VOLTERRA'S INTEGRAL EQUATION OF THE SECOND KIND, WITH DISCONTINUOUS KERNEL, SECOND PAPER

BY

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In an earlier paper † we have considered integral equations of the second kind with variable upper limit ‡ whose kernels in spite of discontinuities were absolutely integrable, and by change of variable we were able to show the existence of solutions in some special cases of equations § where the kernels belonged to a rather general type and were not absolutely integrable. In this paper we take up that type in more detail. In particular, we study the equation

$$u(x)=\phi(x)+\int_a^x \frac{K(x,\xi)}{f(\xi)g(x)}u(\xi)\,d\xi \tag{24}$$

where $K(x,\xi)$, the numerator in the expression for the kernel, is continuous in the triangular region

$$T: \qquad a \leqq \xi \leqq x \leqq b,$$

and $f(x)$, $g(x)$, comprising the denominator, are continuous in the interval

$$t: \qquad a \leqq x \leqq b.$$

References to the Earlier Paper.

$$u(x)=\phi(x)+\int_a^x K(x,\xi)u(\xi)\,d\xi. \tag{1}$$

$$u(x)=\phi(x)+\int_a^x \frac{K(x,\xi)}{f(\xi)}u(\xi)\,d\xi. \tag{9}$$

$$u(x)=\phi(x)+\int_a^x \frac{K(x,\xi)}{f(\xi)g(x)}u(\xi)\,d\xi. \tag{16}$$

* Presented to the Society, September 13, 1909.

† Transactions of the American Mathematical Society, vol. 11 (1910), p. 393. The numbering of equations and sections is continued from that paper, and reference to it in the present paper is made merely by letter and number. For the more important references see below.

‡ Equation (1).

§ Equations (9), (16), (17).

$$(17) \qquad u(x) = \phi(x) + \int_a^x \frac{K(x, \xi)}{g(x) f(\xi) \prod_{i=1}^{b} [\xi - \psi_i(x)]^{\lambda_i}} u(\xi) d\xi .$$

Condition (*A*). A real function of the two variables x, ξ is to be continuous in the triangle $T: a \leqq \xi \leqq x \leqq b$, $b > a \geqq 0$, except on a finite number of curves, each composed of a finite number of continuous pieces with continuously turning tangents. Any vertical portion is to be considered a separate piece, and of such pieces there are to be merely a finite number, $x = \beta_1$, $x = \beta_2$, $\cdots$, $x = \beta_r$. On the other portions of the system of curves there are to be only a finite number of vertical tangents. Hence by any line $x = x_0$, $x_0 \neq \beta_1, \cdots, \beta_r$, the system of curves will be cut in only a finite number of points.

Condition (*B*). In the region t: $a \leqq x \leqq b$, a real function of the single variable x is to be continuous except at a finite number of points, $\gamma_1, \cdots, \gamma_s$, and is to remain finite.

Problem from Hydrostatics. Suppose we are given a tube, lying in a vertical plane along a curve of arbitrary shape, $s = u(x)$, where s is the distance along the curve and x the altitude. Let us fill this tube with a liquid of variable linear density ν, and then regulate its height x in the tube by allowing various amounts to flow out through the bottom. Let us then regard ν as an analytic function of the depth in the liquid, i. e., $\nu = \nu(x - \xi)$.

If

$$h(x) = \frac{\int_0^x \nu(x - \xi) u'(\xi) d\xi}{\int_0^x u'(\xi) d\xi}$$

is the average linear density and $v(x) = u'(x)$, the equation to determine $v(x)$ from $h(x)$ is

$$v(x) = \int_0^x \frac{G(x - \xi) - g'(x)}{g(x)} v(\xi) d\xi ,$$

where

$$\nu(x - \xi) = \alpha + G(x - \xi), \qquad G(0) = 0,$$

and

$$h(x) = \alpha + g(x), \qquad g(0) = 0.$$

9. *Existence theorem for equation* (24).

We have the following existence theorem:

THEOREM. *If in equation* (24)

$$(24) \qquad u(x) = \phi(x) + \int_a^x \frac{K(x, \xi)}{f(\xi) g(x)} u(\xi) d\xi ,$$

1°. (*a*) $K(x, \xi)$ *is continuous in* T, *and* $f(x)$, $g(x)$ *and their first derivatives are continuous in* t;

(*b*) $\partial K(x, \xi)/\partial x$ *and* $\partial K(x, \xi)/\partial \xi$ *satisfy condition* (*A*) *and are finite in* T;

(*c*) $\phi(x)$ *is continuous in* t *except at* $x = a$, *and is such that the function* $\phi(x)g(x)$ *and its first derivative satisfy* B *in* t;

2°. *the function* $f(x)g(x)$ *is greater than zero in* t *except for the value* $x = a$, *and for the value* $x = a$ *vanishes in such a way that*

$$\int_a^x \frac{dx}{f(x)g(x)}$$

is not convergent; *

3°. $$K(a, a) \neq 0;$$

4°. $$\lim_{x=0} \phi(x)g(x) = 0:$$

then under the foregoing conditions,

(*i*) *if* $K(a, a) < 0$ *there exists one solution of* (24) *continuous throughout* t, *and*

(*ii*) *if* $K(a, a) > 0$ *there exists a one parameter family of solutions of* (24) *continuous throughout* t *except possibly for the value* $x = a$. *As* x *approaches* a *each solution remains less than some constant times* $f(x)/(x-a)^\nu$ *where* ν *is any number that satisfies the two conditions*

(*a*) $$1 > \nu > 0,$$

(*b*) $$\nu > 1 - \frac{1}{\left\{\frac{d}{dx}[f(x)g(x)]\right\}_{x=a}}.$$

In regard to ν it is easy to see that the two conditions (*a*), (*b*) under (*ii*) can always be satisfied. For since $f'(x)$ and $g'(x)$ are supposed to be continuous in t and not negative, since that would make $f(x)$ or $g(x)$ negative, $\{d/dx[f(x)g(x)]\}_{x=a}$ will be finite and not negative, and

$$1 - \frac{1}{\left\{\frac{d}{dx}[f(x)g(x)]\right\}_{x=a}}$$

will always be less than 1.

* Otherwise by change of variable $v(x) = u(x)/g(x)$ equation (23) is changed to one that satisfies the conditions of the theorem of section 1. And in that case there would then be one and only one finite $v(x)$.

10. *A theorem that implies the theorem of section* 9.

A theorem that lies closer to the method of proof that we have adopted, and which implies the theorem just stated, is obtained by substituting in the hypothesis for 1° (b) and for 3° the conditions

1′. (b) $\partial K(x, \xi)/\partial x$ *satisfies* A *and is finite in* T;

3′. $K(a, a) \neq 0$; $\lim_{x=a} [K(x, x) - K(a, a)]/(x-a)^{\nu}$ *exists, where* ν *is defined as in section* 9.

Moreover as the method of proof will show, if $K(a, a) < 0$, we may always put $\nu = 0$ without other change in the theorem. Also if $K(a, a) > 0$ and $\{ d/dx(f(x)g(x) \}_{x=a} < 1$ we may take $\nu = 0$. So that for most cases the theorem holds if $\nu = 0$ be substituted throughout.

11. *Approximate Equations.*

The results of this theorem are obtained by means of a method of approximation, for the purpose of which the solution of the simpler equations

$$u(x) = \phi(x) \pm \int_a^x \frac{u(\xi)}{f(\xi)} d\xi \tag{25}$$

is necessary. If (25) has a solution, continuous except for a finite number of points, even when the discontinuities are not finite its derivative will exist and be continuous except for a finite number of points, and will be given by

$$u'(x) = \phi'(x) \pm \frac{u(x)}{f(x)}. \tag{26}$$

Hence any solution of the integral equation (25) will be a solution of the differential equation (26). It remains to be seen under what conditions the solution of the differential equation will be a solution of the integral equation.

Let us write the equations

$$u(x) = \phi(x) - \int_a^x \frac{u(\xi)}{f(\xi)} d\xi, \tag{25′}$$

$$u(x) = \phi(x) + \int_a^x \frac{x(\xi)}{f(\xi)} d\xi. \tag{25″}$$

In the same way we rewrite (26) as

$$u'(x) = \phi'(x) - \frac{u(x)}{f(x)}, \tag{26′}$$

$$u'(x) = \phi'(x) + \frac{u(x)}{f(x)}. \tag{26″}$$

The general solution of (26′) is

$$(27')\qquad u(x)=e^{\int_x^b\frac{dx}{f(x)}}\left[\int_{a'}^x\phi'(\xi)e^{-\int_\xi^b\frac{d(\xi)}{f(\xi)}}d\xi+C\right],$$

and of (26″)

$$(27'')\qquad u(x)=e^{-\int_x^b\frac{dx}{f(x)}}\left[\int_{a'}^x\phi'(\xi)e^{+\int_\xi^b\frac{d\xi}{f(\xi)}}d\xi+C\right],$$

where a' is arbitrarily chosen in the neighborhood of a.

In (27′), since

$$\lim_{\xi=a} e^{-\int_\xi^b\frac{d\xi}{f(\xi)}}=0,$$

we can choose $a'=a$ without loss of generality. So that instead of (27′) we may write

$$(28)\qquad u(x)=e^{\int_x^b\frac{dx}{f(x)}}\left[\int_a^x\phi'(\xi)e^{-\int_\xi^b\frac{d\xi}{f(\xi)}}d\xi+C\right].$$

To see under what conditions this is a solution of (25′) we first notice that the first term of the expression for $u(x)$ in (28) is itself a solution of the equation, if $\phi(a)=0$.

In fact, substituting in the second member of (25′) we have

$$\phi(x)-\int_a^x\frac{u(\xi)}{f(\xi)}d\xi=\phi(x)-\int_a^x\frac{e^{\int_\xi^b\frac{d\xi}{f(\xi)}}}{f(\xi)}\left\{\int_a^\xi\phi'(\xi')e^{-\int_{\xi'}^b\frac{d\xi'}{f(\xi')}}d\xi'\right\}d\xi.$$

If we integrate by parts we obtain

$$\phi(x)+e^{\int_x^b\frac{dx}{f(x)}}\int_a^x\phi'(\xi)e^{-\int_\xi^b\frac{dx}{f(\xi)}}d\xi$$

$$-\lim_{x=a}\left[e^{\int_x^b\frac{dx}{f(x)}}\int_a^x\phi'(\xi)e^{-\int_\xi^b\frac{d\xi}{f(x)}}d\xi\right]-\int_a^x\phi'(\xi)d\xi=u(x)+\phi(a);$$

for, since $|\phi'(\xi)|\leqq N$,

$$\left|e^{\int_x^b\frac{dx}{f(x)}}\int_a^x\phi'(\xi)e^{-\int_\xi^b\frac{d\xi}{f(\xi)}}d\xi\right|\leqq Ne^{\int_x^b\frac{d\xi}{f(\xi)}}\int_a^x e^{-\int_\xi^b\frac{d\xi}{f(\xi)}}d\xi\leqq N(x-a),$$

and

$$\lim_{x=a}e^{\int_x^b\frac{dx}{f(x)}}\int_a^x\phi'(\xi)e^{-\int_\xi^b\frac{d\xi}{f(\xi)}}d\xi=0.$$

Hence $\phi(a)=0$ is a necessary and sufficient condition that the first term of (28) be a solution of (25′).

On the other hand, if $C \neq 0$ the expression (28) cannot be a solution of the equation. For it is necessary that

$$\int_a^x \frac{u(\xi)}{f(\xi)} d\xi$$

be convergent in the neighborhood of a, and since the integral formed from the first part of $u(\xi)$ is convergent, it is necessary that

$$\int_a^x \frac{C e^{\int_\xi^a \frac{d\xi}{f(\xi)}}}{f(\xi)} d\xi$$

be convergent in the neighborhood of a. But this is obviously not the case.

Hence that (25′) have a solution it is necessary and sufficient that $\phi(a) = 0$. The solution is then unique, and is given by

$$u(x) = e^{+\int_x^b \frac{dx}{f(x)}} \int_a^x \phi'(\xi) e^{-\int_\xi^b \frac{d\xi}{f(\xi)}} d\xi. \tag{29}$$

In (27″) we cannot take $a' = a$ because in general to do this would make the integral divergent. Let us then rewrite (27″) as

$$u(x) = -e^{-\int_x^b \frac{dx}{f(x)}} \left[\int_x^{a'} \phi'(\xi) e^{\int_\xi^b \frac{d\xi}{f(\xi)}} d\xi + C \right] \tag{30}$$

and substitute this in the integral equation (25′). We know that the two members resulting have identical derivatives; it remains merely to show that at a given point they take on the same values to make them identical throughout. But to show this is no easier than to substitute directly into the integral equation. The result of the substitution is the same as that obtained by substituting the first term of the expression for $u(x)$, since the second term is obviously a solution of the homogeneous equation. The second member of (25) then becomes

$$\phi(x) + \int_a^x \frac{-e^{-\int_\xi^b \frac{d\xi}{f(\xi)}}}{f(\xi)} \left\{ \int_\xi^{a'} \phi'(\xi') e^{\int_{\xi'}^b \frac{d\xi'}{f(\xi')}} d\xi' \right\} d\xi.$$

If we integrate this by parts we obtain the expression

$$\begin{aligned} \phi(x) &- e^{-\int_x^b \frac{dx}{f(x)}} \int_x^{a'} \phi'(\xi) e^{\int_\xi^b \frac{d\xi}{f(\xi)}} d\xi \\ &+ \lim_{x=a} \left\{ e^{-\int_x^b \frac{dx}{f(x)}} \int_x^{a'} \phi'(\xi) e^{\int_\xi^b \frac{d\xi}{f(\xi)}} d\xi \right\} - \int_a^x \phi'(\xi) d\xi \\ &= u_1(x) + \phi(a) - \lim_{x=a} u_1(x), \end{aligned}$$

where $u_1(x)$ is the first term of $u(x)$, i. e.,

$$u_1(x) = -e^{-\int_x^b \frac{dx}{f(x)}} \int_x^{a'} \phi'(\xi) e^{\int_\xi^b \frac{d\xi}{f(\xi)}} d\xi .$$

We can show that $\lim_{x=a} u_1(x) = 0$ by showing that a quantity η can be determined small enough so that when $(x-a) \leqq \eta$, $|u_1(x)|$ becomes less than or equal to an arbitrarily given H. We have

$$|u_1(x)| \leqq N \left| e^{-\int_x^b \frac{dx}{f(x)}} \int_x^{a'} e^{\int_\xi^b \frac{d\xi}{f(\xi)}} d\xi \right|$$

$$\leqq N e^{-\int_x^b \frac{dx}{f(x)}} \int_x^{s} e^{\int_\xi^b \frac{d\xi}{f(\xi)}} d\xi + N e^{-\int_x^b \frac{dx}{f(x)}} \int_s^{a'} e^{\int_\xi^b \frac{d\xi}{f(\xi)}} d\xi ,$$

where $x \leqq s$. Then

$$|u_1(x)| \leqq N(s-x) + N e^{-\int_x^b \frac{dx}{f(x)}} F(s),$$

where we represent by $F(s)$ the integral

$$\int_s^{a'} e^{\int_\xi^b \frac{d\xi}{f(\xi)}} d\xi .$$

Take $s = H/2N$ thus fixing s by the value of H. This fixes also the value of $NF(s)$. Take now $\eta \leqq s$ and so small that if $x \leqq \eta$,

$$N e^{-\int_x^b \frac{dx}{f(x)}} \leqq \frac{H}{2} .$$

We then have

$$|u_1(x)| \leqq H \qquad (a \leqq x \leqq a + \eta);$$

which was to be proved.

We see therefore that a necessary and sufficient condition that (25″) have a solution is that $\phi(a) = 0$. The general solution is then given by

$$u(x) = -e^{-\int_x^b \frac{dx}{f(x)}} \left[\int_x^{a'} \phi'(\xi) e^{\int_\xi^b \frac{d\xi}{f(\xi)}} d\xi + C \right]. \tag{31}$$

If the function $\phi(x)$ has a zero at a so intense that the expression

$$\int_x^{a'} \phi'(\xi) e^{\int_\xi^b \frac{d\xi}{f(\xi)}} d\xi$$

converges when $x = a$, we have apparently the state of affairs which was treated in section 6 of the earlier paper. To show the agreement of the two results we notice first that the function $f(\xi)$ of (25″) is not the same as that of (9), for in (25″) the function corresponding to $K(x, \xi)$ of (9) being equal to 1 is *not* less than 1, and that in order to reduce our kernel to one that can be treated by the method of sections 1–4 we must replace $f(\xi)$ in (25″) by

$$\bar{f}(\xi) = Hf(\xi), \qquad H < 1,$$

so that the kernel in (25″) becomes $H/\bar{f}(\xi)$. With this change it is seen that the only solution with sufficient intensity of vanishing is

$$u(x) = e^{-\int_x^b \frac{dx}{\bar{f}(x)}} \int_a^x \phi'(\xi) e^{\int_\xi^b \frac{d\xi}{\bar{f}(\xi)}} d\xi.$$

This solution, provided that $\phi'(\xi)$ vanishes as above, continues to hold if we replace $f(x)$ by $\alpha f(x)$ and let α change continually from $+1$ to -1, thus changing (25″) into (25′).

12. *The Method of Approximation.*

Let us return now to equations with kernels of the form $K(x, \xi)/f(\xi)$ still assuming $g(x) \equiv 1$.

By taking the numerical value of $K(a, a)$ into $f(\xi)$ we can write the kernel as

$$\pm \frac{1 + \Gamma(x, \xi)}{f(\xi)},$$

and we are led to the two equations

$$u(x) = \phi(x) - \int_a^x \frac{1 + \Gamma(x, \xi)}{f(\xi)} u(\xi) d\xi, \tag{32'}$$

$$u(x) = \phi(x) + \int_a^x \frac{1 + \Gamma(x, \xi)}{f(\xi)} u(\xi) d\xi, \tag{32''}$$

according as $K(a, a)$ is < 0 or > 0 respectively. Here condition $3°$ of the hypothesis becomes

$$\lim_{x=a} \frac{\Gamma(x, \xi)}{(x - a)^\nu} \text{ exists, } \nu > 0, \ \nu > 1 - \frac{1}{f'(a)},$$

and condition $4°$ becomes

$$\lim_{x=a} \phi(x) = 0.$$

In (32′) let us write

$$u(x) = u_0(x) + u^{(1)}(x),$$

where

$$u_0(x) = \phi(x) - \int_a^x \frac{u_0(\xi)}{f(\xi)} d\xi$$

and consequently

$$u^{(1)}(x) = -\int_a^x \frac{\Gamma(x, \xi)}{f(\xi)} u_0(\xi)\,d\xi - \int_a^x \frac{1 + \Gamma(x, \xi)}{f(\xi)} u^{(1)}(\xi)\,d\xi,$$

or

$$u^{(1)}(x) = \phi_1(x) - \int_a^x \frac{1 + \Gamma(x, \xi)}{f(\xi)} u^{(1)}(\xi)\,d\xi,$$

where

$$\phi_1(x) = -\int_a^x \frac{\Gamma(x, \xi)}{f(\xi)} u_0(\xi)\,d\xi.$$

In the same way we may write

$$u^{(1)}(x) = u_1(x) + u^{(2)}(x),$$

$$u_1(x) = \phi_1(x) - \int_a^x \frac{u_1(\xi)}{f(\xi)}\,d\xi,$$

$$u^{(2)}(x) = \phi_2(x) - \int_a^x \frac{1 + \Gamma(x, \xi)}{f(\xi)} u^{(2)}(\xi)\,d\xi,$$

where

$$\phi_2(x) = -\int_a^x \frac{\Gamma(x, \xi)}{f(\xi)} u_1(\xi)\,d\xi.$$

By continuing the process indefinitely, we have

$$u^{(m)}(x) = u_m(x) + u^{(m+1)}(x),$$

$$u_m(x) = \phi_m(x) - \int_a^x \frac{u_m(\xi)}{f(\xi)}\,d\xi,$$

$$\text{(33)} \qquad u^{(m+1)}(x) = \phi_{m+1}(x) - \int_a^x \frac{1 + \Gamma(x, \xi)}{f(\xi)} u^{(m+1)}(\xi)\,d\xi,$$

$$\phi_{m+1}(x) = -\int_a^x \frac{\Gamma(x, \xi)}{f(\xi)} u_m(\xi)\,d\xi,$$

and (29) gives us

$$\text{(34)} \qquad u_m(x) = e^{\int_x^b \frac{dx}{f(x)}} \int_a^x \phi'_m(\xi)\, e^{-\int_\xi^b \frac{d\xi}{f(\xi)}}\,d\xi.$$

We have also

$$\text{(35)} \qquad \phi'_m(x) = -\frac{\Gamma(x, x)}{f(x)} u_{m-1}(x) - \int_a^x \frac{\Gamma_1(x, \xi)}{f(\xi)} u_{m-1}(\xi)\,d\xi.$$

It is quite easy to find restrictions on the absolute values of $\phi'_m(x)$ and

$u_m(x)$.* If

$$|\phi'_m(x)| \leqq N_m (x-a)^{i_m},$$

it follows that

$$|u_m(x)| \leqq N_m (x-a)^{i_m} e^{\int_x^b \frac{dx}{f(x)}} \int_a^x e^{-\int_\xi^b \frac{d\xi}{f(\xi)}} d\xi .$$

If we integrate by parts and remember that $f(a) = 0$, we have

$$|u_m(x)| \leqq N_m (x-a)^{i_m} \left[f(x) + e^{\int_x^b \frac{dx}{f(x)}} \int_a^x f'(\xi) e^{-\int_\xi^b \frac{d\xi}{f(\xi)}} d\xi \right].$$

Since

$$e^{-\int_\xi^b \frac{d\xi}{f(\xi)}} \leqq e^{-\int_x^b \frac{d\xi}{f(\xi)}},$$

the last expression in the brackets is less than

$$\int_a^x f'(\xi) d\xi = f(x).$$

Hence

$$|u_m(x)| \leqq 2N_m (x-a)^{i_m} f(x). \tag{36}$$

Since a constant A may be chosen so that

$$|\Gamma(x, x)| \leqq A(x-a)^\nu, \qquad |\Gamma_1(x, \xi)| \leqq A, \qquad a \leqq x \leqq R,$$

we have from (35) and (36)

$$\begin{aligned} |\phi'_m(x)| &\leqq A(x-a)^\nu \frac{|u_{m-1}(x)|}{f(x)} + A \int_a^x \frac{|u_{m-1}(\xi)|}{f(\xi)} d\xi \\ &\leqq 2N_{m-1} A (x-a)^{i_{m-1}+\nu} + 2N_{m-1} A \int_a^x (x-a)^{i_{m-1}} dx \\ &\leqq 4N_{m-1} A (x-a)^{i_m+\nu}, \end{aligned}$$

provided that we take R so that $R - a \leqq 1$.

Now $N_0 \leqq N$, where $|\phi'(x)| \leqq N_0$, and therefore

$$\begin{array}{ll} & |u_0| \leqq 2Nf(x), \\ N_1 = 4NA, \; i_1 = \nu, & |u_1| \leqq 2Nf(x)\{4A(x-a)^\nu\}, \\ N_2 = (4A)^2 N, \; i_2 = 2\nu, & |u_2| \leqq 2Nf(x)\{4A(x-a)^\nu\}^2, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ N_m = (4A)^m N, \; i_m = m\nu, & |u_m| \leqq 2Nf(x)\{4A(x-a)^\nu\}^m. \end{array}$$

*In the cases where we can take $\nu = 0$, we define a constant Γ so that if $a \leqq x \leqq R$, $|\Gamma(x, x)| \geqq \Gamma$. We then have $|\phi'_m(x)| \leqq N_m \Gamma^{im}$, etc. By taking R small enough, the series in powers of Γ is shown to be convergent.

13. *A Solution of Equation* (32′).

Let us now form the series

$$U(x) = u_0(x) + u_1(x) + u_2(x) + \cdots. \tag{37}$$

This series is absolutely and uniformly convergent through a region $a \leqq x \leqq R \leqq a + 1$, where R is so small that $4A(R-a)^\nu < 1$. In fact

$$|U(x)| \leqq 2Nf(x)\frac{1}{1-4A(R-a)^\nu},$$

and $U(x)$ vanishes at least to the order of vanishing of $f(x)$.

The continuous function $U(x)$, defined by (37), is a solution of the integral equation (32′). For, write

$$U(x) = u_0(x) + u_1(x) + \cdots + u_n(x) + U^{(n+1)}(x)$$

and substitute in the expression

$$L(x) = u(x) - \phi(x) + \int_a^x \frac{1+\Gamma(x,\xi)}{f(\xi)}u(\xi)d\xi;$$

the result is

$$\begin{aligned} L(x) = U^{(n+1)}(x) - \phi(x) + \int_a^x \frac{1+\Gamma(x,\xi)}{f(\xi)}U^{(n+1)}(\xi)d\xi \\ + u_0(x) + \int_a^x \frac{\Gamma(x,\xi)}{f(\xi)}u_0(\xi)d\xi + \int_a^x \frac{u_0(\xi)}{f(\xi)}d\xi \\ + u_1(x) + \int_a^x \frac{\Gamma(x,\xi)}{f(\xi)}u_1(\xi)d\xi + \int_a^x \frac{u_1(\xi)}{f(\xi)}d\xi \\ \cdots\cdots\cdots\cdots\cdots \\ + u_n(x) + \int_a^x \frac{\Gamma(x,\xi)}{f(\xi)}u_n(\xi)d\xi + \int_a^x \frac{u_n(\xi)}{f(\xi)}d\xi, \end{aligned}$$

or

$$\begin{aligned} L(x) = U^{(n+1)}(x) - \phi_{n+1}(x) + \int_a^x \frac{1+\Gamma(x,\xi)}{f(\xi)}U^{(n+1)}(\xi)d\xi \\ + u_0(x) - \phi(x) + \int_a^x \frac{u_0(\xi)}{f(\xi)}d\xi + u_0(x) - \phi_1(x) + \int_a^x \frac{u_1(\xi)}{f(\xi)}d\xi \\ \cdots\cdots\cdots\cdots\cdots \\ + u_n(x) - \phi_n(x) + \int_a^x \frac{u_n(\xi)}{f(\xi)}d\xi, \end{aligned}$$

or

$$L(x) = U^{(n+1)}(x) - \phi_{n+1}(x) + \int_a^x \frac{1+\Gamma(x,\xi)}{f(\xi)}U^{n+1}(\xi)d\xi.$$

Now

$$|\phi_{n+1}(x)| \leqq \{4A(x-a)^\nu\}^{n+1} N \leqq \{4A(R-a)^\nu\}^{n+1} N,$$

and

$$U^{(n+1)}(x) \leqq 2Nf(x) \sum_{m=n+1}^{\infty} \{4A(x-a)^\nu\}^m \leqq 2Nf(x) \sum_{m=n+1}^{\infty} \{4A(R-a)^\nu\}^m$$

$$\leqq 2Nf(x)\{4A(R-a)^\nu\}^{n+1} \frac{1}{1-4A(R-a)^\nu}.$$

Also

$$\left| \int_a^x \frac{1+\Gamma(x,\xi)}{f(\xi)} U^{(n+1)}(\xi)\,d\xi \right| \leqq (A+1) \int_a^x \frac{|U^{(n+1)}(\xi)|}{f(\xi)}\,d\xi$$

$$\leqq \frac{2N(A+1)\{4A(R-a)^\nu\}^{n+1}}{1-4A(R-a)^\nu} \int_a^x d\xi$$

$$\leqq \frac{2N(R-a)(A+1)\{4A(R-a)^\nu\}^{n+1}}{1-4A(R-a)^\nu}.$$

From these inequalities it follows that

$$|L(x)| \leqq \{4A(R-a)^\nu\}^{n+1} \left[N + \frac{2NB + 2N(R-a)(A+1)}{1-4A(R-a)^\nu} \right],$$

where $B \geqq f(x)$. But the right hand member may be made as small as we please, since $4A(R-a)^\nu < 1$, by taking n large enough. Hence $L(x) \equiv 0$, and $U(x)$ is a solution of the given integral equation (32′).

In order to extend this solution beyond the point R we may write the equation as

$$u(x) = \phi(x) - \int_a^R \frac{1+\Gamma(x,\xi)}{f(\xi)} u(\xi)\,d\xi + \int_R^x \frac{1+\Gamma(x,\xi)}{f(\xi)} u(\xi)\,d\xi,$$

or

$$u(x) = \phi_R(x) + \int_R^x \frac{1+\Gamma(x,\xi)}{f(\xi)} u(\xi)\,d\xi.$$

This is an equation with a continuous kernel, and since

$$\phi_R(x) = \phi(x) - \int_a^R \frac{1+\Gamma(x,\xi)}{f(\xi)} U(\xi)\,d\xi,$$

is also continuous, there is one and only one solution of the equation continuous except for a finite number of discontinuities, and finite, for $R \leqq x \leqq b$. This solution on account of its uniqueness must be independent of the choice of the point R,

$$a < R < a + \frac{1}{(4A)^{1/\nu}}, \qquad a < R < a + 1.$$

We have arrived then by this approximation method at a single continuous solution of the integral equation (32′), which vanishes as x approaches a at least as strongly as $f(x)$.

14. *A Solution of Equation* (32″).

For the solution of (32″) we define

$$U(x) = u_0(x) + u_1(x) + u_2(x) + \cdots,$$

where

$$u_m(x) = \phi_m(x) + \int_a^x \frac{u_m(\xi)}{f(\xi)}\,d\xi, \qquad \phi_m(x) = \int_a^x \frac{\Gamma(x,\xi)}{f(\xi)}\,u_{m-1}(\xi)\,d\xi.$$

From (31) we have as a particular solution

$$u_m(x) = -e^{-\int_x^b \frac{dx}{f(x)}} \int_x^R \phi_m'(\xi)\, e^{\int_\xi^b \frac{d\xi}{f(\xi)}}\,d\xi. \tag{38}$$

We have also

$$\phi_m'(x) = \frac{\Gamma(x,x)}{f(x)}\,u_{m-1}(x) + \int_a^x \frac{\Gamma_1(x,\xi)}{f(\xi)}\,u_{m-1}(\xi)\,d\xi, \tag{39}$$

if the various integrals are convergent. We can in this case also quite easily get restrictions on the absolute values of $\phi_m'(x)$ and $u_m(x)$.

If

$$\phi_m'(x) \leqq N_m(R-a)^{i_m}$$

$$|u_m(x)| \leqq N_m(R-a)^{i_m}\left| e^{-\int_x^b \frac{dx}{f(x)}} \int_x^R e^{\int_\xi^b \frac{d\xi}{f(\xi)}}\,d\xi \right|$$

$$\leqq N_m(R-a)^{i_m}|\bar{u}(x)|,$$

where

$$\bar{u}(x) = -e^{-\int_x^b \frac{dx}{f(x)}} \int_x^R e^{\int_\xi^b \frac{d\xi}{f(\xi)}}\,d\xi.$$

Let us assume for the present, and show later, that ν' and C can be found so that

$$|\bar{u}(x)| \leqq Cf(x)(x-a)^{-\nu'} \qquad (a \leqq x \leqq R',\ 0 \leqq \nu' < \nu),$$

R' to be determined later. When $|f'(a)| < 1$ we can take $\nu' = 0$. We may write, assuming this and taking $R \leqq R'$,

$$|u_m(x)| \leqq CN_m(R-a)^{i_m}(x-a)^{-\nu'}f(x),$$

and from (39)

$$|\phi_m'(x)| \leqq ACN_{m-1}(R-a)^{i_{m-1}}(x-a)^{\nu-\nu'} + ACN_{m-1}(R-a)^{i_{m-1}}\int_a^x \frac{d\xi}{(\xi-a)^{\nu'}},$$

where $\nu - \nu' > 0$. We shall take R near enough to a so that $(R-a) < 1$. Hence

$$|\phi_m'(x)| \leqq ACN_{m-1}(R-a)^{i_{m-1}}(x-a)^{\nu-\nu'}\left(1+\frac{1}{1-\nu'}\right)$$

$$\leqq DCN_{m-1}(R-a)^{i_{m-1}}(x-a)^{\nu-\nu'}$$

$$\leqq DCN_{m-1}(R-a)^{i_{m-1}+\nu-\nu'},$$

where

$$D = A\left(1+\frac{1}{1-\nu'}\right).$$

We have then

$$N_m = DCN_{m-1}, \qquad i_m + i_{m-1} + (\nu - \nu').$$

But since $N_0 = N$, $i_0 = 0$, we have

$$|u_0^{(x)}| \leqq CN(x-a)^{-\nu'}f(x),$$

$$|\phi_1'(x)| \leqq DCN(R-a)^{\nu-\nu'}, \qquad |u_1(x)| \leqq DC^2N(R-a)^{\nu-\nu'}(x-a)^{-\nu'}f(x),$$

$$|\phi_2'(x)| \leqq D^2C^2N(R-a)^{2(\nu-\nu')}, \qquad |u_2(x)| \leqq D^2C^3(R-a)^{2(\nu-\nu')}N(x-a)^{-\nu'}f(x),$$

$$|\phi_m'(x)| \leqq \{DC(R-a)^{\nu-\nu'}\}^m N, \quad |u_m(x)| \leqq \{DC(R-a)^{\nu-\nu'}\}^m CN(x-a)^{-\nu'}f(x).$$

The series

(40) $$U(x) = u_0(x) + u_1(x) + \cdots + u_m(x) + \cdots$$

can now be treated. For if

$$(x-a) \leqq R - a < \left(\frac{1}{DC}\right)^{\frac{1}{\nu-\nu'}}$$

the series for $(x-a)^{\nu'}U(x)$ is absolutely and uniformly convergent, and represents a continuous function of x, $a \leqq x \leqq R$. And

$$|U(x)| \leqq N(x-a)^{-\nu'}f(x)\frac{C}{1-DC(R-a)^{\nu-\nu'}}.$$

$U(x)$ is shown to be a solution of the equation in the same way as for equations of the other type. The inequalities are but slightly different. We have

$$L(x) = U(x) - \phi(x) - \int_a^x \frac{1+\Gamma(x,\xi)}{f(\xi)}U(\xi)\,d\xi$$

$$= U^{(n+1)}(x) - \phi_{n+1}(x) - \int_a^x \frac{1+\Gamma(x,\xi)}{f(\xi)}U^{(n+1)}(\xi)\,d\xi,$$

where

$$|\phi_{m+1}(x)| \leqq \{DC(R-a)^{\nu-\nu'}\}^{n+1}N,$$

$$|U^{(n+1)}(x)| \leqq N(x-a)^{-\nu'}f(x)C\frac{\{DC(R-a)^{\nu-\nu'}\}^{n+1}}{1-DC(R-a)^{\nu-\nu'}},$$

and

$$\left|\int_a^x \frac{1+\Gamma(x,\xi)}{f(\xi)}U^{(n+1)}(\xi)\,d\xi\right| \leqq (A+1)\int_a^x \frac{|U^{(n+1)}(\xi)|}{f(\xi)}\,d\xi$$

$$\leqq \frac{\{DC(R-a)^{\nu-\nu'}\}^{n+1}N(A+1)C}{1-DC(R-a)^{\nu-\nu'}}\,\frac{(x-a)^{1-\nu'}}{1-\nu'},$$

so that

$$|L(x)| \leqq \{DC(R-a)^{\nu-\nu'}\}^{n+1}\left[N+\frac{N(x-a)^{-\nu'}f(x)}{1-DC(R-a)^{\nu-\nu'}}\right.$$

$$\left.+\frac{N(A+1)C(x-a)^{1-\nu'}}{(1-\nu')\{1-DC(R-a)^{\nu-\nu'}\}}\right].$$

The right hand member of this inequality can be made as small as we please for any value x_0 of x, $a \leqq x \leqq R$, by taking n large enough. Hence $L(x) \equiv 0$, $a \leqq x \leqq R$, and $U(x)$ is a solution of (32″).

15. *The Function* $\bar{u}(x)$.

Let us now return and justify the inequalities stated for the function

$$\bar{u}(x) = -e^{-\int_x^b \frac{dx}{f(x)}}\int_x^R e^{\int_\xi^b \frac{d\xi}{f(\xi)}}\,d\xi$$

in section 14. This function is a solution of the equation

$$\bar{u}(x) = x + \int_a^x \frac{\bar{u}(\xi)}{f(\xi)}\,d\xi.$$

We have assumed that $f(x)$ is positive for $a < x \leqq R$ and that

$$\int_a^R \frac{dx}{f(x)}$$

is not convergent. We shall show that $|\bar{u}(x)| \leqq Cf(x)$ when $|f'(a)| < 1$.

Consider first the value of *

$$\frac{1}{f(x)}e^{-\int_x^b \frac{dx}{f(x)}}.$$

* The method used here is essentially the same as that used in evaluating $\lim_{x=a} f(x)/g(x)$ when $\lim_{x=a} f(x) = \infty$, $\lim_{x=a} g(x) = \infty$. See OSGOOD: *Calculus*, p. 236.

This remains finite as x approaches a. We have, in fact,

$$\frac{\dfrac{1}{f(x)}-\dfrac{1}{f(x')}}{e^{\int_x^{b'}\frac{dx}{f(x)}}-e^{\int_{x'}^{b'}\frac{dx}{f(x)}}}=\frac{\dfrac{d}{dx}\left(\dfrac{1}{f(X)}\right)}{\dfrac{d}{dx}\left(e^{\int_x^{b}\frac{dx}{f(x)}}\right)},\qquad a<x<X'<x',$$

or

$$\frac{\dfrac{1}{f(x)}}{e^{\int_x^{b}\frac{dx}{f(x)}}}\left[\frac{1-\dfrac{f(x)}{f(x')}}{1-e^{\int_x^{x'}\frac{dx}{f(x)}}}\right]=\frac{\dfrac{f'(X)}{f(X)}}{e^{\int_X^{x}\frac{dx}{f(x)}}}.$$

Choose $x'=R'$ close enough to the point a so that

$$|f'(x)|<H,\qquad a<x\leqq x',$$

where H is a constant such that $f'(a)<H<1$. There is then a point $x_0<x'$ such that if $a<x\leqq x_0$ the bracket may be written $1+\lambda(x)$, where $|\lambda(x)|\leqq\eta$, η arbitrarily assigned $<1-H$. We have then

$$\frac{\dfrac{1}{f(x)}}{e^{\int_x^{b}\frac{dx}{f(x)}}}(1-\eta)\leqq H\frac{\dfrac{1}{f(X)}}{e^{\int_X^{b}\frac{dx}{f(x)}}},$$

and

$$\frac{\dfrac{1}{f(x)}}{e^{\int_x^{b}\frac{dx}{f(x)}}}<\frac{\dfrac{1}{f(X)}}{e^{\int_X^{b}\frac{dx}{f(x)}}}.$$

Hence for every point $x\leqq x_0$ there must be a point X, $x_0<X\leqq R'$, for which

$$\frac{\dfrac{1}{f(X)}}{e^{\int_X^{b}\frac{dx}{f(x)}}}$$

is greater; otherwise we should get a sequence of points X_1, X_2, $\cdots$ having a limit point $\bar{x}\leqq x_0$. This point, however, determines in the same way a point $\bar{X}$, for which $\bar{X}>\bar{x}$. So there is no limiting point in this sense.

But for all points beyond x_0 the given function

$$\frac{\dfrac{1}{f(x)}}{e^{\int_x^{b}\frac{dx}{f(x)}}}$$

is less than or equal to a certain constant J. Hence

$$\frac{1}{f(x)} e^{-\int_x^b \frac{dx}{f(x)}} \leqq J,$$

and

$$e^{-\int_x^b \frac{dx}{f(x)}} \leqq J f(x) \qquad (a \leqq x \leqq R').$$

Consider now the function

$$\bar{u}(x) = -e^{-\int_x^b \frac{dx}{f(x)}} \int_x^R e^{\int_\xi^b \frac{d\xi}{f(\xi)}} d\xi,$$

and perform an integration by parts. Then

$$\bar{u}(x) = f(x) - e^{-\int_x^b \frac{dx}{f(x)}} \left[f(R) e^{\int_R^b \frac{dx}{f(x)}} \right] + e^{-\int_x^b \frac{dx}{f(x)}} \int_x^R f'(\xi) e^{\int_\xi^b \frac{d\xi}{f(\xi)}} d\xi$$

$$\leqq f(x) \left(1 + J f(R) e^{\int_R^b \frac{dx}{f(x)}} \right) + e^{-\int_x^b \frac{dx}{f(x)}} \int_x^R |f'(\xi)| e^{\int_\xi^b \frac{d\xi}{f(\xi)}} d\xi.$$

We have assumed $R \leqq R'$, whence $|f'(\xi)| < H$.

Defining

$$J' = J f(R) e^{\int_R^b \frac{dx}{f(x)}},$$

we have

$$|\bar{u}(x)| \leqq (1 + \zeta') f(x) + H |\bar{u}(x)|,$$

and since $H < 1$,

$$|\bar{u}(x)| \leqq \frac{(1 + J') f(x)}{1 - H},$$

and this defines the constant C.

We now consider $\bar{u}(x)$ when $f(a) = c$, $|c| \geqq 1$. The constant c must be positive; otherwise $f(x)$ would be negative in the neighborhood of a. We have

$$(c - \delta)(x - a) \leqq f(x) \leqq (c + \delta)(x - a), \qquad a \leqq x \leqq a + \eta,$$

where δ is small, so chosen that $c - \delta > 0$ and neither $c - \delta$ nor $c + \delta$ is equal to 1. Hence

$$\frac{1}{(c + \delta)(x - a)} \leqq \frac{1}{f(x)} \leqq \frac{1}{(c - \delta)(x - a)},$$

and

$$|\bar{u}(x)| = e^{-\int_x^b \frac{dx}{f(x)}} \int_x^R e^{\int_\xi^b \frac{d\xi}{f(\xi)}} d\xi = e^{-\int_x^\eta \frac{dx}{f(x)}} \int_x^R e^{\int_\xi^\eta \frac{d\xi}{f(\xi)}} d\xi$$

$$\leqq e^{-\int_x^\eta \frac{d\xi}{(c+\delta)(x-a)}} \int_x^R e^{\int_\xi^\eta \frac{d\xi}{(c-\delta)(\xi-a)}} d\xi.$$

If we now perform the indicated integrations, we have

$$|\bar{u}(x)| \leqq \left(\frac{x-a}{\eta-a}\right)^{\frac{1}{c+\delta}} \int_x^R \left(\frac{\xi-a}{\eta-a}\right)^{\frac{-1}{c-\delta}} d\xi.$$

If we write

$$c' = \frac{1}{c}, \qquad c' - \delta' = \frac{1}{c+\delta}, \qquad c' + \delta'' = \frac{1}{c-\delta},$$

the right hand member becomes

$$\left(\frac{x-a}{\eta-a}\right)^{c'-\delta'} \int_x^R \left(\frac{\xi-a}{\eta-a}\right)^{-(c'+\delta'')} d\xi,$$

where $c' - \delta' > 0$, $c' - \delta' \neq 1$, $c' + \delta'' \neq 1$. Hence

$$\bar{u}(x) \leqq (\eta - a)^{\delta'+\delta''}(x-a)^{c'-\delta'} \frac{(R-a)^{1-c'-\delta''} + (x-a)^{1-c'-\delta''}}{|1-c'-\delta''|}$$

$$\leqq \frac{(\eta-a)^{\delta'+\delta''}}{|1-c'-\delta''|}\left[(x-a)^{c'-\delta'}(R-a)^{1-c'-\delta''} + (x-a)^{1-\delta'-\delta''}\right],$$

where we may take δ' and δ'' as small as we please. And so if $c = c' = 1$ we may take $\nu' > 0$ as near zero as we please; if $c > 1$, $c' < 1$ we may take $\nu' > 1 - c'$ as near $1 - c'$ as we please. Hence for the general case $c \geqq 1$, we take

$$\nu' = 1 - \frac{1}{c} + \epsilon = 1 - \frac{1}{f'(a)} + \epsilon$$

where ϵ is as small as we please. And thus we prove the inequalities of § 14.

16. *The One Parameter Family of Solutions of Equation* (32″).

The question now is presented to us, since we have obtained a particular solution of (32″) by one choice of R, whether it is not possible to get other solutions by means of different values of R, $R = R_1, R_2, \cdots, R_j$, and whether there is any sort of dependence among the solutions thus resulting. The answer is that all the solutions corresponding to values of R satisfying the conditions laid down for it belong to a linear system

$$V(x) = U(x) + CW_0(x).$$

The difference of any two solutions of the non-homogeneous equation (25″) or the non-homogeneous equation (32″) is a solution of the corresponding homogeneous equation. Now the complete solution of the homogeneous equation correspond- to (25″),

$$w(x) = \int_a^x \frac{w(\xi)}{f(\xi)} d\xi,$$

is

$$w(x) = e^{-\int_x^R \frac{d\xi}{f(\xi)}}.$$

If we let $W(x) = w_1(x) + W^{(1)}(x)$, where $w_1(x) = w(x)$ and substitute in

$$W(x) = \int_a^x \frac{1 + \Gamma(x, \xi)}{f(\xi)} W(\xi)\, d\xi, \tag{41}$$

which is the homogeneous equation corresponding to (32″), the equation to determine $W^{(1)}(x)$ will be

$$W^{(1)}(x) = \int_a^x \frac{\Gamma(x, \xi)}{f(\xi)} w(\xi)\, d\xi + \int_a^x \frac{1 + \Gamma(x, \xi)}{f(\xi)} W^{(1)}(\xi)\, d\xi,$$

or

$$W^{(1)}(x) = \psi(x) + \int_a^x \frac{1 + \Gamma(x, \xi)}{f(\xi)} W^{(1)}(\xi)\, d\xi,$$

where the function

$$\psi(x) = \int_a^x \frac{\Gamma(x, \xi)}{f(\xi)} w(\xi)\, d\xi$$

vanishes when $x = a$, and $\psi'(x)$ remains finite as $x \doteq a$. Hence we get a solution for this equation, keeping the same value for R that was used in getting $U(x)$ in § 14, by forming a function $U_\psi(x)$ based on $\psi(x)$ in the same way that the $U(x)$ of § 14 was based on $\phi(x)$. $U_\psi(x)$ will contain C as a factor to the first degree. Hence

$$W(x) = w_1(x) + U_\psi(x)$$

will be a solution of the homogeneous equation (41), and will contain C as a factor to the first degree. $W(x)$ is not identically zero since it takes on the value C when $x = R$. We may write $W(x)$ as $CW_0(x)$ where $W_0(x)$ takes on the value 1 when $x = R$; and we have thus a system of solutions of the non-homogeneous equation (32″) given by

$$V(x) = U(x) + CW_0(x).$$

Now we may write the general solution of (25″), as we have seen, in the form

$$v(x) = e^{-\int_x^b \frac{dx}{f(x)}} \int_x^R \phi'(\xi) e^{\int_\xi^b \frac{d\xi}{f(\xi)}} d\xi + Ce^{-\int_x^b \frac{dx}{f(x)}}.$$

Suppose we form a series $V(x)$ in the same way we formed the series $U(x)$ of § 17, except that we use the expressions

$$v_m(x) = e^{-\int_x^b \frac{dx}{f(x)}} \int_x^R \phi_m'(\xi) e^{\int_\xi^b \frac{d\xi}{f(\xi)}} d\xi + c_m e^{-\int_x^b \frac{dx}{f(x)}} = \bar{u}_m(x) + c_m e^{-\int_x^b \frac{dx}{f(x)}},$$

instead of the terms $u_m(x)$. If we let w_i be the general term in the expansion of $W_0(x)$ we may write the sum of the first m terms of $V(x)$ as

$$\sum_{i=1}^{m} u_i(x) + c_1 \sum_{i=1}^{m} w_i(x) + c_2 \sum_{i=1}^{m-1} w_i(x) + \cdots + c_{m-1} \sum_{i=1}^{2} w_i(x) + c_m w_1(x).$$

We have already shown the existence of the limits

$$\lim_{m=\infty} \sum_{i=1}^{m} u_i(x), \qquad \lim_{m=\infty} \sum_{i=1}^{m} |w_i(x)|.$$

We now investigate

$$(a) \qquad \lim_{m=\infty} \left\{ \sum_{i=1}^{m} c_i \sum_{j=1}^{m-i+1} w_j(x) \right\}.$$

The expression (a) may be rewritten in the form

$$\lim_{m=\infty} \sum_{p=1}^{m} \sum_{i=1}^{p} c_i w_{p+1-i}.$$

Consider now $c_1 + c_2 + \cdots$ and $w_1 + w_2 + \cdots$. Since $w_1 + w_2 + \cdots$ is absolutely convergent the product of these two series will be the series whose general term is

$$\sum_{i=1}^{p} c_i w_{p+1-i},$$

i. e., the expression (a), provided that the series of c's is convergent.* We see then that if the series of c's is convergent, the series of v's is convergent, V represents a function belonging to the linear system already discovered, and is hence a solution of the integral equation (32″).

On the other hand if the series of v's is convergent for one value of x, $x = R_1 \leqq R$, V is again of the specified linear system. For we may write $v_m(x)$ in the form

$$v_m(x) = e^{-\int_x^{R_1} \frac{dx}{f(x)}} \int_x^{R_1} \phi'_{m-1}(\xi) e^{\int_\xi^{R_1} \frac{d\xi}{f(\xi)}} d\xi + c'_m e^{-\int_x^{R_1} \frac{dx}{f(x)}},$$

so that

$$v_m(R_1) = c'_m, \qquad V(R_1) = \sum_0^\infty c'_m,$$

and the series

$$\sum_0^\infty c'_m = \bar{C}$$

is convergent. The series for $V(x)$ is therefore convergent for $a \leqq x \leqq R_1$,

* *Encyklopädie der Mathematischen Wissenschaften*, vol. I, page 96.

and the function $V(x)$ may be written in the form

$$\bar{U}(x) + \bar{C}\bar{W}(x).$$

Since, however, all solutions of the form

$$U(x) + CW_0(x)$$

are convergent at $x + R'$ they all belong to this family. In fact $W_0(x)$ is a constant times $\bar{W}(x)$, both being solutions of the homogeneous equation, since by the above treatment, putting $\phi(x) = 0$, the family $CW_0(x)$ is contained in the family $\bar{C}\bar{W}(x)$; and $\bar{U}(x)$ is therefore given by $U(x) + C_1 W_0(x)$, where C_1 is some constant. So the two families are identical. We have thus shown that all solutions obtained by taking different values of R in the development of § 14 are members of this family.

17. *Extension of the One Parameter Family beyond R.*

We have still to see how we can extend our solutions beyond the farthest point R. We may write (32″) as

$$u(x) = \phi(x) + \int_a^R \frac{1 + \Gamma(x, \xi)}{f(\xi)} u(\xi)\, d\xi + \int_R^x \frac{1 + \Gamma(x, \xi)}{f(\xi)} u(\xi)\, d\xi$$

or

$$u(x) = \psi_c(x) + \int_R^x \frac{1 + \Gamma(x, \xi)}{f(\xi)} u(\xi)\, d\xi,$$

where

$$\psi_c(x) = \phi(x) + \int_a^R \frac{1 + \Gamma(x, \xi)}{f(\xi)} u(\xi)\, d\xi$$

$$= \phi(x) + \int_a^R \frac{1 + \Gamma(x, \xi)}{f(\xi)} \{U(\xi) + CW_0(\xi)\}\, d\xi.$$

This equation, when the value of C is specified, gives rise to a single solution $u(x)$, since in the region $R \leqq x \leqq b$ the kernel and the function $\psi_c(x)$ are continuous. Moreover this solution has at R the value $\psi_c(R)$ or C and hence joins on continuously to the particular solution that was assumed from a to R.

In particular the solutions $U(x)$ and $U(x) + W_0(x)$ can be so extended, and there will be a one-parameter family of solutions $U(x) + CW_0(x)$, arising from the fact that $W_0(x)$ is a solution of the homogeneous equation. But this family is identical with that obtained by extending each solution separately, since there is only one solution that takes on the value C when $x = R$. Thus the solutions are extended uniquely and continuously to the point b.

18. *The Theorem of section* 10. *A more General Theorem.*

The theorem of § 10 is now easily made complete. For given the equation

$$u(x) = \phi(x) + \int_a^x \frac{K(x, \xi)}{f(\xi) g(x)} u(\xi) d\xi \tag{20}$$

satisfying the conditions of that theorem, we succeed, by making the substitution

$$w(x) = u(x) g(x), \qquad u(x) = \frac{w(x)}{g(x)},$$

in transforming the given equation into the equation

$$w(x) = \phi(x) g(x) + \int_a^x \frac{K(x, \xi)}{[f(\xi) g(\xi)]} w(\xi) d\xi, \tag{42}$$

so that if $u(x)$ is a solution of the first, $w(x)$ is a solution of the second, and vice versa. But this latter equation is of the type of section 12, for which we have solutions continuous in the neighborhood of a and vanishing at a as strongly as $f(x)g(x)$ if $K(a, a) < 0$, or $f(x)g(x)/(x-a)^{\nu'}$ if $K(a, a) > 0$. Hence the solution of the given equation for $K(a, a) < 0$ is continuous at a and in its neighborhood, and for $K(a, a) > 0$ the solutions of the given equation constitute a one-parameter family, continuous in the neighborhood of a except at a and, as we approach a, remaining less in absolute value than $f(x)/(x-a)^{\nu'}$ (since $\nu > \nu'$). It is possible, if $K(a, a) > 0$, to have solutions that become infinite at a.

It is worth noting that by the same sort of substitution,

$$w(x) = \frac{g(x)}{g_1(x)} u(x),$$

we can treat the slightly more general equation

$$u(x) = \phi(x) + \int_a^x \frac{f_1(\xi) g_1(x)}{f(\xi) g(x)} K(x, \xi) u(\xi) d\xi.$$

In fact we have the following theorem:

THEOREM. *In the equation*

$$u(x) = \phi(x) + \int_a^x \frac{f_1(\xi) g_1(x)}{f(\xi) g(x)} K(x, \xi) u(\xi) d\xi,$$

if

1°. (*a*) *$K(x, \xi)$ is continuous in T, and $f(x)$, $g(x)$, $f_1(x)$, $g_1(x)$ and the first derivative of the function $f(x) g(x)/f_1(x) g_1(x)$ are continuous in t;*

(*b*) $\partial K(x, \xi)/\partial x$ *and* $\partial K(x, \xi)/\partial \xi$ *satisfy condition* (A) *and are finite in* T*;*

(*c*) $\phi(x)$ *is continuous in* t *except at* $x = a$, *and is such that the function* $\phi(x) \cdot g(x)/g_1(x)$ *and its first derivatives satisfy condition* (B) *in* t*;*

2°. *the function* $f(x)g(x)$ *is greater than zero in* t *except for the value* $x = a$, *and for the value* $x = a$ *vanishes in such a way that*

$$\int_a^x \frac{f_1(\xi)g_1(x)}{f(\xi)g(x)}\,dx$$

is not convergent;

3°. $K(a, a) \neq 0$;

4°. $\lim_{x=a} \phi(x)\dfrac{g(x)}{g_1(x)} = 0$:

then under the foregoing conditions,

(*i*) *if* $K(a, a) < 0$ *there exists one solution of the equation continuous throughout* t *except perhaps at* a, *and this solution as* x *approaches* a *remains less in absolute value than some constant times* $f(x)/f_1(x)$*; and*

(*ii*) *if* $K(a, a) < 0$ *there exists a one parameter family of solutions continuous throughout* t *except perhaps at* a, *and each solution as* x *approaches* a *remains less in absolute value than some constant times* $f(x)/[f_1(x)(x-a)^\nu]$, *where* ν *is any number that satisfies the conditions*

$$1 > \nu > 0, \qquad \nu > 1 - \frac{1}{\left\{\dfrac{d}{dx}\left[\dfrac{f(x)g(x)}{f_1(x)g_1(x)}\right]\right\}_{x=a}}.$$

19. *Extension of Solutions Beyond a Second Discontinuity of the Kernel.*

The question arises as to whether the solutions can be extended beyond a second vanishing point of $f(x)$ or $g(x)$. In general they cannot, as the following example shows. The equation

$$u(x) = \phi(x) - \int_0^x \frac{(a-\xi)}{\xi(2a-\xi)}\,u(\xi)\,d\xi$$

has the single solution

$$u(x) = e^{\int_x^a \frac{a-x}{x(2a-x)}dx}\int_0^x \phi'(\xi)\,e^{-\int_\xi^a \frac{(a-\xi)}{\xi(2a-\xi)}d\xi}\,d\xi$$

which becomes infinite as x approaches $2a$.

Still, there are large classes of solutions that can be extended. A condition is that the solution itself vanish at the point where $f(x)$ or $g(x)$ vanishes.

20. *Discontinuity at the End of the Interval.*

We have confined ourselves to integral equations (1) where $a \leqq x \leqq b$. When, instead, we have $a \geqq x \geqq b$, so that the discontinuity in the kernel is at the end of the interval, if we make the substitutions

$$x' = a - x, \qquad x = a - x', \qquad\qquad (0 < x' \leqq a - b),$$
$$\xi' = a - \xi, \qquad \xi = a - \xi',$$
$$d\xi' = - d\xi,$$

the equation

$$u(x) = \phi(x) + \int_a^x \frac{K(x, \xi)}{g(x) f(\xi)} u(\xi) d\xi$$

is reduced to

$$u(a - x') = \phi(a - x') - \int_0^{x'} \frac{K(a - x', a - \xi')}{f(a - \xi') g(a - x')} u(a - \xi') d\xi',$$

or

$$\bar{u}(x') = \bar{\phi}(x') - \int_0^{x'} \frac{\bar{K}(x', \xi')}{\bar{f}(\xi') \bar{g}(x')} \bar{u}(\xi') d\xi'.$$

Hence an equation where $K(a, a) > 0$ is changed into an equation of type (32′), and one where $K(a, a) < 0$ into one of type (32″).

21. *A Different Point of View.*

An instructive point of view as regards the nature of the solutions of (32″) developed in section 16, is obtained in the following way. If we had written (32″) in the form

$$u(x) = \phi(x) + \int_a^x \frac{1 + \lambda \Gamma(x, \xi)}{f(\xi)} u(\xi) d\xi,$$

thus introducing the parameter λ,* and sought for a solution as a power series in λ,

$$V(x) = v_0(x) + \lambda v_1(x) + \lambda^2 v_2(x) \cdots,$$

the conditions determining $v_m(x)$ would be precisely those that we have already found, and would yield

$$v_m(x) = \bar{u}_m(x) + c_m e^{-\int_x^b \frac{dx}{f(x)}}.$$

Hence all solutions analytic in λ belong to the linear family

$$U(x) + C W_0(x).$$

* This is analogous to the point of view of M. T. Lalesco in regard to equations of the first kind. T. Lalesco: *Sur l'équation de Volterra*, Journal de Mathematiques, ser. 6, vol. 4 (1908), p. 125. See Transactions of the American Mathematical Society, vol. 11 (1910), p. 395.

The same investigation applied to the equation (32′) shows that there is one and only one solution of the equation that is analytic in λ.

22. *A Restrictive Theorem upon a Class of Solutions.*

We can go, however, somewhat farther in limiting the number of solutions of equations (32′) and (32″). The solutions so far developed satisfy the condition

$$|V(x)| \leqq \text{const}\,\frac{f(x)}{(x-a)^{\nu}}.$$

We can show that any solution which is continuous in the neighborhood of the origin, except possibly at the origin, and which satisfies this condition, is among the solutions already found. We have the following theorem:

THEOREM. *Given the hypotheses of* § 10, 1°–4°, *the solutions there specified are the only ones continuous in the neighborhood of the origin, except possibly at the origin, such that in the neighborhood of the origin each solution satisfies the condition*

$$|u(x)| \leqq \text{const}\,\frac{f(x)}{(x-a)^{\nu}},$$

or, more generally, if

$$\Phi(x) = \int_a^x \frac{K(x,\xi) - K(a,a)}{f(\xi)} u(\xi)\,d\xi,$$

such that

(*a*) $\Phi(x)$ *approaches zero as* x *approaches* a;

(*b*) $\Phi'(x)$ *remains finite as* x *approaches* a.

There are no such solutions if 1°–3° *of section* 10 *hold, but not* 4°.

By the transformation

$$u(x) = \frac{w(x)}{g(x)}$$

the given equation is reduced to the equation (42)

$$w(x) = \phi(x)g(x) + \int_a^x \frac{K(x,\xi)}{f(\xi)g(x)} w(\xi)\,d\xi, \tag{42}$$

and the condition becomes $\lim \Phi(x) = 0$, and $\Phi'(x)$ remains finite, as x approaches a, where

$$\Phi(x) = \int_a^x \frac{K(x,\xi) - K(a,a)}{f(\xi)g(\xi)} w(\xi)\,d\xi.$$

If we assume the first part of the theorem of this section, the second part follows immediately. For since

$$w(x) = \phi(x)g(x) + \Phi(x) + \int_a^x \frac{w(\xi)}{a\dfrac{1}{K(a,a)}f(\xi)g(\xi)}\,d\xi$$

$w(x)$ must satisfy an equation of the type of (25′) or (25″). But we have seen that it is necessary, in order that there be a solution of those equations, that the term not involving the dependent variable have zero as a limit as x approaches a. Hence

$$\lim_{x=a}[\phi(x)g(x)+\Phi(x)]=0.$$

But

$$\lim_{x=a}\Phi(x)=0,$$

so that

$$\lim_{x=a}[\phi(x)g(x)]=0,$$

which was to be proved.

If in (42) we put $\phi(x)$ for $\phi(x)g(x)$, $f(\xi)$ for $f(\xi)g(\xi)$, and $u(x)$ for $w(x)$, we have the equation of § 12, which is then reducible to (32′) or (32″). The condition then becomes

$$\lim_{x=a}\Phi(x)=0, \qquad |\Phi'(x)|\leqq N, \qquad a\leqq x\leqq a' \qquad (a'>a),$$

where

$$\Phi(x)=\int_a^x\frac{\Gamma(x,\xi)}{f(\xi)}u(\xi)\,d\xi.$$

Let us investigate this theorem in regard to equation (32″). Replacing $1+\Gamma(x,\xi)$ by $K(x,\xi)$ we write (32″) as

$$u(x)=\phi(x)+\int_a^x\frac{K(x,\xi)}{f(\xi)}u(\xi)\,d\xi. \tag{43}$$

The equation (25″) then becomes

$$u(x)=\phi(x)+\int_a^x\frac{K(x,\xi)-\Gamma(x,\xi)}{f(\xi)}u(\xi)\,d\xi, \tag{44}$$

and we can obtain a solution of it by a method of approximation from (43).

For, let $U_0^0(x)$ be any chosen solution of (43) that satisfies the conditions of the theorem, and let us try to find a solution of (44) in the form

$$u_0(x)=U_0^0(x)+U_1^0(x)+U_2^0(x)+\cdots$$

where

$$U_m^0(x)=\Phi_m^0(x)+\int_a^x\frac{K(x,\xi)}{f(\xi)}U_m^0(\xi)\,d\xi, \tag{45}$$

and

$$\Phi_m^0(x)=-\int_a^x\frac{\Gamma(x,\xi)}{f(\xi)}U_{m-1}^0(\xi)\,d\xi, \tag{46}$$

$$\Phi_0^0(x)=\phi(x).$$

Then $\Phi_1^0(x) = -\Phi(x)$, as defined in the hypothesis, so that $\Phi_1^0(a) = 0$, and $d\Phi_1^0(x)/dx$ remains finite. For U_1^0 take the solution developed in section 14, on $\Phi_1^0(x)$ as a basis. That analysis gives us

$$|U_1^0(x)| \leqq (x-a)^{-\nu'} \frac{CNf(x)}{1-DC(R-a)^{\nu-\nu'}}{}^{*} \qquad (R \leqq a'),$$

so that

$$|\Phi_2^{0\prime}(x)| \leqq N \frac{DC(R-a)^{\nu-\nu'}}{1-DC(R-a)^{\nu-\nu'}},$$

since

$$\Phi_2^{0\prime}(x) = -\frac{\Gamma(x,x)}{f(x)} U_1^0(x) - \int_a^x \frac{\Gamma_1(x,\xi)}{f(\xi)} U_1^0(\xi)\,d\xi.$$

In the same way

$$(\alpha) \quad |U_m^0(x)| \leqq (x-a)^{-\nu'} \left\{ \frac{DC(R-a)^{\nu-\nu'}}{1-DC(R-a)^{\nu-\nu'}} \right\}^{m-1} \frac{CNf(x)}{1-DC(R-a)^{\nu-\nu'}},$$

and

$$(\beta) \qquad |\Phi_m^{0\prime}(x)| \leqq N \left\{ \frac{DC(R-a)^{\nu-\nu'}}{1-DC(R-a)^{\nu-\nu'}} \right\}^{m-1}.$$

If we denote this parenthesis by ρ we have

$$\sum_{m=1}^{\infty} |\Phi_m^{0\prime}(x)| \leqq \frac{N}{1-\rho}$$

and the series for u_0 is uniformly convergent for $a \leqq x \leqq a'$, since taking R near enough to a we can make ρ as small as we please.

The function u_0 thus developed is shown in the same way as in § 14 to be a solution of the equation (44), which is, of course, equation (25″). Let us now pursue a method of approximation from (44) back again to (43), defining each solution of (44) by a method of approximation from solutions of (43). We define

$$\phi_1(x) = \int_a^x \frac{\Gamma(x,\xi)}{f(\xi)} u_0(\xi)\,d\xi,$$

and in general

$$\phi_n(x) = \int_a^x \frac{\Gamma(x,\xi)}{f(\xi)} u_{n-1}(\xi)\,d\xi, \qquad u_n(x) = \sum_{m=0}^{\infty} U_m^n(x),$$

$$U_m^n(x) = \Phi_m^n(x) + \int_a^x \frac{K(x,\xi)}{f(\xi)} U_m^n(\xi)\,d\xi,$$

$$\Phi_m^n(x) = -\int_a^x \frac{\Gamma(x,\xi)}{f(\xi)} U_{m-1}^n(\xi)\,d\xi, \qquad \Phi_0^n(x) = \phi_n(x).$$

* Or $CNf(x)/(1-DC\Gamma)$ when $\nu = 0$.

The series for $u_n(x)$ is a solution of the equation

$$u_n(x) = \phi_n(x) + \int_a^x \frac{u(\xi)}{f(\xi)}\,d\xi,$$

as may be proved by a method similar to that used in section 14.

We have in fact,

$$\begin{aligned}\phi_1(x) &= \int_a^x \frac{\Gamma(x,\xi)}{f(\xi)} \sum_{m=0}^{\infty} U_m^0(\xi)\,d\xi \\ &= \sum_{m=0}^{j} \int_a^x \frac{\Gamma(x,\xi)}{f(\xi)}\, U_m^0(\xi)\,d\xi + \int_a^x \frac{\Gamma(x,\xi)}{f(\xi)} \sum_{j+1}^{\infty} U_m^0(\xi)\,d\xi,\end{aligned}$$

and the limit of this as j becomes infinite gives us, the last term disappearing on account of the inequalities (α), (β),

$$\phi_1(x) = \sum_{m=0}^{\infty} \int_a^x \frac{\Gamma(x,\xi)}{f(\xi)}\, U_m^0(\xi)\,d\xi,$$

or

$$\phi_1(x) = -\sum_{m=1}^{\infty} \Phi_m^0(x).$$

We see then that if we define

$$U_0^1(x) = -\sum_{m=1}^{\infty} U_m^0(x),$$

it will be a solution of the equation

$$U_0^1(x) = \Phi_0^1(x) + \int_a^x \frac{K(x,\xi)}{f(\xi)}\, U_0^1(\xi)\,d\xi,$$

and will satisfy the inequality

$$|U_0^1(x)| \leqq (x-a)^{-\nu} f(x) \frac{N\rho}{D(1-\rho)}, \qquad \text{since} \qquad (x-a)^{-\nu'} \leqq \frac{(R-a)^{\nu-\nu'}}{(x-a)^{\nu}}.$$

Now if we determine $U_m^1(x)$, $m > 0$, as the function U of section 14 based on $\Phi_m^1(x)$, we have

$$|\Phi_m^{1\prime}(x)| \leqq \frac{N\rho^m}{(1-\rho)},$$

and

$$\sum_{m=1}^{\infty} |\Phi_m^{1\prime}(x)| \leqq \frac{N}{1-\rho} \cdot \frac{\rho}{1-\rho}.$$

And if in general we put

$$U_0^n(x) = -\sum_{m=1}^{\infty} U_m^{n-1}(x),$$

$U_0^n(x)$ will be a solution of its proper equation

$$U_0^n(x) = \Phi_0^n(x) + \int_a^x \frac{K(x, \xi)}{f(\xi)} U_0^n(\xi) d\xi,$$

and

$$(\gamma) \; |U_0^n(x)| \leqq (x-a)^{-\nu} f(x) \frac{N}{D} \cdot \left(\frac{\rho}{1-\rho}\right)^n, \quad |\Phi_m^{n\prime}(x)| \leqq \frac{N}{1-\rho}\left(\frac{\rho}{1-\rho}\right)^{n-1} \rho^m,$$

and

$$(\delta) \qquad |\phi'_{n+1}(x)| \leqq \sum_{m=1}^{\infty} |\Phi_m^{n\prime}(x)| \leqq \frac{N}{1-\rho}\left(\frac{\rho}{1-\rho}\right)^n;$$

formulas that are verified by mathematical induction.

Let us now consider the function

$$V(x) = \lim_{n=\infty} [u_0(x) + u_1(x) + u_2(x) + \cdots + u_n(x)].$$

We have

$$\begin{aligned} u_0(x) + u_1(x) &+ \cdots + u_n(x) \\ &= U_0^0(x) + U_1^0(x) + U_2^0(x) + \cdots \\ &\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &\quad + U_0^n(x) + U_1^n(x) + U_2^n(x) + \cdots, \end{aligned}$$

and this reduces to

$$U_0^0(x) - U_0^{n+1}(x).$$

But by (γ)

$$|U_0^{n+1}(x)| \leqq (x-a)^{-\nu} f(x) \frac{N}{D}\left(\frac{\rho}{1-\rho}\right)^{n+1},$$

whence if we take R so small that $\rho < \frac{1}{2}$, i. e., $\rho/(1-\rho) < 1$, we shall have

$$\lim_{n=\infty} |U_0^{n+1}(x)| \equiv 0.$$

Hence

$$V(x) = U_0^0(x).$$

But since

$$\sum_{n=0}^{\infty} u_n(x)$$

is convergent for $x = R$, $V(x)$ must be one of the linear family

$$V(x) = U(x) + CW_0(x).*$$

Hence any solution that satisfies the conditions of the present theorem is a member of that one-parameter family.

This same analysis applies to equation (32′). In the case of (32′), since each $u_n(x)$ is uniquely determined, there is only one possible solution.

* See section 16.

23. *A Restrictive Theorem upon the Totality of Solutions.*

So far we have considered only finite solutions, or at most solutions that become infinite at $x = a$ to an order not greater than the first. It is possible, however, to limit the totality of solutions as to character.

THEOREM. *Let the kernel of* (1) *be in the form*

$$\frac{K(x, \xi) f_1(\xi)}{f(\xi) g(x)}*$$

where

1°. (*a*) *$K(x, \xi)$ is continuous in T*,† *and $f(x)$, $g(x)$ and $f_1(x)$ are continuous in t;*

(*b*) *$\partial K(x, \xi)/\partial\xi = K_2(x, \xi)$ satisfies A, and is finite in T;*

(*c*) *$\phi(x)$ is continuous in t except perhaps at a, and is such that the function $\phi(x) g(x)$ is continuous at a.*

2°. *The function $f(x) g(x)$ vanishes at most a finite number of times in t.*

3°. *On any horizontal line $\xi = \xi_0$ cutting T there is at least one point in T for which $K(x, \xi) \neq 0$.*‡

Then all the solutions of (1) *continuous in t except for a finite number of points are such that the function $u(x) g(x)$ remains continuous in t, except possibly at $x = b$.* §

This conclusion is equivalent to the statement that there can be no solutions, continuous in t except possibly for a finite number of points, of the equation

$$u(x) = \phi(x) + \int_a^x K(x, \xi) \frac{f_1(\xi)}{f(\xi)} u(\xi) d\xi,$$

and not continuous in all points of t except possibly $x = b$. In fact equations of the first form may be reduced to those of the second by change of independent variable (see section 18).

For the proof of this theorem we need the following lemma, which is stated in slightly more general form than necessary:

Let $h(x)$ be a function of x continuous for $A < x \leqq B$ except for a finite number of finite discontinuities, such that

$$\int_A^B h(x) dx$$

converges.

---

* This redundant form of expression for the kernel is used on account of condition 3°.

† If we replace the triangle T by the square S: $a \leqq \xi \leqq b$, $a \leqq x \leqq b$ this theorem holds for the equation with constant limits, $u(x) = \phi(x) + \int_a^b K(x, \xi) u(\xi) d\xi$.

‡ An example that does not satisfy the condition 3° is given by BÔCHER, *Introduction to the Study of Integral Equations*, p. 17. The equation is $u(x) = \int_0^x \xi^{x-\xi} u(\xi) d\xi$, which has as solutions $u(x) = kx^{x-1}$, which are non-integrable.

§ This exception was not noted in a preliminary paper, Bulletin of the American Mathematical Society, 16, (1909), p. 130.

Let $r(x)$ be a function such that $r(x)$ and $r'(x)$ are continuous for $A \leqq x \leqq B$ except for a finite number of finite discontinuities.

Then

$$\int_A^B r(x)h(x)dx$$

converges.

The statement is easily verified by integration by parts. If $B' > A$ is any point before the second discontinuity of $r(x)$,

$$\int_A^B r(x)h(x)dx = \int_{B'}^B r(x), h(x)dx + r(B')\int_A^{B'} h(x)dx - \int_A^{B'} r'(x)\int_A^x h(s)dsdx,$$

whence if in the region $A \leqq x \leqq B$ we define P, P' and Q by the relations

$$P \geqq |r(x)|, \qquad P' \geqq |r'(x)|, \qquad Q\begin{cases} \geqq \left|\int_A^x h(s)ds\right|, \\ \geqq \left|\int_{B'}^B |h(x)|dx\right|, \end{cases}$$

we shall have

$$\left|\int_A^B r(x)h(x)dx\right| \leqq PQ + P'Q|B' - A|.$$

To apply this lemma to the conditions of the theorem it is convenient to extend our integrations outside the oblique boundary of T. For this purpose let us define

$$K(x, \xi) = K(x, x), \qquad \xi > x,$$

by which means a continuous extension across the boundary line $\xi = x$ is provided. Beyond this boundary we shall have $K_2(x, \xi) = 0$.

Let us suppose that $x = a$ is the first point where $u(x)$ is not continuous. On the line $\xi = a$ in T there is a point in the neighborhood of which $K(x, a) \neq 0$. Let $x = x_0$ be a point for which $K(x, a) \neq 0$ and for which

$$\int_a^{x_0} K(x_0 \xi)\frac{f_1(\xi)}{f(\xi)}u(\xi)d\xi$$

is convergent. Then also

$$\int_a^R K(x_0, \xi)\frac{f_1(\xi)}{f(\xi)}u(\xi)d\xi$$

is convergent, $a \leqq R \leqq x_0$. Let us fix R less than a_1, any second supposed discontinuity of $u(x)$, and near enough to a so that $K(x_0, \xi) \neq 0$, $a \leqq \xi \leqq R$, and $f(\xi) \neq 0$, $a < \xi \leqq R$.

In the integral

$$\int_a^R K(x_0 \xi) \frac{f_1(\xi)}{f(\xi)} u(\xi)$$

take the integrand as the function $h(\xi)$ of the lemma. As $r(x)$ choose the function

$$\frac{K(x, \xi)}{K(x_0 \xi)} \qquad (a \leqq x \leqq R),$$

which for a fixed value of x is a continuous function of ξ. The value of $r'(\xi)$ is

$$r'(\xi) = \frac{K(x_0 \xi) K_2(x, \xi) - K(x, \xi) K_2(x_0 \xi)}{[K(x_0, \xi)]^2},$$

and accordingly remains finite. Consequently the conditions of the lemma are satisfied. Since

$$r(\xi) h(\xi) = K(x, \xi) \frac{f_1(\xi)}{f(\xi)} u(\xi),$$

we have the convergence of the integral

$$\int_a^R \frac{K(x, \xi) f_1(\xi)}{f(\xi)} u(\xi)\, d\xi \qquad (a \leqq x \leqq R),$$

which carries with it the convergence of the integral

$$u(x) - \phi(x) = \int_a^x K(x, \xi) \frac{f_1(\xi)}{f(\xi)} u(\xi)\, d\xi \qquad (a \leqq x \leqq R).$$

Since the numbers P, P', Q of the lemma can be defined as independent of x, it follows that

$$\lim_{x=a} \int_a^x K(x, \xi) \frac{f_1(\xi)}{f(\xi)} u(\xi)\, d\xi = 0,$$

and therefore, since $\phi(x)$ is continuous at $x = a$, $u(x)$ is also.

In the same way the integral

$$\int_{a_1}^x K(x, \xi) \frac{f_1(\xi)}{f(\xi)} u(\xi)\, d\xi$$

may be treated, where x is in the neighborhood of a_1 $(a_1 \neq b)$, the next succeeding possible discontinuity of u. The integrals

$$\int_a^{a_1} K(x, \xi) \frac{f_1(\xi)}{f(\xi)} u(\xi)\, d\xi, \qquad \int_a^x K(x, \xi) \frac{f_1(\xi)}{f(\xi)} u(\xi)\, d\xi$$

are continuous functions of x at $x = a_1$. And therefore, since $\phi(x)$ is continuous, $u(x)$ must be continuous at $x = a_1$ also.

24. *Continuity of Certain Functions Represented by Definite Integrals.*

In particular the above method of proof establishes the continuity throughout t except at b of any integral of the form

$$\int_a^x G(x, \xi)\frac{f_1(\xi)}{f(\xi)}u(\xi)\,d\xi,$$

where $G(x, \xi)$ is continuous in T, $\partial G/\partial\xi$ satisfies (A) and is finite in T, and $u(x)$ is a solution of the equation

$$u(x) = \phi(x) + \int_a^x K(x, \xi)\frac{f_1(\xi)}{f(\xi)}u(\xi)\,d\xi,$$

and therefore where $u(x)$ is a solution of the equation

$$u(x) = \phi(x) + \int_a^x K(x, \xi)\frac{f_1(\xi)}{f(\xi)g(x)}u(\xi)\,d\xi,$$

under the conditions imposed in section 23. Also if $G(x, \xi)$ and $\partial G/\partial\xi$ merely satisfy (A) and are finite in T, the integral is continuous in t except at a finite number of points, and remains finite except possibly at $x = b$.

25. *Application to Analytic Kernels.*

If we let $g(x) \equiv 1, f(x) \equiv 1$ the equation (1) becomes

$$u(x) = \phi(x) + \int_a^x f_1(\xi)K(x, \xi)u(\xi)\,d\xi.$$

We know that there is *one and only one continuous solution of this equation.* The theorem of *section* 23 tells us that *this is the only solution continuous except for a finite number of points.*

Under this special case is included the integral equation with analytic kernel; for any function of the two variables analytic in T can be written in the form

$$f_1(\xi)K(x, \xi)$$

where $K(x, \xi)$ satisfies the conditions of section 23. If $\phi(x)$ is also analytic the unique solution is the known analytic solution.*

26. *Equations for which section* 9 *gives the Complete Solution.*

Let us make the hypothesis:

The kernel of (1) *shall be in the form*

$$\frac{K(x, \xi)}{f(\xi)g(x)},$$

* If the kernel of the integral equation with constant coefficients is analytic in S and if $\phi(x)$ is continuous in t, the continuous solutions are the only solutions continuous except for a finite number of points.

where

1°. (*a*) $K(x, \xi)$ *is continuous in* T, *and* $f(x)g(x)$ *and first derivatives are continuous in* t;

(*b*) $\partial K(x, \xi)/\partial x$ *and* $\partial K(x, \xi)/\partial \xi$ *satisfy* A, *and are finite in* T;

(*c*) $\phi(x)$ *is continuous in* t *except perhaps at* a, *and is such that the function* $\phi(x)g(x)$ *and its first derivative satisfy* (B);

2°. *the function* $f(x)g(x)$ *is greater than zero in the neighborhood of* a, *and vanishes at* a *in such a way that*

$$\int_a^x \frac{dx}{f(x)g(x)}$$

is not convergent;

3°.
$$K(a, a) \neq 0;$$

4°.
$$\lim_{x=a} \phi(x)g(x) = 0.$$

Under these conditions the requirements of the theorems of sections 9 or 10 and 23 are satisfied in the neighborhood of the origin. There are some cases where the solutions specified in section 9 are the only possible solutions of the equation continuous except for a finite number of points. To investigate this let us impose the further condition that

$$K_{12}(x, \xi) = \frac{\partial^2 K}{\partial x \partial \xi}$$

shall satisfy A *and be finite in* T.* It is easy to show that if in addition (i) $K(x, \xi)$ *is constant when* $\xi = x$ *or* (ii) *it is known that the derivative of* $u(x)g(x)$ *remains finite as* x *approaches* a, *then all solutions continuous in* t *except for a finite number of points belong to the system of section* 9.

We have the equations (see section 22),

$$\Phi(x) = \int_a^x \frac{K(x, \xi) - K(a, a)}{f(\xi)} u(\xi)\, d\xi,$$

$$\Phi'(x) = \frac{K(x, x) - K(a, a)}{f(x)} u(x) + \int_a^x \frac{K_1(x, \xi)u(\xi)}{f(\xi)} d\xi \quad (x \neq a),$$

where

$$K_1(x, \xi) = \partial K(x, \xi)/\partial x.$$

In (i)

$$K(x, x) - K(a, a) \equiv 0,$$

and since the integrals

$$\int_a^x \frac{K(x, \xi) - K(a, a)}{f(\xi)} u(\xi)\, d\xi, \qquad \int_a^x \frac{K_1(x, \xi)}{f(\xi)} u(\xi)\, d\xi,$$

* This condition was left out in the preliminary paper mentioned before.

are both convergent and remain finite at $x = a$ (see section 24), $\Phi(x)$ and $\Phi'(x)$ both have the limit 0 as x approaches a. Hence the conditions of the theorem of section 22 are satisfied.

In (*ii*) we have

$$u(x)g(x) = \phi(x)g(x) + \int_a^x \frac{K(x, \xi)}{f(\xi)} u(\xi)d\xi,$$

$$\frac{d}{dx}\{u(x)g(x)\} = \frac{d}{dx}\{\phi(x)g(x)\} + \frac{K(x, x)}{f(x)} u(x) + \int_a^x \frac{K_1(x, \xi)}{f(\xi)} u(\xi)d\xi \quad (x \neq a),$$

$$\frac{d}{dx}\{u(x)g(x)\} - \frac{d}{dx}\{\phi(x)g(x)\} - \int_a^x \frac{K_1(x, \xi)}{f(\xi)} u(\xi)d\xi = \frac{K(x, x)}{f(x)} u(x) \quad (x \neq a),$$

where

$$K_1(x, \xi) = \frac{\partial K(x, \xi)}{\partial x}.$$

The first member of this last equation remains finite as x approaches a; hence the second member must remain finite. And since $K(x, \xi)$ is continuous and $K(a, a) \neq 0$, the function $u(x)/f(x)$ and therefore

$$[K(x, x) - K(a, a)] \frac{u(x)}{f(x)}$$

remains finite as x approaches a. Hence by section 24

$$\lim_{x=a} \Phi(x) = 0,$$

and $\Phi'(x)$ remains finite as x approaches a. So the conditions of the theorem of section 22 are satisfied.

The solutions specified in section 9 are therefore the only solutions continuous except for a finite number of points.

27. *Equations Whose Kernels are Analytic Functions of x, ξ Divided by a Power of ξ. Treatment by Power Series.*

Let us now consider in the neighborhood of the origin equations where the kernel is in the form of an analytic function,

$$K(x, \xi) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij} x^i \xi^j,$$

divided by an integral power of ξ, ξ^μ. This kernel is no less general than that in which the denominator is of the form $x^\nu \xi^\mu$, since the second case can be reduced to the first by a change of dependent variable. We shall suppose $\phi(x)$ also to be analytic at $x = 0$.

Let us seek analytic solutions of this equation,

$$(48) \qquad u(x) = \phi(x) + \int_0^x \frac{K(x, \xi)}{\xi^\mu} u(\xi) d\xi,$$

in the neighborhood of the origin. Substitute

$$u(x) = \sum_{i=0}^{\infty} b_i x^i, \qquad \phi(x) = \sum_{i=0}^{\infty} g_i x^i,$$

$$\frac{K(x, \xi)}{\xi^\mu} = \frac{1}{\xi^\mu} \sum_{p=\lambda}^{\infty} \sum_{i=0}^{p} a_{i,p-i} x^i \xi^{p-i} \qquad (\mu > \lambda \geqq 0).$$

The equation then becomes

$$\sum_{i=0}^{\infty} b_i x^i = \sum_{i=0}^{\infty} g_i x^i + \int_0^x \sum_{r=\lambda}^{\infty} \sum_{p=\lambda}^{r} b_{r-p} \xi^{r-p-\mu} \sum_{i=0}^{p} a_{i,p-i} x^i \xi^{p-i} d\xi.$$

Let us formally determine the coefficients b_i. Our operations will be justified by a proof of the convergence of the resulting series for $u(x)$.

We have

$$\sum_{i=0}^{\infty} b_i x^i = \sum_{i=0}^{\infty} g_i x^i + \int_0^x \sum_{r=\lambda}^{\infty} \sum_{r=\lambda}^{r} b_{r-p} \sum_{i=0}^{p} a_{i,p-1} x^i \xi^{r-\mu-i} d\xi.$$

We may assume that $K(x, \xi)$ contains no factor ξ. Hence in order that the integral be convergent it is necessary that $u(x)$ contain x^μ as a factor. Under these conditions we get

$$(48') \qquad \sum_{i=0}^{\infty} b_i x^i = \sum_{i=0}^{\infty} g^i x^i + \sum_{r=\mu}^{\infty} \sum_{p=\lambda}^{r-\mu} b_{r-p} \sum_{i=0}^{p} \frac{a_{i,p-i}}{r-\mu-i+1} x^{r-\mu+1},$$

and

$$(48'') \qquad b_{r-\mu+1} = g_{r-\mu+1} + \sum_{p=\lambda}^{r-\mu} b_{b-p} \sum_{i=0}^{p} \frac{a_{i,p-i}}{r-\mu-i+1}.$$

Or, writing

$$r - \lambda = m, \qquad r = \lambda + m,$$

we have

$$b_{m+\lambda-\mu+1} = g_{m+\lambda-\mu+1} + \sum_{p=\lambda}^{m+\lambda-\mu} b_{m+\lambda-p} \sum_{i=0}^{p} \frac{a_{i,p-i}}{m+\lambda-\mu-i+1}.$$

To determine b_m we have the equation

$$-b_m \left(\sum_{i=0}^{\lambda} \frac{a_{i,\lambda-1}}{m+\lambda-\mu-i+1} \right) = g_{m+\lambda-\mu+1} - b_{m+\lambda-\mu+1}$$
$$+ \sum_{p=\lambda+1}^{m+\lambda-\mu} b_{m+\lambda-p} \sum_{i=0}^{p} \frac{a_{i,p-1}}{m+\lambda-\mu-i+1},$$

or, on writing $s = m + \lambda - p$,

$$(49)\qquad -b_m\left(\sum_{i=0}^{\lambda}\frac{a_{i,\lambda-i}}{m+\lambda-\mu-i+1}\right) = g_{m+\lambda-\mu+1} - b_{m+\lambda-\mu+1} + \sum_{s=\mu}^{m-1} b_s \sum_{i=0}^{m+\lambda-s}\frac{a_{i,m+\lambda-s-1}}{m+\lambda-\mu-i+1}.$$

Since $b_m = 0$ for $m < \mu$, g_m must be zero for $i \leqq \lambda$.

We have seen that if $\lambda = 0$, i. e., $K(0,0) \neq 0$ the solutions are developable into a series of functions which have *essential singularities* at the origin. And so in general we may expect the above series for $u(x)$ to have no circle of convergence; this in fact can be shown by particular cases, the coefficients b_m being comparable to $m!$ in magnitude. In the special case when $\mu - \lambda = 1$ however, the convergence of the series may be shown. The equation to determine b_m becomes

$$(50)\qquad b_m\left(1-\sum_{i=0}^{\lambda}\frac{a_{i,\lambda-i}}{m-i}\right) = g_m + \sum_{s=\mu}^{m-1} b_s \sum_{i=0}^{m+\lambda-s}\frac{a_{i,m+\lambda-s-i}}{m-i},$$

or

$$b_m = \frac{g_m}{1-\sum_{i=0}^{\lambda}\frac{a_{i,\lambda-i}}{m-i}} + \frac{\sum_{s=0}^{m-1} b_s \sum_{i=0}^{m+\lambda-s}\frac{a_{i,m+\lambda-i}}{m-i}}{1-\sum_{i=0}^{\lambda}\frac{a_{i,\lambda-i}}{m-i}}.$$

Let us take $m \geqq m_0$ great enough so that

$$\sum_{i=0}^{\lambda}\frac{|a_{i,\lambda-i}|}{m_0-i} \leqq \frac{1}{m_0-\lambda}\sum_{i=0}^{\lambda}|a_{i,\lambda-i}| \leqq \tfrac{1}{2}.$$

Then we shall have

$$|b_m| < 2|g_m| + 2\sum_{s=\mu}^{m-1}|b_s|\sum_{i=0}^{m+\lambda-s}\frac{|a_{i,m+\lambda-s-i}|}{m-i} \qquad (m \geqq m_0),$$

and therefore

$$|b_m| < 2|g_m| + 2\sum_{s=\mu}^{m-1}|b_s|\sum_{i=0}^{m+\lambda-s}|a_{i,m+\lambda-s-i}|.$$

Let us define

$$c_s = \sum_{i=0}^{\lambda+s}|a_{i,\lambda+s-i}|.$$

Then c_s is the sum of the absolute values of the terms of the sth diagonal in the rectangular array of $K(x,\xi)$, and

$$\sum_{s=0}^{\infty} c_s x^s$$

is a power series with a radius of convergence $\rho \neq 0$. With this definition we can write our last inequality as

$$|b_m| < 2|g_m| + 2\sum_{s=\mu}^{m-1}|b_s|c_{m-s} \qquad (m > m_0),$$

or, since $b_0 = b_1 = \cdots = b_{\mu-1} = 0$,

$$|b_m| < 2|g_m| + 2\sum_{s=0}^{m-1}|b_s|c_{m-s} \qquad (m > m_0).$$

Let us define

$$b'_m = 2|g_m| + 2\sum_{s=0}^{m-1} b'_s c_{m-s},$$

$$b'_0 = b'_1 = \cdots = b'_{m_0} > |b_j| \qquad (j = 0, 1, \cdots, \mu, \cdots, m_0),$$

assuming for the present that the coefficients in the series for b_m up to the m_0th term have been defined. They are, of course, defined uniquely beyond that.

Then $|b_m| < b'_m$. In fact $|b_0| < b'_0$, $|b_1| < b'_1$, $\cdots$, $|b_{m_0}| < b'_{m_0}$, and if $|b_0|, \cdots, |b_{m-1}|$ are respectively less than $b'_0, \cdots, b'_m$ then $|b_m| < b'_m$, as we see from the explicit formulas.

Accordingly we have merely to show the convergence of the power series

$$\sum_{m=0}^{\infty} b'_m x^m.$$

But we shall show, and the convergence of the given series follows from this fact, that if

$$\sum_{m=0}^{\infty} \gamma_m x^m \qquad \text{and} \qquad \sum_{m=0}^{\infty} \kappa_m x^m \qquad (\gamma_m > 0,\ \kappa_m > 0),$$

are convergent throughout the circle of radius $\rho > 0$, there is a circle of radius $\rho' > 0$ within which the series

$$\sum_{m=0}^{\infty} \beta_m x^m$$

is convergent, a series whose coefficients are defined by the relation

$$\beta_m = \gamma_m + \sum_{s=0}^{m-1} \beta_s \kappa_{m-s}.$$

For if $R < \rho$ and $R < 1$, then

$$\gamma_m < \frac{M}{R^m},$$

where M is some constant, and there exists a constant N such that

$$\kappa_m < \frac{N}{R^m}, \qquad \kappa_{m-s} < \frac{N}{R^m},$$

whence

$$\beta_m \leqq \frac{M}{R^m} + \frac{N}{R^m}\sum_{s=0}^{m-1} \beta_s.$$

Now if we define

$$(51) \qquad \beta'_0 = \beta_0, \qquad \beta'_m = \frac{M}{R^m} + \frac{N}{R^m}\sum_{s=0}^{m-1} \beta'_s,$$

we shall have $\beta_m < \beta'_m$. But by (51)

$$\beta'_m = \left(\frac{1+N}{R}\right)\beta'_{m-1}.$$

Therefore if $\rho' < R/(1+N)$, the power series defined by (51) is convergent for $|x| \leqq \rho'$, and the series

$$\sum_{m=0}^{\infty} \beta_m x^m$$

is convergent throughout the same circle.

It remains to see under what conditions the terms b_i, $i < m_0$, in (50) will not be defined. This will be when for some value of m

$$1 - \sum_{i=0}^{\lambda} \frac{a_{i,\,\lambda-1}}{m-i} = 0.$$

Otherwise, namely in the general case, there is one and only one analytic solution. Since $\mu - \lambda = 1$, it is necessary that

$$g_0 = g_1 =, \cdots, = g_{\mu-1} = 0.$$

There cannot be more than $\lambda + 1$ integral values of m for which

$$1 - \sum_{i=0}^{\lambda} \frac{a_{i,\,\lambda-1}}{m-1} = 0.$$

Let these k values be $m_1, \cdots, m_k$. Then there is a one-parameter family of solutions of the homogeneous equation if we take $b_0, b_1, \cdots, b_{m_k-1} = 0$ and choose b_{m_k} arbitrarily. In general these are all the analytic solutions of the homogeneous equation. For suppose b_l $(l < m_k)$ is the first coefficient not zero; let b_{m_j} be the next coefficient for which

$$1 - \sum_{i=0}^{\lambda} \frac{a_{i,\,\lambda-1}}{m-1} = 0.$$

Then substituting in the equations (50) $g_m = 0$, we see that the condition

$$\sum_{i=0}^{m-1} b_s \sum_{s=\mu}^{m+\lambda-s} \frac{a_{i,\,m_j+\lambda-s-i}}{m_j - 1} = 0$$

must be satisfied. If this condition is satisfied for one value of b_e it is satisfied for all, and if it is not satisfied for any chosen value it is satisfied for none, since the succeeding coefficients up to b_{m_j} are uniquely determined from b_l and linearly dependent on it. Kernels can be chosen so that this condition shall be satisfied and equally well so that the condition shall fail to be satisfied. Of course if $\mu > m_k$ there can be no analytic solution of the homogeneous equation, other than $u \equiv 0$, since $b_0 \cdots b_{\mu-1} = 0$.

If there is a particular analytic solution of the non-homogeneous equation, the general analytic solution will be the sum of this particular solution and the solutions developed above for the homogeneous equation. A necessary and sufficient condition that there should be a particular solution is that b's can be found satisfying (50). The equations (50) include among them the equations

$$g_{m_j} + \sum_{i=0}^{m_j-1} b_s \left(\sum_{s=\mu}^{m_j+\lambda-s} \frac{a_{i,\, m_j+\lambda-s-i}}{m_j - i} \right) = 0 \qquad (j = 1, 2, \cdots, k),$$

which can be regarded as equations restricting certain coefficients in the development of $\phi(x)$. It is sufficient for a particular solution if, in addition to the general restriction $g_0, g_1, \cdots, g_\lambda = 0$, also $g_0, g_1, \cdots, g_{m_k} = 0$.

Then $b_0, b_1, \cdots, b_\mu, \cdots, b_{m_k}$ may all be chosen zero, and a particular solution thereby uniquely determined.

28. *Case where $\mu - \lambda > 1$; Asymptotic Developments.*

As we have already noticed, the series are in general divergent when $\mu - \lambda > 1$. In the cases that fall under the treatment of the present paper, that is, in all cases where $\lambda = 0$, these divergent series still have meaning.

To interpret these divergent series let us put

$$u(x) = \sum_{i=0}^{n} b_i x^i + u_n(x),$$

where the b's have the meaning already assigned to them, and where $u_n(x)$ is to be determined. Substituting in the equation

$$(48) \qquad u(x) = \phi(x) + \int_a^x \frac{K(x, \xi)}{\xi^\mu} u(\xi)\, d\xi,$$

we get

$$\sum_{i=0}^{n} b_i x^i + u_n(x) = \sum_{i=0}^{\infty} g_i(x) + \int_a^x \frac{K(x, \xi)}{\xi^\mu} \sum_{i=0}^{n} b_i \xi^i\, d\xi + \int_a^x \frac{K(x, \xi)}{\xi^\mu} u_n(\xi)\, d\xi,$$

or, to determine $u_n(x)$,

$$(52) \qquad u_n(x) = \phi_n(x) + \int_a^x \frac{K(x, \xi)}{\xi^\mu} u_n(\xi)\, d\xi,$$

where

$$\phi_n(x) = -\sum_{i=0}^{n} b_i x^i + \sum_{i=0}^{\infty} g_i x^i + \int_a^x \frac{K(x, \xi)}{\xi^\mu} \sum_{i=0}^{n} b_i \xi^i\, d\xi.$$

But from (48′) and (48″) we know, putting $\lambda = 0$, that the function

$$\psi_n(x) = -\sum_{i=0}^{n-\mu+1} b_i x^i + \sum_{i=0}^{n-\mu+1} g_i x^i + \sum_{r=\mu}^{n} \sum_{p=0}^{r-\mu} b_{r-p} \sum_{i=0}^{p} \frac{a_{i,\, p-i}}{r - \mu - i + 1} x^{r-\mu+1}$$

is identically zero. But the function

$$\phi_n(x) - \psi_n(x) = \sum_{i=n-\mu+2}^{\infty} g_i x^i + \sum_{r=n+1}^{\infty} \sum_{p=0}^{r-n-1} b_{r-p} \sum_{i=0}^{p} \frac{a_{i,\,p-i}}{r-\mu-i+1} x^{r-\mu+1}$$

vanishes to the order $n-\mu+2$. Hence $\phi_n(x)$ vanishes to the order $n-\mu+2$, and since we have a simple power series, $\phi_n'(x)$ vanishes to the order $n-\mu+1$. Now by the development of sections 11–15, we know that there are solutions of (52). In that development we may take $\nu = 1$; and any solution, since $|\phi_n'(x)| \leqq Nx^{n-\mu+1}$, will satisfy a condition

$$|u_n(x)| \leqq Nx^{n-\mu+1} Cx^{\mu-\nu'} \leqq CNx^{n+1-\nu'} \leqq CNx^n \qquad (\text{since } \nu' < 1).$$

Now

$$u(x) = \sum_{i=0}^{n-1} b_i x^i + b_n x^n + u_n x = \sum_{i=0}^{n-1} b_i x^i + u_{n-1}(x).$$

Hence

$$u_{n-1}(x) = b_n x^n + u_n(x),$$

and

$$|u_{n-1}(x)| \leqq \text{const}\, x^n.$$

Hence

$$u(x) - \sum_{i=0}^{n} b_i x^i \leqq \text{const}\, x^{n+1}.$$

This is the meaning of the power series development. It is asymptotic.*

29. *Equations of the First Kind.*

The treatment of integral equations of the first kind

$$(53) \qquad 0 = \phi(x) + \int_0^x K(x, \xi) u(\xi)\, d\xi$$

with analytic kernels, is analogous to that of section 28. There is no gain in generality in having a denominator ξ^n in the kernel, because such a denominator may be removed by the transformation $u(\xi) = \xi^n v(\xi)$.

Equation (53) can be reduced to an equation of the form of (48) where, with some exceptions, $\mu - \lambda = 1$. Or from (48) we get at once the equations for the determination of the coefficients of the analytic expansion of the solution by putting $\mu = 0$, and omitting the 1 in the parenthesis of the left hand member. The equations then are

$$(54) \quad -b_m \left(\sum_{i=0}^{a_{i,\,\lambda-i}} \frac{a_{i,\,\lambda-i}}{m+\lambda+1-i} \right) = g_{m+\lambda+1} + \sum_{s=0}^{m-1} b_s \left(\sum_{i=0}^{m+\lambda-s} \frac{a_{i,\,m+\lambda-s+i}}{m+\lambda+1-i} \right).$$

* BOREL, *Leçons sur les séries divergentes*, page 35.

Consider now the equation

$$(55)\qquad \sum_{i=0}^{\lambda}\left(\frac{a_{i,\lambda-i}}{m+\lambda+1-i}\right)=0,$$

which cannot have more than λ integral roots. If the left hand member does not vanish for any integral value of m, and if its product by m does not vanish as m becomes infinite, the power series defined by (54) is convergent, as may be easily shown, and a necessary and sufficient condition that it be a solution of the equation (53) is that

$$g_0, g_1, \cdots, g_\lambda = 0.$$

If there are integral values of m that satisfy (55) the considerations that enter are those that we have treated on pages 467–8. The condition that

$$\lim_{m=\infty} m\sum_{i=0}^{\lambda}\frac{a_{i,\lambda-i}}{m+\lambda+1-i} \neq 0$$

is equivalent to the condition

$$(56)\qquad \sum_{i=0}^{\lambda} a_{i,\lambda-1} \neq 0.*$$

30. *A Problem in Hydrostatics.*

Let us now return to the example from hydrostatics given in the earlier paper. The equation to be solved is

$$v(x)=\int_0^x \frac{G_1(x-\xi)-g'(x)}{g(x)}v(\xi)\,d\xi,$$

where $g(x)$ vanishes at least to the first order when $x=0$. For convenience let us consider $g(x)$ as an analytic function, and let us suppose $\alpha_1-g'(0)\neq 0$, where

$$G_1(x-\xi)=\alpha_1+2\alpha_2(x-\xi)+3\alpha_3(x-\xi)^2+\cdots.$$

By the transformation

$$w(x)=g(x)v(x),$$

the above equation is reduced to

$$w(x)=\int_0^x\frac{G_1(x-\xi)-g'(x)}{g(\xi)}w(\xi)\,d\xi.$$

Since we are considering only continuous solutions $v(x)$, i. e., only tubes lying on curves which do not become horizontal at the bottom, we must have $w(x)/g(x)$ continuous. Hence both $w(x)$ and $w'(x)$ are continuous, and the

* See first paper, page 395–396 ; $a_{i,\lambda-i}=A_i$ in the condition of T. LALESCO.

only possible solutions will be given by those of the theorem of section 9. Hence if in the neighborhood of $x=0$, $g(x)$ has the opposite sign from $\alpha_1-g'(0)$ there will be no solutions; and if $g(x)$ has the same sign as $\alpha_1-g'(0)$ there will be an infinite number of solutions.

A problem leading to an equation somewhat similar to this is the question of filling the tube with a second liquid, whose density at a point $h(\xi)$ is a function of its height *from the bottom*, so that the tube shall have the same weight filled with the second liquid as it has filled with the first, to the same height. The equation then is

$$\int_A^x h(\xi)ds = \int_A^x \nu(x-\xi)ds,$$

or

$$\int_0^x h(\xi)u'(\xi)d\xi = \int_0^x \nu(x-\xi)u'(\xi)d\xi,$$

which yields by differentiation

$$h(x) = \frac{1}{u'(x)}\frac{d}{dx}\int_0^x \nu(x-\xi)u'(\xi)d\xi,$$

or

$$h(x) = \alpha + \int_0^x \frac{G_1(x-\xi)u'(\xi)}{u'(x)}d\xi.$$

If being given $h(x)$ we try to find the curve, the equation is

$$0 = \int_0^x \{\nu(x-\xi) - h(\xi)\} v(\xi)d\xi,$$

where $v(\xi) = u'(\xi)$; whence, differentiating, we find

$$[\alpha - h(x)]\, v(x) = \int_0^x G_1(x-\xi)v(\xi)d\xi,$$

or

$$v(x) = \int_0^x \frac{G_1(x-\xi)v(\xi)}{\alpha - h(x)}d\xi.$$

If $\alpha - h(x)$ is continuous, with continuous first derivative, there are no solutions of this equation continuous except at a finite number of points, except those for which $v(x)[\alpha - h(x)]$ remains finite. If $h(\alpha) \neq \alpha$, $v(x) = 0$ is the only solution possible. We assume that $\alpha_1 \neq 0$.

The equation belongs to the special class where the numerator of the kernel is the sum of a constant and a function which vanishes identically when $\xi = x$. Hence the only possible solutions are those given in section 9.

If in the neighborhood of $x = 0$, $\alpha - h(x)$ has the opposite sign from α_1 there will be no solutions except $v(x) \equiv 0$. Hence, if the density of the first liquid increases as we descend from the surface, the density of the second must decrease as we go up from the bottom.

If $a - h(x)$ has the same sign as α_1 there will be an infinite number of solutions such that

$$u(x)\,[a - h(x)]$$

remains finite as x approaches 0 (§ 9). In fact, any solution is less in absolute value than const. $/(x - a)^\nu$, where

$$1 > \nu > 1 - \frac{1}{|h'(x)|}$$

(§ 14). And if $|h'(x)| < 1$ all the solutions are finite, i. e., the curves start up from the bottom with a positive slope. If $h(x)$ is an analytic function, the power series development given in section 28 will have all its coefficients zero, since $\alpha_1 \neq 0$.

Of course, if $v(x)$ vanishes when $x = a$, the solution will represent an imaginary curve; for with real curves $|ds| \geqq |dx|$, i. e.,

$$\left|\frac{ds}{dx}\right| \geqq 1.$$

HARVARD UNIVERSITY,
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