MULTIPLE CORRESPONDENCES DETERMINED BY THE RATIONAL
PLANE QUINTIC CURVE*

BY

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§ 1. Conjugate Quintic Curves.

The occurrence of rational curves in pairs is a well-known fact: thus, given a rational curve \( \rho_n^p \), of order \( n \), in a space of \( p \) dimensions, there is uniquely determined, to within a collineation, a curve \( \rho_n^{n-p-1} \), of order \( n \), in a space of \( n - p - 1 \) dimensions, by requiring all hyperplane sections of either curve to be apolar to the hyperplane sections of the other.† We call two curves associated in this way conjugate curves.

If the rational curve \( \rho_n^p \) is regarded as the projection of the norm-curve \( \rho_n^n \) in a space \( S_n \) of \( n \) dimensions, from an \( S_{n-p} \), the interpretation of this fact is immediate. An \( S_{n-1} \) in \( S_n \) meets \( \rho_n^n \) in \( n \) points which may be regarded as given by a binary form of order \( n \): dually, a point of \( S_n \) determines on \( \rho_n^n \) a set of \( n \) points, which may be given by a second binary form. The condition of apolarity of the two forms is precisely the condition of incidence of point and \( S_{n-1} \).

All \( S_{n-1} \)'s having \( n \)-point contact with \( \rho_n^n \) meet \( S_{n-p-1} \) in the hyperplanes of a curve \( \tau_{n-p}^n \) of class \( n \). The curves obtainable by projection from \( S_{n-p-1} \) and section by \( S_{n-p-1} \) are conjugate curves.

In this paper we shall deal with the case \( n = 5, p = 2 \), the rational plane quintic. If our curve \( \tau_5^5 \) is given parametrically by

\[
(1) \quad x_i = (\alpha_i t)^5 \quad (i = 0, 1, 2),
\]

and \( (b_it)^5 \) are three linearly independent quintics apolar to the \( \alpha \)'s, the conjugate quintic of (1), which we take for convenience as a curve of lines, may be given by

\[
(2) \quad \eta_i = (b_i t)^5 \quad (i = 0, 1, 2).
\]

The quintic (1) is equally the conjugate quintic of (2). Regarding (1) and (2) as situated in two independent planes \( \pi_x \) and \( \pi_y \), we shall prove the existence

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* Presented to the Society, December 28, 1910.
† W. F. Meyer, Apolarität und rationale Kurven, p. 9.

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of certain multiple correspondences between these planes, and point out the significance of these correspondences for the curves (1) and (2).

We shall make use of the norm-curve in five dimensions to prove the existence of these correspondences and to deduce their characteristic properties.

A set of the fundamental involution of (1) is defined on (2) by tangent lines through a point \( y \) of \( \pi_y \); similarly, a set of the fundamental involution of (2) is defined on (1) by its points of intersection with a line \( \xi \) of \( \pi_x \).

In the first part of this paper we shall take (1) as

\[ \xi_i = (\alpha_i t)^5, \]

where \( \xi_i \) are the coördinates of the line marked out on a plane \( \pi_1 \) by the \( S_4 \) having 5-point contact with the norm-curve at the point \( t \), and (2) as

\[ y_i = (b_i t)^6 \]

where the \( y_i \) are the coördinates of the projection of the point \( t \) of the norm-curve from \( \pi_1 \) on a plane \( \pi_2 \). In the second part we shall recur to the representation given by (1) and (2) in order to regard the fundamental correspondence as one of point to point.

The point of view of this paper is closely analogous to that of STAHル,* though STAHル does not make explicit use of the correspondence which we call \( T \). It is possible to extend many of the theorems given to rational curves in general. BERZOLARI† has obtained some of these extensions.

§ 2. The Rational Norm-quintic.

A norm-curve, \( R \), in \( S_6 \) may be given parametrically as follows:

(3) \[ x_0 = 1, \quad x_1 = - t, \quad x_2 = t^2, \quad x_3 = - t^3, \quad x_4 = t^4, \quad x_5 = - t^5, \]

or, in \( S_6 \)'s,

(4) \[ \xi_0 = t, \quad \xi_1 = 5t^4, \quad \xi_2 = 10t^3, \quad \xi_3 = 10t^2, \quad \xi_4 = 5t, \quad \xi_5 = 1. \]

We shall call an \( S_p \) having \( (p + 1) \)-point contact with \( R \) simply an \( S_p \) of \( R \).

Any point \( x \) of \( S_6 \) carries five \( S_6 \)'s of \( R \). We have from (4) that the parameters of these \( S_6 \)'s are given by

(5) \[ (xt)^5 = x_0 t^5 + 5x_1 t^4 + \cdots + x_6 = 0. \]

Dually, any \( S_4, \xi \), of \( S_6 \) meets \( R \) in five points whose parameters are given by the binary quintic

(6) \[ (\xi t)^5 = \xi_0 - \xi_1 t + \cdots \xi_5 t^5 = 0. \]

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The apolarity-condition of (5) and (6) is the incidence-condition of point and $S_4$: 

$$\xi x = \xi_0 x_0 + \cdots + \xi_5 x_5 = 0.$$ 

The binary quintics (5) and (6) we may call, briefly, the quintics $x$ and $\xi$, respectively, or merely $x$ and $\xi$.

The curve $R$ has three developables: $D_1$, the locus of lines of $R$; $D_2$, the locus of planes of $R$; and $D_3$, the locus of spaces of $R$. The locus $D_1$ is a 2-way spread (or simply a 2-way) of order 8; the locus $D_2$, a 3-way of order 9; and $D_3$, a 4-way of order 8. To these we may add $D_0$, the points of $R$, and $D_4$, the $S_4$'s of $R$.

The general rational quintic curve in an $S_p$ is obtainable from $R$ in points by projecting $R$ on $S_p$ from an $S_{p-1}$; in $S_{p-1}$'s, by cutting the $S_4$'s of $R$ by $S_p$. The points of the latter curve form the curve of intersection with $S_p$ of $D_{S-p}$. The curve has stationary points (apparent cusps) at the points of intersection with $S_p$ of $D_{S-p}$.

§ 3. The Lines $l$.

Consider a point $x$ of $S_5$. Quintics apolar to the quintic $x$ are cut out of $R$ by $S_4$'s on $x$. The $\infty^2$ quartics apolar to $x$ are cut out of $R$ by spaces on $x$ and meeting $R$ four times. The unique cubic apolar to $x$—the so-called canonizant of $x$—is cut out of $R$ by a unique plane on $x$ and trisecant to $R$.

From the above we have at once:

(a) If $R$ is projected from a plane $\pi_1$ on a plane $\pi_2$, sets of the fundamental involution of the curve so obtained are quintics $x$, where $x$ is chosen on $\pi_1$. Line-sections of the curve are quintics $\xi$ where $\xi$ is made to contain $\pi_1$. The two rational quintics obtainable from $R$ by projection from $\pi_1$ and section by $\pi_1$ are conjugate curves.

The general line, $p$, of $S_5$ carries a unique space quadrisecant to $R$; in the theory of binary forms this means that there is a unique quartic apolar to all quintics of a pencil. It is evident from the fact that apolarity relations are linear that if there are two quartics apolar to all quintics of a pencil, every quartic of the pencil built on the two is apolar to every quintic of the pencil of quintics. In $S_5$ the corresponding theorem is that if a line carries two spaces quadrisecant to $R$, it carries an infinity of such spaces. Lines having this property are called by Marletta lines $l$.

Marletta points out the following facts:

(b) Any two $S_4$'s quadrisecant to $R$ meet in a line $l$.

† G. Marletta, l. c., p. 95.
It follows that there are \( \infty^6 \) lines \( l \) in \( S_5 \), a line \( l \) being determined by any selected pencil of quartics on \( R \).

(c) Lines \( l \) on a space of \( S_5 \) are in a \( (3, 1) \) congruence.

(d) Spaces quadrisecant to \( R \) and on a line \( l \) are on a quadric 4-way.

We may add to these:

(e) Lines meeting \( R \) are lines \( l \).

Let the given line meet \( R \) in a point \( t \). From this line \( R \) is projected upon an \( S_3 \) into a rational quartic curve \( \rho t \); the \( \infty^1 \) lines trisecant to the latter curve are the traces on the \( S_3 \) of \( \infty^1 S_3 \)'s on \( l \) and meeting \( R \) in three points and the point \( t \). The pencil of quartics determined on \( R \) by \( l \) is here a fixed point \( t \) taken with a pencil of cubics. Pencils of cubics on \( R \) are thus put into \((1, 1)\) correspondence with lines through any point \( t \) of \( R \).

(f) Lines on planes trisecant to \( R \) are lines \( l \).

For let a plane \( \pi \) meet \( R \) in the points \( t_1, t_2, t_3 \). Then an \( S_3 \) may be put on \( t_1 t_2 t_3 \) and any point, \( t_4 \), of \( R \); thus the plane \( \pi \) and hence any line on it carries \( \infty^1 S_3 \)'s quadrisecant to \( R \).

§4. The Spread of Lines Bisecant to \( R \).

The norm-curve \( R \) has \( \infty^2 \) bisecant lines; there are \( \infty^1 \) points on each line, and therefore \( \infty^3 \) points on all lines bisecant to \( R \). It follows that these points are on a 3-way, which we call \( g_6 \). Any space, \( \sigma \), quadrisecant to \( R \) meets \( g_6 \) in 6 lines—the 6 lines of the 4-point in which \( \sigma \) meets \( R \). But these 6 lines are the complete intersection of \( g_6 \) and \( \sigma \). If not, let there be a point \( a \) on \( \sigma \) and not on one of these lines. Then the \( S_4 \) on \( a \) and on the line through \( a \) bisecant to \( R \) will meet \( R \) in 6 points, but this is impossible.* Hence

(g) *The 3-way \( g_6 \) is of order six.

The following is obvious from § 2:

(h) *The spread \( g_6 \) is the locus of quintics \( x \) having an apolar quadratic, i.e., whose canonizant vanishes identically.

The coefficients of the canonizant of \((xt)^5\) represent four linearly independent cubic spreads on \( g_6 \). Also,

(i) *The spread \( g_6 \) is the locus of cyclic quintics \( x \), i.e., quintics reducible to the form \( \alpha t_1^5 + \beta t_2^5 \), this quintic having the apolar quadratic \( t_1 t_2 = 0 \).

The intersection of two spreads in \( S_5 \) will be indicated by writing them consecutively: thus, \( \alpha \) being a space in \( S_5 \), \( g_6 \alpha \) is the sextic curve in which \( g_6 \) meets \( \alpha \), and \( g_6 \pi \) is six points in a plane \( \pi \).

* A similar argument may be used to show that the spread of lines bisecant to the rational norm-curve in \( S_6 \) is a 3-way of order \( \frac{1}{2} (n - 1)(n - 2) \), which is the number of nodes of the rational plane \( n \)-ic, as it should be. Cf. Castelnuovo, *Studio dell' involuzione sulle curve'rationali mediante la loro curva normale dello spazio a n dimensioni*, Atti del R. Istituto Veneto, series 6, vol. 4 (1886), p. 1167.
(j) The sextic \( g_6 \alpha \) admits \( 5 \)-planes.

For any one of the \( 5 \)-s on \( \alpha \) meets \( R \) in five points; the ten lines joining these five points two and two meet \( \alpha \) in the ten points of a \( 5 \)-plane inscribed in \( g_6 \alpha \). The ten lines of this \( 5 \)-plane are given by planes on three of the five points, and the planes, by spaces on four of the five points.

(k) Any line trisecant to \( g_6 \alpha \) is the trace on \( \alpha \) of a plane trisecant to \( R \).

For, let a line \( p \) meet \( g_6 \) three times. \( p \) then meets three lines, \( p_1, p_2, p_3 \), each bisecant to \( R \) — let these three lines meet \( R \) in the points \( a_1, b_1; a_2, b_2; a_3, b_3 \), respectively. On \( p \) and \( p_i \), we may then put an \( S_4 \) meeting \( R \) six times, if \( a_i, b_i \) are all distinct points. It follows that they cannot be all distinct. Let say \( b_1 = b_2 \). Then \( p_1 \) and \( p_2 \) meet and \( p_1, p_2, p \) must be on a plane trisecant to \( R \), and the theorem is proved.

By (j), \( g_6 \alpha \) is a sextic curve in space admitting \( 5 \)-planes. Such a curve is known to possess the following properties:*

(i) It is a curve of genus 3.

(ii) Planes of a group of 5 inscribed in the curve are planes of a fixed norm-curve, \( r^2 \).

(iii) The groups of inscribed 5-planes mark on \( r_5^2 \) sets of an involution \( I_{1,4} \).†

From theorems (f) and (ii) we have

(iv) The \( (3, 1) \) congruence of lines \( l \) in a space \( \alpha \) is the congruence of axes of \( r^2 \).

The curve \( R \) determines on any space \( \alpha \) the following:

(v) The rational curve \( D_2 \alpha \) of order nine and class five, having stationary point at the eight points \( D_1 \alpha \).

The planes of \( D_2 \alpha \) are the planes \( D_4 \alpha \).

The curve \( D_2 \alpha \), of class five, given in \( \alpha \) is sufficient to determine all geometrical forms in \( D \) obtainable from \( R \) and \( \alpha \); \( R \) and \( \alpha \) completely characterize \( D_2 \alpha \) and all its concomitant forms. If an invariant of \( D_2 \alpha \) vanishes \( \alpha \) must have some special position with reference to \( R \).

(vi) The curve \( g_6 \alpha \), covariantly associated with \( D_2 \alpha \) in a manner which will be pointed out in the following section. This curve \( g_6 \alpha \) is obviously the locus of a point \( x \) of \( \alpha \) such that the five parameters of planes of \( D_2 \alpha \) through \( x \) form a cyclic quintic.

(vi) The curve \( r_5^2 \), the locus of 5-planes inscribed in \( g_6 \alpha \).

(vi) The locus of lines trisecant to \( g_6 \alpha \); a ruled surface of order eight having \( g_6 \alpha \) as a triple curve.

We shall call this surface \( \Phi_a \).


† \( I_{1,4} \): An involution of five things, one of which determines four, a pencil of quintics.
§ 5. A Cremona Transformation Between Two Spaces.

It is of interest to characterize briefly a cremona transformation determined by \( R \) between two spaces \( \alpha \) and \( \alpha' \) of \( S_5 \). Points of \( g_6\alpha \) and \( g_6\alpha' \), sextic curves of genus 3, are singular points of this transformation—a property which it possesses in common with the cubo-cubic cremona transformation determined by three bilinear forms.

In \( S_5 \) a plane meets a space, in general, in a point. Given \( R \) and the two spaces \( \alpha \) and \( \alpha' \), we have through the general point \( x \) of \( \alpha \) a unique plane \( \pi_x \) trisecant to \( R \).* \( \pi_x \) meets \( \alpha' \) in a point \( x' \), which is unique when \( x \) is given. We have thus a cremona transformation \( (x, x') \) between the spaces \( \alpha \) and \( \alpha' \). Let us call this transformation \( W \).

Any point \( x \), of \( g_6\alpha \) is on a line bisecant to \( R \). The point \( x \) then carries \( \infty^1 \) planes trisecant to \( R \). The curve \( R \) is projected from the bisecant line on \( x \) by planes of a cubic 3-way cone which meets \( \alpha' \) in a cubic curve. Hence to the points of \( g_6\alpha \) correspond cubic curves in \( \alpha' \), and, similarly, to the points of \( g_6\alpha' \) correspond cubic curves in \( \alpha \).

Points of \( g_6\alpha \) and \( g_6\alpha' \) are triple singular points of \( W \).

Through a line of \( \Phi_\alpha \), there is a plane trisecant to \( R \); hence to any point of such a line corresponds the unique point, \( x \), in which this plane meets \( \alpha \).

There are \( \infty^1 \) such planes meeting \( \alpha' \) in the \( \infty^1 \) lines of \( \Phi_{\alpha'} \). Their locus is a 3-way which meets \( \alpha \) in a curve \( K_\alpha \) of simple singular points of \( W \). This 3-way is met by any \( S_4 \) on \( \alpha' \) in \( \Phi_{\alpha'} \) and in 10 planes, a total intersection of order \( 10 + 8 = 18 \). Hence

The curve \( K_\alpha \) is of order 18. Its genus is 3, since it is in one-one correspondence with the lines of \( \Phi_\alpha \), and these, in turn, are by the 5-planes inscribed to \( g_6\alpha' \), in one-one correspondence with the latter curve.

A line of \( \Phi_\alpha \) is a fundamental line in \( \alpha \), i.e., a line such that all of its points have only one correspondent. It contains three triple singular points, its intersections with \( g_6\alpha \). Hence it is represented in \( \alpha' \) by a point, taken with three fundamental cubic curves, of total order 9. Hence

The transformation \( W \) is of order 9.

Let us call the locus of cubic curves in \( \alpha' \) corresponding to points of \( g_6\alpha \), \( G_{\alpha'} \) and the similarly determined surface in \( \alpha \), \( G_\alpha \). We desire to know the order of these surfaces. A plane \( \pi_\alpha \) in \( \alpha \) has as correspondent a nonic surface \( \pi_{\alpha'} \) in \( \alpha' \). \( \pi_{\alpha'} \) has \( K_{\alpha'} \) as a simple curve, \( g_6\alpha' \) as a triple curve. \( \pi_{\alpha'} \) is sent by \( W \) into:

the plane \( \pi_\alpha \) of degree 1,
the surface \( \Phi_\alpha \) of degree 8,
the surface \( G_\alpha \), taken 3 times, of degree 72,

making a total of degree 81.

*See § 3, first paragraph.
It follows that $G_2$ is of order 24.

Certain general deductions can be made from the above observations, namely,
- Planes trisecant to $R$ and meeting a given line are on a 3-way of order 9.
- Planes trisecant to $R$ and meeting a given plane are on a 4-way of order 9.
- Planes trisecant to $R$ and meeting a given space in a line are on a 3-way of order 18.

§6. Osculants.

First osculants of $R$ as a curve of points are the curves $D_1\xi$, where $\xi$ is any $S_4$ of $R$.* Dually, first osculants of $R$ as a curve of $S_4$'s are the perspections of the spaces of $R$ from point $x$ on $R$. Similarly, mixed cubic osculants of $R$ are perspections of its planes from its bisecant lines; mixed conic osculants, perspections of its lines from its trisecant planes.†

For a rational curve, projections of osculants are osculants of projections; dually, sections of developables of osculants are osculants of sections.

Let $\pi$ be any plane of $S_6, \alpha$, any space on $\pi$, and $\xi$, any $S_4$ on $\alpha$. There is determined on $\pi$ the rational curve $D_3\pi$ of class 5. The 6 points $g_6\pi$ may be called the cyclic points of $D_3\pi$, since tangents to the latter curve from one of these points touch in five points whose parameters form a quintic reducible to the form $\alpha t_1^5 + \beta t_2^5 = 0$. The $S_4 \xi$ meets $R$ in five points 1, 2, 3, 4, 5—a set of the fundamental involution of $D_3\pi$.

On 1, 2, 3, 4, 5 there are the ten lines 12, ten planes 123, and five spaces 1234. The osculant 1 of $D_3\alpha$ has stationary points at $12\alpha, 13\alpha, 14\alpha, 15\alpha$. These are four points of $g_6\alpha$. We have then

The curve $g_6\alpha$ is the locus of stationary points of quartic osculants of $D_3\alpha$.

The 3-way $r^3_a\pi$ is the locus of planes of tetrahedra of stationary points of quartic osculants of $D_3\alpha$.

The second osculant 12 of $D_3\pi$ has stationary points at the points $123\pi, 124\pi, 125\pi$. For $D_5\pi$ we have

(p) Any set of the fundamental involution of $D_3\pi$, say 1, 2, 3, 4, 5, determines ten mixed cubic osculants $O_{ijk}$ of $D_3\pi$. There exist five lines of $\pi$, $\mu_1, \mu_2, \mu_3, \mu_4, \mu_5$, such that the 3-points $\mu_1, \mu_2, \mu_3, \mu_4, \mu_5$ are the stationary points of the osculant $O_{ijk}$. That is, the stationary points of cubic osculants of pairs of points selected from a set of the fundamental involution of $D_3\pi$ are ten points of a 5-line.

§7. The Curves $r^3_a\pi$ ‡ and $\Phi_5\pi$.

The (3, 1) congruence of lines $l$ in a space $\alpha$ is the congruence of axes of $r^3_a\pi$. The class of this congruence being one, there is a unique line $l$ on any plane $\pi$. Hence,

* Cf. Berzolari, l. c., p. 7.
† Let $\xi = (\alpha t)^* be the parametric equations of a rational curve: $\xi = (\alpha_1 t_1) \cdots (\alpha_d t_d) (\alpha_1 t_1)^{n-k}$ is a mixed osculant; $\xi = (\alpha_1 t_1)^k (\alpha_2 t_2)^{n-k}$ is a pure osculant.
‡ We indicate by $r^3_a \pi$ the curve of lines in which planes of $r^3_a$ meet $\pi$.
The curve $r_\alpha \cdot \pi$ has the unique line $l$ on the $\pi$ as double tangent.

Any two spaces $\alpha$ and $\alpha'$ on $\pi$ are contained in a common $S_4$, $\xi$. Hence, the two curves $r_\alpha \cdot \pi$ and $r_{\alpha'} \cdot \pi$ have, besides the line $l$ on $\pi$, five lines $\mu_1^{(\alpha)}$ in common, $\mu_1^{(\alpha)}$ being the five lines in which spaces on four out of the five points $R \xi$ meet $\pi$.

The pencil of $S_4$'s, $\xi$, on $\alpha$, a space containing $\pi$, cut out of $R$ a pencil of the fundamental involution of $D_3 \pi$. This leads to the following

Given the curve $D_3 \pi$ and any set, $f$, of the fundamental involution on this curve, stationary points of cubic osculants of $f$-airs of points chosen from the roots of $f$ are ten points of a 5-line. If $f$ vary in a pencil the locus of the 5-lines so determined is a cubic curve having the covariant line $l$ on $\pi$ as double tangent.

Given two lines $p$ and $p'$ of $\pi$, there is a unique space $\sigma$ quadrisecant to $R$ on $p$, and on $\sigma$ and $\pi$ there is a unique $S_4$, $\xi$. Similarly $p'$ determines a space, $\sigma'$, and an $S_4$, $\xi'$. $\xi$ and $\xi'$ meet in an $S_3$, $\alpha$, containing $\pi$. The curve $r_\alpha \cdot \pi$ is thus uniquely determined by requiring it to touch $p$ and $p'$. Hence *

The curves $r_\alpha \cdot \pi$ are in a linear system of $\infty^2$ cubics having the double tangent $l$.

Again, given two points $x$ and $x'$ of $\pi$, there are on $x$ and $x'$ respectively two planes $\pi_1$ and $\pi'$ trisecant to $R$. On $\pi_1$ and $\pi$ there is an $S_4$, $\xi$, and on $\pi'$ and $\pi$, an $S_4$, $\xi'$. The two $S_4$'s, $\xi$ and $\xi'$ meet in a space $\alpha$ on $\pi$. The surface $\Phi_x$ contains the lines $\pi_1 \alpha$ and $\pi_1 \alpha$, and $\Phi_x \pi$ is hence on $x$ and $x'$. There is thus a unique curve $\Phi_x \pi$ on two general points of $\pi$, and we have this theorem:†

The curves $\Phi_x \pi$ are in a linear two-fold system. All curves of this system have as triple points the six cyclic points $g_6 \pi$ of $D_3 \pi$.

§ 8. The Correspondences $T$ and $U$.

Projecting $R$ from a plane $\pi_\xi$ on a plane $\pi$, we obtain a rational quintic curve $\rho_\pi^\xi$. The $S_4$'s on $\pi_\xi$ cut out of $R$ line-sections of $\rho_\pi^\xi$ and points of $\pi_\xi$ determine on $R$ sets of the fundamental involution of $\rho_\pi^\xi$, i. e. the curves $D_3 \pi_\xi$ and $\rho_\pi^\xi$ are conjugate quintics.

We determine a correspondence $T$ between the lines $\xi$ of $\pi_\xi$ and the lines $\eta$ of $\pi_\eta$ in the following manner: A line $\xi$ of $\pi_\xi$ carries in general a unique space $\sigma_\xi$ quadrisecant to $R$; the only case in which $\sigma_\xi$ is not uniquely determinate is when $\xi$ is the line $l$ on $\pi_\xi$. On $\sigma_\xi$ and $\pi_\xi$ there is an $S_4$ which meets $\pi_\eta$ in a line $\eta$. $T$ is defined as the correspondence $(\xi, \eta)$.

Conversely, given a line $\eta$ of $\pi_\eta$ there is a unique $S_4$ on $\eta$ and $\pi_\xi$. This $S_4$ meets $R$ in five points; on any four of these five points there is an $S_3$, $\sigma$, quadrisecant to $R$. Each of these five spaces meets $\pi_\xi$ in a line $\xi$ corresponding to $\eta$.

* Bertini, Geometria proiettiva degli iperspazi, p. 224.
† Bertini, l. c.
T is a \((5, 1)\) correspondence.

Comparing with theorem \((d)\) what we have just said, we have the following:

The line \(l\) on \(\pi\) is a singular line of \(T\); to \(l\) correspond lines of a conic in \(\pi_l\);

the line \(l\) is the only singular line of \(T\) in either plane.

Any space \(\alpha\) on \(\pi_l\) marks a point \(y_\alpha\) on \(\pi_\eta\). All \(S_\alpha\)'s on \(\alpha\) give lines \(\eta\) on \(y_\alpha\).

To these lines correspond by \(T\) lines of the cubic curve \(r^3 \cdot \pi_l\). Hence

The correspondence \(T\) is cubic. To points \(y_\alpha\) of \(\pi_\eta\) correspond lines of the curves \(r^3 \cdot \pi_l\).

There is a second correspondence, \(U\), between \(\pi_l\) and \(\pi_\eta\) which we obtain by associating with a line, \(\eta\), of \(\pi_\eta\), the ten points of intersection of its corresponding lines \(\xi\). The line \(\eta\) is unique when a general point \(x\) of the plane \(\pi_l\) is given, since there is on \(x\) a unique plane trisecant to \(R\), and on this plane \(\pi_l\) there is an \(S_\alpha\) which meets \(\pi_\eta\) in the line \(\eta\). To lines on a point \(y_\alpha\) of \(\pi_\eta\) correspond by \(U\) points of a curve \(\Phi_\alpha \pi_\xi\). Hence

The relation \(U\) is an octave \(10:1\) line-point correspondence between the planes \(\pi_l\) and \(\pi_\eta\). It has the six triple singular points \(g_x \pi_l\).

The latter statement follows from the fact that the curves \(\Phi_\alpha \pi_\xi\) have \(g_x \pi_\xi\) as triple points. To a cyclic point in \(\pi_l\) correspond by \(U\), therefore, lines of a cubic curve in \(\pi_\eta\). This cubic is necessarily rational.

\section{9. \(T\) and \(U\) Considered Dually.}

We have now to study \(T\) and \(U\) from the point of view of the general theory of correspondences. It will be more convenient to discuss \(T\) as a point-point correspondence, and hence to consider dually the facts which we have obtained. We give briefly the dual statement of the essential facts, without reference to the arguments from hyperspace which we have been using hitherto.

Given a rational quintic curve, \(\rho^5\), in a plane, \(\pi_x\), its conjugate quintic, \(r^5\), is determined to within projections. We consider \(r^5\) as a curve of lines and regard it as situated in a plane \(\pi_y\); the curves \(\rho^5\) and \(r^5\) then assume the forms \((1)\) and \((2)\) respectively (§1), and are in \((1, 1)\) correspondence through the parameter \(t\).

Tangents to \(r^5\) from a point \(y\) define on \(r^5\) and hence on \(\rho^5\) a set of the fundamental involution of \(\rho^5\). The inflexional lines of mixed cubic osculants of pairs of parameters chosen from this set are ten lines of a 5-point, \(x_1, \ldots, x_5\). To \(y\) correspond by the correspondence which we have called \(T\) the five points \(x_i\). Given a point \(x\), its \(y\) is uniquely determined. If \(y\) move on a line \(\eta\) in \(\pi_y\), the corresponding points \(x_i\) move on a rational cubic, \(c_x\), in \(\pi_x\). All curves \(c_x\) are in a linear two-fold system: they have a common node \(O\), which is a rational covariant point of \(\rho^5\), and is characterized by the fact that the pencil of binary quintics cut out of \(\rho^5\) by lines on \(O\) has a pencil of apolar quartics; in other words, these quintics are the first polars of a binary sextic (§3).
To proceed: A curve $c_x$ is in natural $(1, 1)$ correspondence with $\rho^x$ and the groups $x_i$ on $c_x$ as a rational support are the pencil of the fundamental involution of $\rho^x$ which determines it. This pencil is cut out on any curve $c_x$ by other curves of the system. In fact, the common parameter on all curves $c_x$ is the parameter of a line in the pencil with vertex at $O$. Otherwise we would obtain from any two curves $c_x$ two projective pencils with vertex at $O$ and these two pencils would have five self-corresponding elements.

A rational plane quartic, $\rho^4$, has an important covariant conic defined in a manner similar to the curves $c_x$. It is the locus of vertices of triangles of stationary lines of cubic osculants of the quartic. The pencil of quartics in the fundamental involution of $\rho^4$ give a single infinity of 4-points of this conic. Stahl calls this conic the conic $K$ of $\rho^4$.

If $\gamma$ is a line of $\rho$ with parameter $t_0$, $T\gamma$ is the Stahl conic $K$ of the osculant of $\rho^5$ at $t_0$, taken with a line through $O$. For the mixed cubic osculant $t_0 t_1$ is the osculant $t_1$ of the osculant $t_0$. We have, then,

All conics $K$ of first osculants of $\rho^5$ are on the point $O$.

If $\xi$ is a line in $\pi_5$ on $O$, $T\xi$ is a line of $\rho^5$, the conic $TO$ factoring out of the transform of $\xi$. The conic $TO$ and $\rho^5$ are in $(1, 1)$ correspondence with lines on $O$, directions around $O$ corresponding to points of $TO$; we may therefore choose for each the parameter of a line of this pencil. The line $t$ of $\rho^5$ is then on the point $t$ of $TO$. Hence

The conic $TO$ is the perspective conic of $\rho^5$.\[†\]

In the correspondence $U$ a point $y$ is made to correspond to the ten lines joining the five points $x_i$ corresponding to $y$ by $I$. $U\gamma$ is a curve of class 8 having the cyclic lines of $\rho^5$ as triple tangents. These are triple singular lines of $U$. This correspondence has no singular points in $\pi_5$.

The locus $T\xi$ is a cubic curve $C_\xi$ in $\pi_5$, necessarily rational, since it is in $(1, 1)$ correspondence with $\xi$. It follows that $\xi$ contains a single pair of associated points, $x_1, x_2$, corresponding to the node $y$ of $T\xi$. The transform $U\xi$ is the point $y$. The points $x_1$ and $x_2$ are the neutral pair of the $I_{2,1}$, of points in which $\xi$ is met by curves of the net $c_x$.

Since the curve $\rho^5$ has six double tangents, it follows that there are six curves $c_x$ which are made up of three lines. Call the double tangents of $\rho^5$, $\beta_i$. Then,

$$T\beta_i = \alpha_{i1} \alpha_{i2} \alpha_i;$$

where $\alpha_{i1}$, $\alpha_{i2}$ are two lines on $O$ and $\alpha_i$ is a cyclic line of $\rho^5$: the latter statement follows from the fact that $U\alpha_i$ is indeterminate. Cubics $c_x$ meet $\alpha_{i1} \alpha_{i2} \alpha_i$ in groups of 5 points $x_i$. Two of these, $x_1$ and $x_2$, say, are on $\alpha_{i1}$ and $\alpha_{i2}$

respectively. The others then are on the line $\alpha_i$ and form an $I_1,2$. The points $x_1, x_2$, varying, determine on $\alpha_1$ and $\alpha_2$ two projective ranges and these are in turn projective to the corresponding points $y$ of $\beta_1$. Lines joining corresponding points $x_1, x_2$, on $\alpha_1, \alpha_2$, touch a conic $q_i$. This conic from its mode of generation touches $\alpha_n, \alpha_1$, and all lines $\alpha_j$ where $j \neq i$. Now $Uq_i$ is $\beta_i$. But $q_i$ is related projectively to $\beta_i$ by $U$ in such a way that to the line $\alpha_j$ corresponds the point where $\beta_i$ meets $\beta_j$. This is a characteristic property of the singular lines of a quintic cremona line-line transformation with 6 double singular lines. Furthermore, given two sets of lines having this property, the transformation is uniquely determined. Then given a line $\xi$ of $\pi_x$, and $\eta$, its transform in $\pi_y$, the points $\xi\alpha_i$ are projective to $\eta\beta_i$. We may state this in the following theorem:

The cyclic lines of $\rho^5$ and the double lines of $\tau^6$ are double singular lines of a quintic cremona line-line transformation between the planes $\pi_x$ and $\pi_y$.

A line $\xi$ on $O$ is transformed by $T$ into a line $\eta$ of $\tau^6$; since there is one variable point $x$ on $\xi$, the points of $\xi$ are related projectively to the points of $\eta$, and in such a way that $\xi\alpha_i$ are projective to $\eta\beta_i$. Hence, This quintic transformation sends lines on $O$ into lines of $\tau^6$.

The transforms of points $y$ of $\pi_y$ are a net of quintic curves with $\alpha_i$ as double tangents. To the 5 lines of $\tau^6$ on $y$ correspond the five lines of a quintic of this net on $O$. This gives at once a determination of the fundamental involution of $\rho^5$ in the pencil of lines about $O$:

Given two lines $t_1$ and $t_2$ on $O$, there is a unique curve of class five on $t_1$ and $t_2$ and having the $\alpha_i$ as double lines. The other three lines of this curve, say $t_3, t_4, t_5$ on $O$ form with $t_1$ and $t_2$ a set of the fundamental involution of $\rho^5$.

§10. The Involution of Points $x_i$.

Points $x_1, \ldots, x_6$ having the same correspondent $y$ by $T$ are groups of an involution, $I$, in $\pi_x$. The correspondence $x_i, x_j$ is obviously $(4, 4)$. Since

$$T^2 \xi = \xi I \xi,$$

and $T^2 \xi$ must be a nonic curve, it follows that $I \xi$ is an octavic curve. Hence The involution $I$ is octavic.

The only singular point of $I$ is $O$. We have obviously

$$IO = Tc_0,$$

$c_0$ being the perspective conic of $\tau^6$ in $\pi_y$. The curve $IO$ is therefore a sextic and it follows that $I \xi$ must have a 6-fold point at $O$. The curve $I \xi$ has besides the three nodes $a_8, a_4, a_5, a_1, \ldots, a_6$ being a group of $I$ and $a_1, a_2$ being the unique pair of $I$ on $\xi$.

* A. B. Coble, these Transactions, vol. 9 (1908), p. 398.
The curve \( P^2 \) is of order 48. It can be nothing but \( IO \) taken eight times. There is an involution of points \( x_2, x_3, x_4, x_5 \) on \( IO \), \( x \) being indefinitely near to \( O \). We should therefore expect \( IO \) to appear three times in \( P^2 O \); to make up the necessary eight times \( IO \) must pass five times through \( O \). Hence,

The curve \( IO \) is a Jonquières sextic with 5-fold point at \( O \).

The Jacobian of the net of curves \( c_x \) is a Jonquières curve, \( J \), with 5-fold point at \( O \). Note also that \( J \) is the coincidence curve of the involution \( I \).

The curve \( TJ \) is the curve \( r^5 \) taken in points, an octavie.

Further, \( IJ \) is a curve of order 12, \( J \) factoring once and \( IO \) 5 times out of the transform of \( I \) by \( J \).

Any question connected with the osculants of \( r^5 \) must be intimately associated with the transformations which we have been discussing. We give a few examples of this statement.

Let \( x_1, \ldots, x_5 \) be a set of the involution \( I \); further let \( x_1 \) and \( x_2 \) be indefinitely near to each other, and hence to \( J \). Let

\[
t_1 = t_2, \quad t_3, \quad t_4, \quad t_5
\]

be the corresponding parameters on \( r^5 \)—a set of the fundamental involution. The pure cubic osculant \( t_1 \) has a triangle of inflexional lines \( x_2, x_4, x_5 \). Hence

The curve \( IJ \) is the locus of vertices of triangles of inflexional lines of pure cubic osculants of \( r^5 \).

The curve \( I^2 J \) is \( J \) taken six times; \( IJ \) twice; and \( IO \) six times to make up the necessary order of \( I^2 J \), 96. Hence,

The transform \( IJ \) passes six times through \( O \).

If a rational cubic have a cusp, two inflexional tangents have become the cusp tangent, the third is the single proper inflexional tangent of the curve. Consider the osculant \( t_4, t_5 \) with inflexional triangle \( x_1, x_2, x_3 \). The points \( x_1 \) and \( x_2 \) are indefinitely near; the lines \( x_1 x_3, x_2 x_3 \) are indefinitely near. This osculant must have a cusp; \( x_1 x_2 \) is the inflexional tangent and \( x_3 \) is the cusp. Hence

The curve \( IJ \) is also the locus of cusps of cuspidal mixed cubic osculants of \( r^5 \). Six such osculants have cusps at \( O \).

The line \( x_1 x_2 \) is the inflexional tangent of the three cuspidal osculants \( t_4s, t_5s, t_6s \). The locus of this line must be a rational curve, since it is in \((1,1)\) correspondence with \( J \). Call this locus \( \Gamma \). Cubics \( c_x \) meet a line \( a_\xi \) in a pencil of binary cubics; there are four points on \( a_\xi \) at which a cubic of the pencil has a double root; hence,

The locus \( \Gamma \), defined above, has the cyclic lines of \( r^5 \) as four-fold tangents.

Now \( UT \) is \( r^5 \) taken in points, a curve of order eight. If \( m \) is the class of \( \Gamma \) we have

\[
8m - 3 \cdot 4 \cdot 6 = 8,
\]
whence $m = 10$. Hence

The locus of inflexional lines of cuspidal cubic osculants of $\rho^5$ is a rational curve of class ten having the cyclic lines of $\rho^5$ as 4-fold lines. The transform of this curve by $U$ is the curve $\rho^6$, taken in points.

§ 11. Perspective Curves of $\rho^5$.

It will be seen at once that any Jonquières curve with multiple point at $O$ in $\pi_x$ is in one-one correspondence with lines on $O$ and is transformed by $T$ into a rational curve perspective to $\rho^5$. Thus lines of $\pi_x$ give $\alpha^2$ perspective cubics of $\rho^5$, that is,

The curves $C_v$ are perspective cubics of $\rho^5$.

The transforms by $T$ of the $\alpha^{2m}$ curves $j^{(m)}$ of order $m$ with $(m - 1)$-fold point at $O$, are the $\alpha^{2m}$ perspective $(m + 2)$-ics of $\rho^5$.* The points of contact of the perspective curves with $\rho^5$ are the transforms of the points of intersection of these curves $j^{(m)}$ with $J$.†

Some interesting properties of systems of perspective curves are easily obtainable from this point of view. For instance, curves $c_v$ on a point $y$ break up into 5 systems, the transforms of pencils of lines on the 5 points $x_i$. The contacts of curves of each system are in a pencil. The locus of nodes of a system $x_i$ is the curve $Ux_i$, a rational octavic. There are 25 perspective cubics of $\rho^5$ on two points.

It is known that, if a rational sextic plane curve is given, and a quadratic involution on it, the joins of corresponding pairs of this involution touch a curve of class 5. A curve related in this way to $\rho^5$ is $TC$, where $C$ is any conic of $\pi_x$. Now a rational sextic with a given involution on it has 19 constants, the number involved in our scheme if we include the conic $C$. Presumably all rational sextics related to $\rho^5$ in this way are obtainable as transforms of conics of $\pi_x$. The contacts of these curves with $\rho^5$ are the transforms of the meets of conics of $\pi_x$ with $J$. These groups of 12 points are in an $I_5, 7$.

A word as to the possibility of extension of this method seems advisable. Curves of a net of Jonquières curves $c^{(m)}_x$, of order $m$, in a plane $\pi_x$, and with fixed multiple point $O$, may be put into homographic correspondence with the lines of a second plane $\pi_y$. It is easily seen that an $m$-ic point-point $(2m - 1, 1)$ correspondence, $T^{(m)}$, is thus determined between $\pi_x$ and $\pi_y$.

To degenerate curves of the net $c^{(m)}_x$ correspond by $T^{(m)}$ lines of a curve $r^{2m-1}$ of class $2m - 1$ in $\pi_y$.

The constants are exactly right for $r^{2m-1}$ to be a general rational curve of class $2m - 1$; in fact there is an argument from hyperspace which develops the apparatus and proves that this curve is general.

* W. STAHL, loc. cit.
The jacobian of the net \( c_z^{(m)} \) is a Jonquières \((3m - 3)\)-ic with \((3m - 4)\)-fold point at \(O\). Call this \( J^{(3m-3)} \). Then \( T^{(m)} J^{(3m-3)} \) is \( r^{2m-1} \) taken in points.

The curve \( T^{(m)}O \) is the unique perspective \((m - 1)\)-ic of \( r^{2m-1} \). The curves \( T^{(m)} \xi, \xi \) being a line of \( \pi_z \), are the \( \alpha^2 \) perspective \( m \)-ics of \( r^{2m-1} \). We see, then, that the properties of systems of perspective curves which we have stated for the rational quintic are extensible to all rational curves of odd order.

Somewhat similar theorems are true for curves of even order; for, by making the curves \( c_z^{(m)} \) pass through a second point \( O' \), the class of \( r^{(2m-1)} \) is reduced by one.

§ 12. Determination of \( \rho^5 \) and \( r^5 \) from the Curves \( c_z \).

Given the net of curves \( c_z \) with a common node \( O \), the transformation \( T \) is, at once

\[
y_i = (c; x)^3,
\]

where \( (c; x)^3 \) are three linearly independent curves of the net, and \( r^5 \) is the locus of lines \( \eta \) whose transforms are degenerate cubics.

Stahl* obtains the general rational space quintic curve from a norm-curve \( R^3 \) and a binary octavic \((at)^8 \) on it, in the following manner: A cubic polar \((at)^5 (at)^3 \) considered as a binary cubic on \( R^3 \) defines a point \( x \) to which is associated the parameter \( \tau \). The locus of \( x \) is the rational quintic. This is the point of view of § 4 of this paper, the curve \( r^5 \) being the Stahl cubic of \( D_2 \alpha \). The involution \( I_1,4 \) on \( R^3 \) is the involution of quintics apolar to \((at)^8 \).

By projection from a point of space the curve \( R^3 \) becomes a cubic \( c_x \) of the quintic \( \rho^5 \) which we obtain; hence \( \rho^5 \) is determined from any of the curves \( c_z \) (non-degenerate), in the following manner. On \( c_x \) is a pencil of binary quintics, an \( I_1,4 \), its intersection-groups with the other cubics of the system. This pencil has a unique apolar octavic \((at)^8 \). Given a cubic polar \((a \tau)^5 (at)^3 \), of \( \alpha \), a point \( x \) is determined by requiring line-sections of \( c_x \) on the point \( x \) to be apolar to this cubic in \( t \). Then \( x \) is the point \( \tau \) of \( \rho^5 \).

The net \( c_z \) is determined uniquely by the lines \( \alpha \) and the point \( O \). On all lines \( \alpha \) except \( \alpha_i \) there is a conic \( q_i \). The line \( \alpha_i \), taken with the tangents, \( \alpha_{i1}, \alpha_{i2} \), from \( O \) to \( q_i \) is a cubic. The six cubics thus obtainable are in a net, and the net \( c_z \) and hence the quintic \( \rho^5 \) are uniquely determined. This leads to the following theorem:

* W. Stahl, loc. cit.