A CONDITION THAT A FUNCTION IN A PROJECTIVE SPACE BE RATIONAL*

BY

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HURWITZ † has shown that a function of n complex variables which is in general analytic and has no other than unessential singularities is a rational function of the n arguments. The infinite region is that of the theory of functions. A point $(z_1, \cdots, z_n)$ lies at infinity if at least one of its coördinates corresponds to the point $z_k = \infty$ in the complex z_k-plane; and such a point $(z_1, \cdots, z_n)$ is projected into the finite region by subjecting the individual coördinate or coördinates which become infinite to a linear transformation:

$$Z_k = \frac{1}{z_k},$$

or, more generally,

$$Z_k = \frac{\alpha_k z_k + \beta_k}{\gamma_k z_k + \delta_k}, \qquad \alpha_k\delta_k - \beta_k\gamma_k \neq 0, \qquad \gamma_k \neq 0.$$

Thus the points of the space of the theory of functions can be represented by n pairs of binary homogeneous variables:

$$[(\zeta_1, \zeta_1'), (\zeta_2, \zeta_2'), \cdots, (\zeta_n, \zeta_n')],$$

the transformations being n independent binary linear transformations:

$$\left.\begin{aligned} \mathsf{Z}_k &= \alpha_k\zeta_k + \beta_k\zeta_k' \\ \mathsf{Z}' &= \gamma_k\zeta_k + \delta_k\zeta_k' \end{aligned}\right\} \qquad \begin{vmatrix} \alpha_k & \beta_k \\ \gamma_k & \delta_k \end{vmatrix} \neq 0, \qquad k = 1, 2, \cdots, n.$$

A function is said to be analytic in a point at infinity if, on making the above transformation for as many of the coördinates separately as become infinite, the transformed function is analytic in the new variables at the corresponding point, save as for a removable singularity (hebbare Unstetigkeit). A function

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† A. HURWITZ, Journal für Mathematik, vol. 95 (1883) p. 201. The theorem was stated without proof by WEIERSTRASS.

has an unessential singularity at a point (finite or at infinity), if in the neighborhood of this point, while not remaining finite, it can be expressed as the quotient of two functions, each of which is analytic in the point in question.

§ 1.

In the present paper it is shown that a theorem similar to the above holds for a complex n-fold projective space. Before stating the theorem in precise form we will consider some preliminary definitions and conceptions.

Let a point of the projective space in question be represented by the complex, homogeneous coördinates $x_0, x_1, \cdots, x_n$. These coördinates never vanish simultaneously, and infinite values of the coördinates never enter into consideration. Let $F(x_0, x_1, \cdots, x_n)$ be a function single-valued and analytic throughout a complex $(n+1)$-dimensional continuum $\mathfrak{R}_1$ lying in the above complex $(n+1)$-dimensional space $\mathfrak{R}$ of the variables $(x_0, x_1, \cdots, x_n)$, and let F be homogeneous of dimension 0; i. e., if $(x_0, x_1, \cdots, x_n)$ is a point of $\mathfrak{R}_1$, then all the points $(\lambda x_0, \lambda x_1, \cdots, \lambda x_n)$, where λ is any complex number except 0, shall also lie in $\mathfrak{R}_1$, and furthermore

$$F(x_0, x_1, \cdots, x_n) = F(\lambda x_0, \lambda x_1, \cdots, \lambda x_n).$$

Let $(a_0, a_1, \cdots, a_n)$ be a point of $\mathfrak{R}_1$. Then one of the a's (a_k, let us say) is different from 0, while on the other hand F is analytic throughout a certain neighborhood of this point:

$$(1) \qquad |x_l - a_l| < h, \quad l = 0, 1, 2, \cdots, n, \quad 0 < h < |a_k|.$$

If we set $\lambda = x_k^{-1}$, then for the above neighborhood

$$F(x_0, x_1, \cdots, x_n) = F\left(\frac{x_0}{x_k}, \frac{x_1}{x_k}, \cdots, \frac{x_{k-1}}{x_k}, 1, \frac{x_{k+1}}{x_k}, \cdots, \frac{x_n}{x_k}\right).$$

For such values of

$$w_l = \frac{x_l}{x_k}, \qquad l = 0, 1, \cdots, k-1, k+1, \cdots, n$$

as correspond to the points $(x_0, x_1, \cdots, x_n)$ of the neighborhood (1),

$$F(w_0, w_1, \cdots, w_{k-1}, 1, w_{k+1}, \cdots, w_n)$$

is seen to be an analytic function of the n arguments w.

Finally, let $(a_0, a_1, \cdots, a_n)$ be a point of $\mathfrak{R}$, in which F is not defined. Again, let $a_k \neq 0$, and consider the neighborhood (1), h being suitably restricted. Then F shall be defined in general in the points of this region, and shall be equal to the quotient

$$F(x_0, \cdots, x_n) = \frac{Q(x_0, \cdots, x_n)}{P(x_0, \cdots, x_n)},$$

where P, Q are each analytic in (a) and P vanishes there, P and Q being relatively prime in (a).* All points of (1) in which $P \neq 0$ shall belong to the domain of definition $\mathfrak{R}_1$ of F.

If now we write

$$F(x_0, x_1, \cdots, x_n) = F\left(\frac{x_0}{x_k}, \frac{x_1}{x_k}, \cdots, \frac{x_{k-1}}{x_k}, 1, \frac{x_{k+1}}{x_k}, \cdots, \frac{x_n}{x_k}\right)$$
$$= F(w_0, \cdots, w_{k-1}, 1, w_{k+1}, \cdots, w_n),$$

this latter function can be shown to be capable of being written as the quotient of two functions:

$$F = \frac{H(w_0, \cdots, w_{k-1}, w_{k+1}, \cdots, w_n)}{G(w_0, \cdots, w_{k+1}, w_{k-1}, \cdots, w_n)},$$

each analytic in the point

$$w_l = \frac{a_l}{a_k}, \qquad l = 0, \cdots, k-1, k+1, \cdots, n.$$

The function G vanishes in the point in question, and G and H are relatively prime there.

§ 2.

We are now in a position to state our theorem.

THEOREM. *The function $F(x_0, x_1, \cdots, x_n)$ is a rational function of its arguments.*

To prove the theorem, we make a transformation of the projective space R whose points are determined by the homogeneous variables $x_0, x_1, \cdots, x_n$ on an n-dimensional space of the theory of functions, S. Let the points of the latter space be given by the homogeneous variables

$$[(\zeta_1, \zeta_1'), (\zeta_2, \zeta_2') \cdots, (\zeta_n, \zeta_n')]$$

and let

$$z_k = \frac{\zeta_k}{\zeta_k'}, \qquad k = 1, 2, \cdots, n.$$

Then we set

$$\begin{aligned}
\rho x_0 &= \zeta_1 \zeta_2 \cdots \zeta_n, \\
\rho x_1 &= \zeta_1' \zeta_2 \cdots \zeta_n + \zeta_1 \zeta_2' \zeta_3 \cdots \zeta_n + \cdots + \zeta_1 \zeta_2 \cdots \zeta_{n-1} \zeta_n', \\
\rho x_2 &= \zeta_1' \zeta_2' \zeta_3 \cdots \zeta_n + \zeta_1' \zeta_2 \zeta_3' \cdots \zeta_n + \cdots + \zeta_1 \zeta_2 \cdots \zeta_{n-1}' \zeta_n', \qquad (A) \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
\rho x_n &= \zeta_1' \zeta_2' \cdots \zeta_n',
\end{aligned}$$

*Cf. *Encyklopädie der mathematischen Wissenschaften*, II B 1, § 45.

where ρ is an arbitrary complex number distinct from 0. Thus to every point of S corresponds one point and only one of R, though an infinite number of points in $\mathfrak{R}$ present themselves. If $(x_0, x_1, \cdots, x_n)$ is one of the latter points, then the others are $(\lambda x_0, \lambda x_1, \cdots, \lambda x_n)$, where $\lambda \neq 0$ is any complex number.

The inverse of (A) is easily written down. Consider the equation in

$$z = \zeta / \zeta' :$$

$$(B_1) \qquad x_0 - x_1 z + x_2 z^2 - \cdots \pm x_n z^n = 0,$$

or in homogeneous form:

$$(B_2) \qquad x_0 \zeta'^n - x_1 \zeta'^{n-1} \zeta + x_2 \zeta'^{n-2} \zeta^2 - \cdots \pm x_n \zeta^n = 0.$$

$(x_0, x_1, \cdots, x_n)$ being a point of $\mathfrak{R}$, not all the x's can vanish, and hence the equation (B_2) determines precisely n (distinct or coincident) pairs of values, — or more properly the ratios of the letters in each pair:

$$(\zeta_1, \zeta_1'), \quad (\zeta_2, \zeta_2'), \quad \cdots, \quad (\zeta_n, \zeta_n'),$$

and these in turn, when arranged in all $n!$ possible ways, yield $n!$ points $(z_1, z_2, \cdots, z_n)$ in the space S.

We have, then, in (A) and (B) a $(1, n!)$ transformation of S on R, and an $(\infty, n!)$ transformation of S on $\mathfrak{R}$.

On applying this transformation to the function $F(x_0, \cdots, x_n)$, the latter is carried over into a function

$$\Phi(z_1, z_2, \cdots, z_n),$$

single-valued at all points of S where it is defined. Moreover, Φ is, as we will presently show, analytic in S where it is defined, and it has no other than unessential singularities in S. It is, therefore, by the theorem cited at the beginning of this paper, a rational function of $z_1, \cdots, z_n$. It is, furthermore, symmetric in these variables, and consequently F is rational in $x_0, x_1, \cdots, x_n$.

To complete the proof, consider first a point $(a) = (a_0, a_1, \cdots, a_n)$ of $\mathfrak{R}$ in which $a_n \neq 0$. Let $(b) = (b_1, \cdots, b_n)$, where $z_i = b_i$, be one of the points of S into which (a) is carried by the transformation (B_1). This point will lie in the finite part of S. To every point (x) of a certain neighborhood of (a), and hence to every point $(w) = (w_0, w_1, \cdots, w_{n-1})$ of a certain neighborhood of $(w) = (c) = (c_0, c_1, \cdots, c_{n-1})$, where $c_l = a_l / a_n$, will correspond one or more points (z) of a certain neighborhood of (b); and conversely, to each point (z) of a neighborhood of (b) will correspond one point (w) of a neighborhood of (c). Finally, for all pairs of corresponding points we shall have

$$w_0 = \frac{x_0}{x_n} = z_1 z_2 \cdots z_n,$$

$$w_1 = \frac{x_1}{x_n} = z_2 z_3 \cdots z_n + z_1 z_3 \cdots z_n + \cdots + z_1 z_2 \cdots z_{n-1},$$

$$w_2 = \frac{x_2}{x_n} = z_3 z_4 \cdots z_n + \cdots + z_1 z_2 \cdots z_{n-2},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$w_{n-1} = \frac{x_{n-1}}{x_n} = z_1 + z_2 + \cdots + z_n.$$

Since $F(w_0, w_1, \cdots, w_{n-1}, 1)$ is analytic in $(c) = (c_0, c_1, \cdots, c_{n-1})$, or else is the quotient of two functions of (w) each analytic in (c), and since the above equations express w_i as a function of $z_1, \cdots, z_n$, analytic in $(z) = (b)$ and equal there to c_i, it follows that $\Phi(z_1, \cdots, z_n)$ is analytic in (b), or else is the quotient of two functions of (z) each analytic in (b).

It remains merely to consider the points of $\mathfrak{R}$ for which $a_n = 0$. The corresponding points (z) lie in the infinite region of S. Let

$$a_n = a_{n-1} = \cdots = a_{n-l+1} = 0, \quad a_{n-l} \neq 0,$$

and let (b) be one of the points of S which correspond to (a). Then l of the ζ''s will vanish at (b); suppose these are $\zeta_1', \cdots, \zeta_l'$:

$$\zeta_1' = \zeta_2' = \cdots = \zeta_l' = 0; \quad \zeta_i' \neq 0, \quad l < i; \quad \text{also} \quad \zeta_k \neq 0, \quad k \leqq l.$$

We now set

$$\zeta_k = 1, \quad \zeta_k' = z_k', \quad k = 1, \quad \cdots, l;$$

$$\zeta_i' = 1, \quad \zeta_i = z_i, \quad i > l,$$

and let

$$w_0 = \frac{x_0}{x_{n-l}}, \cdots, \quad w_{n-l-1} = \frac{x_{n-l-1}}{x_{n-l}}, \quad w_{n-l+1} = \frac{x_{n-l+1}}{x_{n-l}}, \cdots, \quad w_n = \frac{x_n}{x_{n-l}}.$$

Then each w will be the quotient of two polynomials in $z_1', \cdots, z_l', z_{l+1}, \cdots, z_n$, and the denominator polynomial will take on the value 1 in the point

$$(z_1', \cdots, z_l', \quad z_{l+1}, \cdots, z_n) = (0, \cdots, 0, b_{l+1}, \cdots, b_n) = (b).$$

Since $F(w_0, \cdots, w_{n-l-1}, 1, w_{n-l+1}, \cdots, w_n)$ is analytic in the point $(w) = (c)$ corresponding to the present $(x) = (a)$, or else is the quotient of two such functions, the case is precisely similar to the one above discussed, and Φ is analytic in (b), or is the quotient of two functions each analytic in (b). This completes the proof.

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