vanish, can be expressed as the sum of the product of m distinct linear forms, plus the square of an arbitrarily chosen linear form, multiplied by a "satellite" form of order $m-2$ whose coefficients are, except for the factor D^{-1}, integral rational seminvariants of the original form f_{3m}.*

§ 2. *A class of ternary seminvariants.*

Let us write the general ternary quantic in homogeneous variables as follows:

$$f_{3m} = a_{0x}^m + a_{1x}^{m-1}x_3 + a_{2x}^{m-2}x_3^2 + \cdots + a_{m0}x_3^m,$$

where

$$a_{ix}^{m-i} \equiv a_{i0}x_1^{m-i} + a_{i1}x_1^{m-i-1}x_2 + a_{i2}x_1^{m-i-2}x_2^2 + \cdots + a_{im-i}x_2^{m-i} \quad (i=1, 2, \cdots, m).$$

Then write

$$\frac{a_{1x}^{m-1}}{a_{0x}^m} = \frac{a_{1x}^{m-1}}{\prod_{k=1}^{m}(r_2^{(k)}x_1 - r_1^{(k)}x_2)} = \sum_{k=1}^{m}\frac{\alpha_k}{r_2^{(k)}x_1 - r_1^{(k)}x_2} \quad (a_{00}=r_2^{(1)}r_2^{(2)}\cdots r_2^{(m)});$$

and we have in consequence, assuming that $D \neq 0$, and writing

$$a'^{m}_{0r^{(k)}} = \left[\frac{\partial a^m_{0x}}{\partial x_1}\right]_{x_1=r_1^{(k)},\ x_2=r_2^{(k)}}, \qquad a''^{m}_{0r^{(k)}} = \left[\frac{\partial a^m_{0x}}{\partial x_2}\right]_{x_1=r_1^{(k)},\ x_2=r_2^{(k)}},$$

the results

$$\alpha_k = r_2^{(k)} a^{m-1}_{1r^{(k)}} / a'^{m}_{0r^{(k)}} = - r_1^{(k)} a^{m-1}_{1r^{(k)}} / a''^{m}_{0r^{(k)}}. \tag{1}$$

Hence also

$$a''^{m}_{0r^{(k)}} = - \frac{r_1^{(k)}}{r_2^{(k)}} a'^{m}_{0r^{(k)}}. \tag{2}$$

The discriminant of a^m_{0x} can be expressed in the following form:

$$D = \prod_{j=1}^{m} a'^{m}_{0r^{(j)}} / a_{00} (-1)^{\frac{1}{2}m(m-1)}, \tag{3}$$

and therefore

$$\alpha_k = \frac{r_2^{(k)} a^{m-1}_{1r^{(k)}} a'^{m}_{0r^{(1)}} a'^{m}_{0r^{(2)}} \cdots a'^{m}_{0r^{(k-1)}} a'^{m}_{0r^{(k+1)}} \cdots a'^{m}_{0r^{(m)}}}{a_{00} (-1)^{\frac{1}{2}m(m-1)} D}, \tag{4}$$

and in like manner we get

$$\prod_{k=1}^{m} \alpha_k = a^{m-1}_{1r^{(1)}} a^{m-1}_{1r^{(2)}} \cdots a^{m-1}_{1r^{(m)}} / (-1)^{\frac{1}{2}m(m-1)} D. \tag{5}$$

The numerator of the right hand member of this last equality is evidently the resultant (say R_m) of a^m_{0x} and a^{m-1}_{1x}.

Consider next the two differential operators

$$\Delta_1 = m a_{00} \frac{\partial}{\partial a_{10}} + (m-1) a_{01} \frac{\partial}{\partial a_{11}} + \cdots + a_{0m-1} \frac{\partial}{\partial a_{1m-1}},$$

$$\Delta_2 = m a_{0m} \frac{\partial}{\partial a_{1m-1}} + (m-1) a_{0m-1} \frac{\partial}{\partial a_{1m-2}} + \cdots + a_{01} \frac{\partial}{\partial a_{10}};$$

and particularly their effect when applied to a^{m-1}_{1x}. We get [see (2)]

$$\Delta_1 a^{m-1}_{1r^{(k)}} = a'^{m}_{0r^{(k)}}, \qquad \Delta_2 a^{m-1}_{1r^{(k)}} = a''^{m}_{0r^{(k)}} = - \frac{r_1^{(k)}}{r_2^{(k)}} a'^{m}_{0r^{(k)}}, \tag{6}$$

and from these relations we deduce the following:

$$\Delta_1 \prod_{k=1}^{m} \alpha_k = \frac{\Delta_1 R_m}{(-1)^{\frac{1}{2}m(m-1)} \underline{|1} D} = a_{00} \Sigma \frac{a^{m-1}_{1r^{(1)}} a^{m-1}_{1r^{(2)}} \cdots a^{m-1}_{1r^{(m-1)}}}{a'^{m}_{0r^{(1)}} a'^{m}_{0r^{(2)}} \cdots a'^{m}_{0r^{(m-1)}}}, \tag{7}$$

or, from (1)

$$\frac{\Delta_1 R_m}{(-1)^{\frac{1}{2}m(m-1)} \underline{|1} D} = \Sigma \alpha_1 \alpha_2 \cdots \alpha_{m-1} r_2^{(m)}. \tag{8}$$

In (7) the symmetric function Σ is to be read with reference to the r's, the superscripts of the r's replacing the subscripts usual in a symmetric function. Let us now operate with Δ_2 on both members of (7). This gives

$$\frac{\Delta_1\Delta_2 R_m}{(-1)^{\frac{1}{2}m(m-1)}\underline{|1}\,D} = a_{00}\Sigma\,\frac{a^{m-1}_{1r^{(1)}}a^{m-1}_{1r^{(2)}}\cdots a^{m-1}_{1r^{(m-2)}}}{a'^{m}_{0r^{(1)}}a'^{m}_{0r^{(2)}}\cdots a'^{m}_{0r^{(m-2)}}}\left(-\frac{r_1^{(m-1)}}{r_2^{(m-1)}}-\frac{r_1^{(m)}}{r_2^{(m)}}\right).$$

Let Σ_h represent an elementary symmetric function of the two groups of homogeneous variables r_1, r_2 which involves h distinct letters of each group, viz: $r_i^{(m-j+1)}$ $(j=1, 2, \cdots, h)$. Then we have

$$(9)\qquad \frac{\Delta_1\Delta_2 R_m}{(-1)^{\frac{1}{2}m(m-1)}\underline{|1}\,\underline{|1}\,D} = \Sigma[(-1)\alpha_1\alpha_2\cdots\alpha_{m-2}\Sigma_2 r_1^{(m-1)}r_2^{(m)}].$$

We are now in position to prove by induction the following fundamental formula:

$$(10)\quad \frac{\Delta_1^{m-s-t}\Delta_2^t R_m}{(-1)^{\frac{1}{2}m(m-1)}\underline{|m-s-t}\,\underline{|t}\,D}$$
$$= \Sigma[(-1)^t\alpha_1\alpha_2\cdots\alpha_s\Sigma_{m-s}r_1^{(s+1)}r_1^{(s+2)}\cdots r_1^{(s+t)}r_2^{(s+t+1)}\cdots r_2^{(m)}]$$
$$(s=0, 1, \cdots, m;\ t=0, 1, \cdots, m-s),$$

where the outer summation covers all subscripts from 1 to m, superscripts of the r's counting as subscripts in the symmetric function. Representing by $J_{m-s-t,t}$ the left hand member of this equality we have from (6)

$$\Delta_2 J_{m-s-t,\,t} = \Sigma\Bigg((-1)^{t+1}\frac{a^{m-1}_{1r^{(1)}}a^{m-1}_{1r^{(2)}}\cdots a^{m-1}_{1r^{(s-1)}}}{a'^{m}_{0r^{(1)}}a'^{m}_{0r^{(2)}}\cdots a'^{m}_{0r^{(s-1)}}}$$
$$\times\, r_2^{(1)}r_2^{(2)}\cdots r_2^{(s)}\frac{r_1^{(s)}}{r_2^{(s)}}\Sigma_{m-s}r_1^{(s+1)}\cdots r_1^{(s+t)}r_2^{(s+t+1)}\cdots r_2^{(m)}\Bigg).$$

This equals

$$\Sigma(-1)^{t+1}\alpha_1\alpha_2\cdots\alpha_{s-1}S,$$

where S is a symmetric function each term of which involves $t+1$ letters r_1 and $m-s-t$ letters r_2. The number of terms in an elementary symmetric function of any number of groups of homogeneous variables equals the number of permutations of the letters occurring in any one term when the subscripts (here superscripts) are removed. Hence the number of terms in Σ_{m-s} is

$$\frac{\underline{|m-s}}{\underline{|m-s-t}\,\underline{|t}},$$

and the number of terms in S is

$$(m-s+1)\underline{|m-s}/\underline{|t}\,\underline{|m-s-t}.$$

But the number of terms in $\Sigma_{m-s+1}(r_1^{(s)}r_1^{(s+1)}\cdots r_1^{(s+t)}r_2^{(s+t+1)}\cdots r_2^{(m)})$ is

$$\underline{|m-s+1}/\underline{|m-s-t}\,\underline{|t+1}.$$

Hence

$$S=(t+1)\Sigma_{m-s+1},$$

and so

$$\frac{\Delta_2 J_{m-s-t,\,t}}{t+1}=\Sigma\left[(-1)^{t+1}\alpha_1\alpha_2\cdots\alpha_{s-1}\Sigma_{m-s+1}\right].$$

This result, with (9), completes the inductive proof of formula (10).

Now the functions $J_{m-s-t,\,t}$ are evidently simultaneous invariants of the binary forms a_{0x}^m, $a_{0x}'^m$, $a_{0x}''^m$, a_{1x}^{m-1}. We shall show in the next section that the expressions

$$I_{m-s-t,\,t}\equiv Da_{st}-DJ_{m-s-t,\,t}\quad(s=2,3,\cdots,m;\ t=0,1,\cdots,m-s)$$

are, in reality, seminvariants of the form f_{3m} as a whole.

§ 3. *Structure of a ternary form.*

The structure of the right hand member of the fundamental equality (10) shows at once that the general (factorable or non-factorable) quantic f_{3m} $(D\neq 0)$ can be reduced to the following form:

$$(11)\qquad f_{3m}=\prod_{k=1}^{m}(r_2^{(k)}x_1-r_1^{(k)}x_2+\alpha_k)+\sum_{s=2}^{m}\sum_{t=0}^{m-s}(a_{st}-J_{m-s-t,\,t})x_1^{m-s-t}x_2^t.$$

This gives explicitly the general "satellite" form of f_{3m}, with coefficients expressed rationally in terms of the coefficients of f_{3m}. It may be written

$$(12)\qquad \begin{aligned} D\mu_{m-2}&=\sum_{s=2}^{m}\sum_{t=0}^{m-s}\left(Da_{st}-\frac{\Delta_1^{m-s-t}\Delta_2^{t}R_m}{(-1)^{\frac{1}{2}m(m-1)}\underline{|m-s-t}\,\underline{|t}}\right)x_1^{m-s-t}x_2^t\\ &=\sum_{s=2}^{m}\sum_{t=0}^{m-s}I_{m-s-t,\,t}\,x_1^{m-s-t}x_2^t.\end{aligned}$$

Now the coefficients $I_{m-s-t,\,t}$ are seminvariants of f_{3m}. To fix ideas let $m=3$ and write the usual set of ternary operators,

$$\Omega_{x_1x_2}=a_{01}\frac{\partial}{\partial a_{00}}+2a_{02}\frac{\partial}{\partial a_{01}}+3a_{03}\frac{\partial}{\partial a_{02}}+a_{11}\frac{\partial}{\partial a_{10}}+2a_{12}\frac{\partial}{\partial a_{11}}+a_{21}\frac{\partial}{\partial a_{20}},$$

$$\Omega_{x_2x_1}=3a_{00}\frac{\partial}{\partial a_{01}}+2a_{01}\frac{\partial}{\partial a_{02}}+a_{02}\frac{\partial}{\partial a_{03}}+2a_{10}\frac{\partial}{\partial a_{11}}+a_{11}\frac{\partial}{\partial a_{12}}+a_{20}\frac{\partial}{\partial a_{21}},$$

$$\Omega_{x_3x_1}=a_{20}\frac{\partial}{\partial a_{30}}+2a_{10}\frac{\partial}{\partial a_{20}}+3a_{00}\frac{\partial}{\partial a_{10}}+Da_{11}\frac{\partial}{\partial a_{21}}+2a_{01}\frac{\partial}{\partial a_{11}}+Da_{02}\frac{\partial}{\partial a_{12}},$$

etc. . . . .

Then I_{10} is annihilated by $\Omega_{x_2x_1}$ but not by $\Omega_{x_1x_2}$, I_{01} is annihilated by $\Omega_{x_1x_2}$ but not by $\Omega_{x_2x_1}$, and I_{00} is annihilated by $\Omega_{x_1x_2}$ but not by $\Omega_{x_3x_1}$. In general $I_{m-s-t,\,t}$ fails of annihilation when operated upon by a general operator $\Omega_{x_ix_j}$ which contains a partial derivative with respect to a_{st}. We have now proved theorem 2.

§ 4. *The semi-discriminants.*

A necessary and sufficient condition that f_{3m} should degenerate into the product of m distinct linear factors is that μ_{m-2} should vanish identically. Hence, since the number of coefficients in μ_{m-2} is $\frac{1}{2}m(m-1)$, these equated to zero give a minimum set of conditions in order that f_{3m} should be factorable in the manner stated. As previously indicated we refer to these seminvariants as a set of semi-discriminants of the form f_{3m}. They are

$$(13) \qquad I_{m-s-t,\,t} = Da_{st} - \frac{\Delta_1^{m-s-t}\Delta_2^{t}R_m}{(-1)^{\frac{1}{2}m(m-1)}\underline{|t}\,\underline{|m-s-t}} \quad \left(\begin{matrix} s=2,\,3,\,\cdots,\,m; \\ t=0,\,1,\,\cdots,\,m-s \end{matrix}\right).$$

They are obviously independent since each one contains a coefficient (a_{st}) not contained in any other. They are free from adventitious factors, and each one is of degree $2m-1$.

In the case where $m=2$ we have

$$I_{00} = -a_{20}\begin{vmatrix} 2a_{00} & a_{01} \\ a_{01} & 2a_{02} \end{vmatrix} + \begin{vmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & 0 \\ 0 & a_{10} & a_{11} \end{vmatrix}.$$

This is also the *ordinary* discriminant of the ternary quadratic.

The three semi-discriminants of the ternary cubic have been computed by the author by another method.* Corresponding results for the case $m=4$ have not been published. They are the following:

$$I_{00} = \tfrac{1}{27}a_{40}(4i_1^3 - J_1^2) - R_4,$$

where

$$i_1 = a_{02}^2 - 3a_{01}a_{03} + 12a_{00}a_{04},$$

$$J_1 = 27a_{01}^2a_{04} + 27a_{00}a_{03}^2 + 2a_{02}^3 - 72a_{00}a_{02}a_{04} - 9a_{01}a_{02}a_{03},$$

$$R_4 = \begin{vmatrix} a_{10} & a_{11} & a_{12} & a_{13} & 0 & 0 \\ 0 & a_{10} & a_{11} & a_{12} & a_{13} & 0 \\ 0 & 0 & a_{10} & a_{11} & a_{12} & a_{13} \\ a_{01}a_{10}-a_{00}a_{11} & a_{02}a_{10}-a_{00}a_{12} & a_{03}a_{10}-a_{00}a_{13} & a_{04}a_{10} & 0 & 0 \\ a_{00} & a_{01} & a_{02} & a_{03} & a_{04} & 0 \\ 0 & a_{00} & a_{01} & a_{02} & a_{03} & a_{04} \end{vmatrix},$$

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*American Journal of Mathematics, vol. 32 (1910), p. 89.

the other members of the set being obtained by operating upon R_4 with powers of Δ_1, Δ_2:

$$\Delta_1 = 4a_{00}\frac{\partial}{\partial a_{10}} + 3a_{01}\frac{\partial}{\partial a_{11}} + 2a_{02}\frac{\partial}{\partial a_{12}} + a_{03}\frac{\partial}{\partial a_{13}},$$

$$\Delta_2 = 4a_{04}\frac{\partial}{\partial a_{13}} + 3a_{03}\frac{\partial}{\partial a_{12}} + 2a_{02}\frac{\partial}{\partial a_{11}} + a_{01}\frac{\partial}{\partial a_{10}},$$

according to the formula

$$I_{4-s-t,\,t} = a_{st}D - \frac{\Delta_1^{4-s-t}\Delta_2^{t}R_4}{\underline{|4-s-t}\,\underline{|t}} \quad (s=2,3,4;\ t=0,1,\cdots,4-s).$$

§ 5. *Proof of Theorem* 1.

The factors of a_{0x}^m being assumed distinct we can always solve $I_{m-s-t,\,t}=0$ for a_{st}, the result being obviously rational in the coefficients occurring in a_{0x}^m, a_{1x}^{m-1}. This proves theorem 1, as far as the case $D \neq 0$ is concerned. Moreover by carrying the resulting values of a_{st} ($s=2,3,\cdots,m$; $t=0,1,\cdots,m-s$) back into f_{3m} we get the general form of a ternary quantic which is factorable into linear forms. In the result a_{0x}^m, a_{1x}^{m-1} are perfectly general (the former, however, subject to the negative condition $D \neq 0$), whereas

$$(-1)^{\frac{1}{2}m(m-1)}Da_{jx}^{m-j} \equiv \frac{\Delta_1^{m-j}R_m}{\underline{|m-j}}x_1^{m-j} + \frac{\Delta_1^{m-j-1}\Delta_2R_m}{\underline{|m-j-1}\,\underline{|1}}x_1^{m-j-1}x_2 + \cdots + \frac{\Delta_2^{m-j}R_m}{\underline{|m-j}}x_2^{m-j}$$
$$(j=2,3,\cdots,m).$$

Assume next that $D=0$. Then there are two cases to consider. First, a_{0x}^m has multiple factors but f_{3m} as a whole has no multiple linear ternary factors, and a mere interchange of subscripts of the variables (x_1, x_2, x_3) transforms f_{3m} into a new quantic whose binary a_{0x}^m has no multiple factors. For this new quantic, then, $D \neq 0$. Secondly, f_{3m} as a whole may have repeated ternary linear factors. Let there be one such D factor, of multiplicity two. Then $J_{m-s-t,\,t}$, which we now write in the briefer form

$$J_{m-s-t,\,t} = \frac{\Delta_{st}R_m(a_{00}, a_{01}, \cdots)}{D(a_{00}, a_{01}, \cdots)},$$

is indeterminate. In fact, in this case, two of the α_i in the right hand member of (10), say α_1, α_2, are to be replaced by one and the same quantity

$$\alpha_{12} \equiv \frac{r_2^{(1)}\dfrac{\partial}{\partial r_1^{(1)}}a_{1r^{(1)}}^{m-1}}{\dfrac{\partial}{\partial r_1^{(1)}}a_{0r^{(1)}}^{\prime\, m}}.^*$$

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*See Bulletin of the American Mathematical Society, vol. 17 (1911), p. 451, theorem in § 2.

Then it is not difficult to show that the corresponding true value* of $J_{m-s-t,\,t}$, and hence of a_{st}, is

$$a_{st} = \frac{\Delta_{st}\dfrac{\partial}{\partial a_{0i}} R_m(a_{00}, \cdots)}{\dfrac{\partial}{\partial a_{0i}} D(a_{00}, \cdots)} \qquad (i = \text{any number of the set } 0, 1, \cdots, m).$$

Likewise, when f_{3m} contains a linear factor of multiplicity three, three of the α_i in (10), say α_1, α_2, α_3 are to be replaced by the same quantity, viz.:

$$\alpha_{123} \equiv \frac{r_2^{(1)} \dfrac{\partial^2}{\partial r_1^{(1)2}} a^{m-1}_{1r^{(1)}}}{\dfrac{\partial^2}{\partial r_1^{(1)2}} a'^{m}_{0r^{\,1)}}}.$$

and then we get

$$a_{st} = \frac{\Delta_{st}\dfrac{\partial^2}{\partial a_{0i}^2} R_m(a_{00}, \cdots)}{\dfrac{\partial^2}{\partial a_{0i}^2} D(a_{00}, \cdots)} \qquad (i = \text{any number of the set } 0, 1, \cdots, m).$$

Similarly in the case of a factor of f_{3m} of multiplicity higher than three. Hence in these cases also a_{st} is rational in the coefficients of a_{0x}^m, a_{1x}^{m-1}.

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* Cf. SALMON, *Modern Higher Algebra* (fourth edition), p. 90.