QUATERNION DEVELOPMENTS WITH APPLICATIONS*

BY

JAMES BYRNIE SHAW

INTRODUCTION.

This paper is an extension of the quaternion algebra along lines analogous to those of a preceding paper.† Certain functions of one or more quaternions are studied in detail, all of them independent of the orientation of the unit axes, 1, i, j, k. The notation is functional and seems preferable to the use of brackets such as those used by Hamilton,‡ who, indeed, called his notation temporary. We seem to have here a natural method for treating several functions that occur frequently in analysis. For example, if we represent quaternion numbers by vectors in four-dimensional space, as was done by Stringham§ and Hathaway,|| these functions are expressions for geometrical relations which are independent of the particular system of rectangular axes used. These expressions are useful in electrodynamics and in relativity problems. They include the Hamiltonian [ab], (abc), [abc], (abcd), and the equivalent forms used by Joly,‡‡ and Schrutka von Recht- enstamm.**

The development of these formulae enable us also to formulate certain differential operators of frequent occurrence, which, from their nature, produce expressions, each invariant under orthogonal transformation in the sense that after the transformation has been performed on the various quaternions of which it is a function, the result is the transform of the initial function (§ 8). Certain of these forms are also invariant under any linear homogeneous transformation. Similar forms were used by MacMahon+++ in the study of the ternary and quaternary invariants and covariants under orthogonal transformations. The study of differential parameters and their syzygies is made

---

* Presented to the Society (Chicago), April 8, 1910, and April 28, 1911.
§ These Transactions, vol. 2 (1901), pp. 183–214.
|| These Transactions, vol. 3 (1902), pp. 46–59.

Trans. Amer. Math. Soc. 18 279
simpler by the use of these forms, and the symbolic invariants of Maschke * appear as a natural part of the development. Two differential operators are introduced which are the extensions of the quaternion $\mathbf{V}$ to space of four dimensions and to curvilinear three-dimensional space.

Up to the present time, the greater part of the development of the quaternion algebra has been in terms of scalars and vectors rather than in terms of quaternions proper. This often introduces prolixity in the development. It is hoped that the present paper will present some fruitful extensions along the more general line. In the expansion of the forms appearing here it will be evident that they are of the nature of determinants and pfaffians. That other forms might be studied is of course self-evident, but these are the ones of most frequent occurrence in the functions of geometry and physics. However the developments below are on the formal side and valid irrespective of the interpretation of the quaternions.

**Part I. Fundamental Formulæ.**

1. Throughout the paper italic letters will be used to represent quaternions, which may occasionally reduce to scalars or vectors. The conjugate of a quaternion $q$ will usually be denoted by $q'$, but the conjugate of an expression, as $qrs$, will be denoted by $K \cdot qrs$.

2. We notice that the quaternions $uv'$ and $v'u$ are important, particularly on account of the relations shown in the formulæ

$$uv' = S \cdot uv' + V\mathbf{s}v - V\mathbf{s}u - VV\mathbf{u}v,$$

which resolves $uv'$ into three parts whose products have vanishing scalars; and if $l$ and $m$ are scalars,

$$uv' (lu + mv) = u (2lSuv' + mT^2v) - ltT^2u = (lu + mv) v'u,$$

which shows that any quaternion of the form $lu + mv$ is converted into another of the same form by left-hand multiplication by $uv'$ or by right-hand multiplication by $v'u$. Also $uv'$ and $v'u$ differ only in their axes, having the same tensors and angles. The quaternions $uv'$ and $vu'$ however differ only in having opposite angles. If $r = lu + mv$, we have $uv'r - rv'u = 0$.

3. We define first the two functions

$$I \cdot uv = \frac{1}{2} (uv' + vu'), \quad A_2 \cdot uv = \frac{1}{2} (uv' - vu').$$

The first of the two functions is obviously a scalar, since it is the sum of a quaternion and its conjugate, and the second is a vector. Indeed,

$$I \cdot uv = S \cdot uv' = T^2v \cdot S \cdot \frac{u}{v}, \quad A_2 \cdot uv = Vuv' = T^2v \cdot V \cdot \frac{u}{v}.$$

*These Transactions, vol. 1 (1900), pp. 197-204; vol. 7 (1906), pp. 69-80.
From the definitions we have at once

$$I \cdot uv = I \cdot vu = I \cdot u' v' = I \cdot v' u', \quad A_2 \cdot uv = -A_2 \cdot vu.$$  

The latter expression is alternating and therefore vanishes if \( u \) is a scalar multiple of \( v \). If we write \( \rho = UVV'u'v' \), we find with no difficulty that

$$A_2 \cdot u' v' = \rho \cdot A_2 \cdot uu \cdot \rho^{-1}.$$  

Hence \( A_2 \cdot uv \) differs from \( A_2 \cdot u' v' \) only in direction. By expanding the two sides of the equations, we find that

$$u^{-1} A_2 \cdot uv \cdot u = A_2 \cdot v' u', \quad v^{-1} \cdot A_2 \cdot uv \cdot v = -A_2 \cdot u' v'.$$

The two forms used by Joly are, in this notation, as follows:

$$\langle uv \rangle = vSu - uSv = \frac{1}{2} (A_3 \cdot u' v' - A_2 \cdot uv),$$

$$[uv] = VV'u'V = \frac{1}{2} (-A_2 \cdot u' v' - A_2 \cdot uv).$$

The single form given by Hamilton, \([uv]\), is twice the corresponding one given by Joly, that is, is \( 2 VV'u'V \).

If \( I \cdot uv = 0 \) we shall speak of \( u \) and \( v \) as orthogonal. The solution of the two equations which express that \( r \) is orthogonal to both \( u \) and \( v \) is

$$r = x (uv' - v' u) + y (uv' uv' - uv' v' u),$$

where \( x \) and \( y \) are arbitrary scalars; for, this value of \( r \) satisfies the two equations, and \( r \) can depend only upon two arbitrary parameters.

4. Passing now to three quaternions, we define the functions

$$A_1 \cdot uvw = \frac{1}{2} (uv' w + wv' u), \quad A_3 \cdot uvw = \frac{1}{2} (uv' w - wv' u).$$

It is evident that \( A_1 \cdot uvw = A_1 \cdot wvu \). Also

$$A_1 \cdot uvw = \frac{1}{2} (vw' u + wv' u - vv' w - vv' u + uu' w + uv' w)$$

$$= uI \cdot vw - vI \cdot wu + wI \cdot uv.$$  

Again from the definition it is obvious that

$$A_3 \cdot uvw = -A_3 \cdot wvu.$$

Since we have identically \( 2uI \cdot vw = 2I \cdot vw \cdot u \), we have by expansion the identity

$$uv' w + uw' v = vw' u + wv' u,$$

from which by transposition, and by reapplying the result, we arrive at

$$uv' w - wv' u = vw' u - uw' v = wu' v - vu' w.$$
Therefore, returning to the form $A_3$, we have

$$A_3 \cdot uvw = A_3 \cdot vwu = A_3 \cdot wuv = - A_3 \cdot wvu = - A_3 \cdot uwv$$

$$= - A_3 \cdot vwu.$$ 

This function is therefore an alternating function of $u, v$, and $w$. It vanishes if they are linearly connected. Joly's form $[uvw]$ is in this notation $A_3 \cdot u'v'w'$. His form $(uvw)$ is $SA_3 \cdot u'v'w'$. We may expand in the form

$$A_3 \cdot uvw = \frac{1}{6} \left[ uv'w - u'vw + vw'u - vwu' + wu'v - wvu' \right]$$

$$= \frac{1}{3} \left[ uA_2 \cdot v'w' + vA_2 \cdot w'u' + wA_2 \cdot u'v' \right].$$

This form shows that the function is of the nature of a determinant.* Again,

$$I \cdot uA_3 \cdot uvw = 0 = I \cdot vA_3 \cdot uvw = I \cdot wA_3 \cdot uvw.$$

For, if we expand $I \cdot uA_3 \cdot uvw$ we have

$$\frac{1}{2} \left[ u'uv'w - u'v'w + w'vu' - u'vw' \right] = \frac{1}{2} \left[ u'uI \cdot vw - u'I \cdot vw - u' \right] = 0.$$

A similar proof holds for $v$ and $w$. Hence the solution of the three equations expressing the orthogonality of $r$ to $u, v, w$, is $r = xA_3 \cdot uvw$, where $x$ is an arbitrary scalar.

We see also from these equations and the expanded form of $A_3 \cdot uvw$ that we may resolve $uv'w$ into two orthogonal parts:

$$uv'w = A_3 \cdot uvw + A_3 \cdot uvw.$$

If $A_1 \cdot uvw = 0$, we have $uv'w + uv'w = 0$, whence

$$uv'vw'v + uv'wv'w = 0 = uv'v + uv'w + uv'w.$$

Adding the two sides of this double equality we have

$$[uv'vw'v + uv'wv'w] u = 0.$$

This gives easily $uv'wI \cdot vw = 0$. Similarly we find that $uv'wI \cdot uvw = 0$. Hence either we must have $I \cdot vw = 0 = I \cdot uvw$ or else $uv'w = 0 = uv'w$. The latter cannot hold unless $T^2u$, or $T^2v$, or $T^2w$ vanishes. Then if the quaternions are real we must have, when $A_1 \cdot uvw = 0$, the three equations $I \cdot uv = 0 = I \cdot uw = I \cdot vw$, and the three quaternions form an orthogonal system.

We find that†

$$TA_1 \cdot uvw = \sqrt{IuvwI^2v + IvwI^2w + IwvI^2w - 2IuvIvwIwu}$$

$$= TA_1 \cdot uvw = \ldots.$$

† The period following $I$ and $A$ will be omitted when no confusion can result, just as it is after $S$ and $V$ in treatises on Quaternions.
Since \( T^2 uv' w = 1 \) when \( u, v, \) and \( w \) are unit quaternions, and
\[
T^2 uv' w = T^2 A_1 \cdot uvw + T^2 A_3 \cdot uvw,
\]
we might call \( TA_1 \cdot uvw \) the cosine of the triple \( u, v, w \) and \( TA_3 \cdot uvw \) the sine. The latter for geometric reasons could also be called the Staudtian. Moreover,
\[
T^2 A_3 \cdot uvw = I \cdot A_3 \cdot uvwA_3 \cdot uvw = \begin{vmatrix} I_{uu} & I_{uv} & I_{uw} \\ I_{vu} & I_{vv} & I_{vw} \\ I_{wu} & I_{wv} & I_{ww} \end{vmatrix}.
\]

5. If we write out each quaternion in terms of 1, \( i, j, k \) we find that
\[
A_3 \cdot uvw = \begin{vmatrix} 1 & i & j & k \\ u_0 & u_1 & u_2 & u_3 \\ v_0 & v_1 & v_2 & v_3 \\ w_0 & w_1 & w_2 & w_3 \end{vmatrix}.
\]
From this we have at once
\[
I \cdot tA_3 \cdot uvw = \begin{vmatrix} 1 & u_1 & v_2 & w_3 \end{vmatrix}.
\]
Also from the properties of determinant multiplication we have the important formula
\[
I \cdot A_3 \cdot abcA_3 \cdot uvw = \begin{vmatrix} I_{au} & I_{av} & I_{aw} \\ I_{bu} & I_{bv} & I_{bw} \\ I_{cu} & I_{cv} & I_{cw} \end{vmatrix}.
\]
We may also easily verify that
\[
A_3 \cdot bcA_3 \cdot uvw = \begin{vmatrix} I_{bu} & I_{bv} & I_{bw} \\ I_{cu} & I_{cv} & I_{cw} \end{vmatrix}.
\]
Hence we have at once
\[
I \cdot aA_3 bA_3 uvw = -IA_3 abcA_3 uvw.
\]

6. Some useful formulae are
\[
A_2 \cdot A_2 uvA_2 wx = -A_2 uvIwx + A_2uxIvw + A_2 vwxIu - A_2 vxIuw,
\]
\[
A_2 \cdot uA_3 vwx = -A_2 vwxIu - A_2 wxIuv - A_2 xvlIuw,
\]
\[
A_3 uvIw \cdot u' = uA_3 u' v' w',
\]
\[
A_3 \cdot A_2 uvA_2 vwxIu = T^2 A_3 uvw.
\]
7. For four quaternions we define the three functions

\[ I \cdot uvwx = IuA_1 vw = IuvIwx - IuwIxz + IuxIwv \]

\[ = \frac{1}{2} (uv' wx' + ux' vw' + vw' xu' + vu' xw') = \frac{1}{2} (S \cdot uv' wx' + S \cdot vw' xu') \]

\[ A_4 \cdot uvwx = -IuA_3 vw = \frac{1}{2} (uv' wx' - ux' vw' + xw' vu' - vw' xu') \]

\[ = \frac{1}{2} (S \cdot uv' wx' - S \cdot vw' xu') \]

\[ A_2 \cdot uvwx = \frac{1}{2} (uv' wx' - xw' vu') = V \cdot uv' wx'. \]

Evidently in \( Iuvwx \) we may permute the four quaternions cyclically, or may reverse the order. \( A_4 \) is an alternating function. In \( A_2 \) the order may be reversed by changing the sign. \( Iuvwx \) is a pfaffian, for it is the square root of a skew symmetric determinant of even order; indeed we have by actually squaring the expanded form,

\[ (Iuvwx)^2 = \begin{vmatrix} 0 & Iuv & Iuw & Iux \\ -Iuv & 0 & Iwv & Ixz \\ -Iuw & -Iwv & 0 & Iwx \\ -Ixu & -Ixv & -Ixz & 0 \end{vmatrix}. \]

8. If now we define a function \( A_6 \cdot uvwxy \) in the same manner as above it will be alternating and vanish, since any five quaternions are linearly connected. Indeed the vanishing of this expression gives us the important identity

\[ uA_4 uvwxy - vA_4 uvwxy + wA_4 uvwxy - xA_4 uvwxy + yA_4 uvwxy = 0. \]

Operating by \( I \cdot z(\cdot) \), where \( z \) is any quaternion, we find the important identity

\[ zA_4 uvwxy = -A_3 wxvIzw + A_3 vxyIwx - A_3 vwyIxz + A_3 vwzIyz. \]

We define further

\[ A_1 \cdot uvwxy = \frac{1}{3} (uv' wx' y + yx' uw' u), \]

\[ A_3 \cdot uvwxy = \frac{1}{3} (uv' wx' y - yx' uw' u). \]

Functions like these and the even numbered functions \( I, A_2, A_4 \) may be defined for any number of quaternions, and are useful in reductions.

As previously stated, all these forms are invariant or pseudo-invariant under the substitution of \( aqb \) for \( q \), where \( Ta = Tb = 1 \). This is evident by mere substitution in the definitions. For example,

\[ A_2 uv = a' A_2 (aub) (avb) \cdot a, \quad A_1 uvw = a^{-1} \cdot A_1 (aub) (avb) (awb) \cdot b^{-1}, \]

\[ A_3 \cdot uvw = a' \cdot A_3 (aub) (awb) \cdot b', \]

while \( I \cdot uv, I \cdot uvwx, A_4 \cdot uvwxy \) are invariant.

The functions \( A_3 uvw \) and \( A_4 uvwxy \) are invariant under any linear homogeneous substitution, in the sense that such a substitution merely multiplies the function by the determinant of the substitution.
9. Let us suppose that the coordinates of a point in space of four dimensions are $w, x, y, z$, corresponding to the real or scalar axis and the vector axes $i, j, k$. We then define the differentiating operator $* D$ as follows:

$$D = \frac{\partial}{\partial w} + i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z}.$$ 

In physical applications it would be desirable to define this operator directly in physical terms, and in geometry in geometrical terms, but for the present formal developments this definition seems simpler. It is obvious that if the quaternion $q$ becomes $q + dq$, where

$$dq = dw + idx + jdy + kdz,$$

then the operator

$$I \cdot dqD = dw \frac{\partial}{\partial w} + dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y} + dz \frac{\partial}{\partial z}$$

gives the differential change in $Q$, any function of the position $q$, due to the differential change $dq$. If $Q$ is a scalar function of position and its levels given by $Q = c$, then the change in $Q$ which is greatest will be normal to these levels and will be given by $DQ$ itself. Let

$$Q = W + iX + jY + kZ.$$ 

Then

$$I \cdot DQ = \frac{\partial W}{\partial w} + \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}.$$ 

We might call this the divergence of the function $Q$. Its vanishing gives an equation well known in hydrodynamics, the equation of continuity. In this case we take $Q = \rho (1 + \sigma)$ where $\sigma$ is the velocity, $\rho$ the density, $w$ the time.

If we combine $D$ with itself we have

$$I \cdot DD = \frac{\partial^2}{\partial w^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}, \quad I \cdot DD' = \frac{\partial^2}{\partial w^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}.$$ 

The first is the extension of the Laplacian to four dimensions and leads to what may be called four-dimensional potential functions. We have an example of the vanishing of these differential forms in the differential equation of wave motion, which if we set $w = at$ may be written

$$I \cdot DD \cdot u = 0, \quad \text{or} \quad I \cdot DD' \cdot u = 0,$$

according as the term containing the differential of the time is negative or positive.

*This is the operator lor of Minkowski, Mathematische Annalen, vol. 68 (1909–10), pp. 472–525.
The first partial derivatives of a scalar function $Q$ as to the coordinates, are the coordinates of the quaternions $1, i, j, k$ in $DQ$. The second partial derivatives would be the coefficients of the dyads in $DIDQ$. In fact, the scalar invariants of the linear quaternion differential operator $DI \cdot D ( )$, namely $M_1, M_2, M_3, M_4$, lead not only to the Laplacian, but to other forms of operators, all of which are invariant under orthogonal transformations of the axes $1, i, j, k$.

We have

$$
A_2 DQ = i \left( \frac{\partial W}{\partial x} - \frac{\partial X}{\partial w} + \frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y} \right) + j \left( \frac{\partial W}{\partial y} - \frac{\partial Y}{\partial w} + \frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z} \right) + k \left( \frac{\partial W}{\partial z} - \frac{\partial Z}{\partial w} + \frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x} \right),
$$

$$
A_2 D' Q' = i \left( -\frac{\partial W}{\partial x} + \frac{\partial X}{\partial w} - \frac{\partial Y}{\partial z} + \frac{\partial Z}{\partial y} \right) + j \left( -\frac{\partial W}{\partial y} + \frac{\partial Y}{\partial w} - \frac{\partial Z}{\partial x} + \frac{\partial X}{\partial z} \right) + k \left( -\frac{\partial W}{\partial z} + \frac{\partial Z}{\partial w} - \frac{\partial X}{\partial y} + \frac{\partial Y}{\partial x} \right).
$$

We may say that if $A_2 DQ = 0$, $Q$ is leftward irrotational, if $A_2 D' Q' = 0$, then $Q$ is rightward irrotational. The expressions $A_2$ we may speak of as left curl and right curl. If both vanish we may speak of $Q$ as irrotational. The rotations here are of course planar and not axial. As an example of the use of these forms, let $Q = \varphi - A$ be a quaternion potential function, $\varphi$ being a scalar potential, and $A$ a vector potential. Then

$$
A_2 DQ = \nabla \varphi + \partial A/\partial w + V \nabla A = -x + y,
$$

$$
A_2 D' Q' = -\nabla \varphi - \partial A/\partial w + V \nabla A = x + y.
$$

Thence

$$
2x = A_2 D' Q' - A_2 \cdot DQ, \quad 2y = A_2 D' Q' + A_2 DQ.
$$

From the identity $A_3 DQD = 0$ we have, taking scalars, $S \cdot VDVQVD = 0$, or

$$
S\nabla y = 0.
$$

Taking vectors we have

$$
V (A_2 DQ \cdot D + DA_2 D' Q') = 0, \quad VD' (-x + y) + VD (x + y) = 0,
$$

that is,

$$
V \nabla x + \frac{\partial y}{\partial w} = 0.
$$

Both equations are thus included in $A_3 DQD = 0$, an identity.

Setting $2DA_2 \cdot Q'D' = A_2 DQ \cdot D - DA_2 D' Q' = 2q$, we have, by taking scalars,

$$
S\nabla x = -S q.
$$

† These invariants are coefficients of the characteristic equation; cf. Joly, *Manual of Quaternions*. 
Taking vectors we have
\[ \nabla \nabla y + \frac{\partial x}{\partial w} = -\nabla q. \]

These two equations arise thus from a single equation.

Now if we set in these formulæ \( w = \sqrt{-1} \cdot ct \), and in the first place set \( x = -\sqrt{-1} \cdot E, y = H, S q = \sqrt{-1} \cdot \rho \), we have the common equations of the electro-magnetic field.* If we set in the next place \( x = -\sqrt{-1} \cdot E, y = B, \) or \( x = H, y = -\sqrt{-1} \cdot D \), respectively, we arrive at the vectors called by Minkowski † electric force of rest, and magnetic force of rest,

\[ R = \frac{1}{2} (A_2 D Q \cdot r - r A_2 D' Q'), \quad S = A_3 D Q r, \]

provided \( r \) is the velocity given by
\[ \sqrt{-1} \cdot r = (1 - \sqrt{-1} \cdot q) / \sqrt{(1 - q^2)}. \]

If we start with such equations as the familiar ones
\[ \nabla E - \frac{\partial E}{\partial t} = \rho \sigma, \quad S \nabla E = -\rho, \quad \nabla \nabla E + \frac{\partial H}{\partial t} = 0, \quad S \nabla H = 0, \]
we may reduce them to the present forms. We have first
\[ \nabla H - \frac{\partial E}{\partial t} = \rho \sigma, \quad -E \nabla + \frac{\partial H}{\partial t} = \rho. \]

Adding these, we have the single equivalent equation \( DH - ED = q \). If \( H = \nabla A \), and \( E = \nabla \varphi + \partial A / \partial t \) this reduces with little trouble to
\[ D' ID' Q - IDD' \cdot Q' = 2q. \]

Formulæ for the extensions of Stokes' theorem and Green's theorem will be found in the reference to Joly below.‡

When we operate on \( T q^{-2} \) or \( (qq')^{-1} \) the operator \( IaD \) will give the hyperspherical harmonics, and the results are susceptible of interpretations similar to those for three dimensions. The introduction of doublets follows analogous lines. These applications can only be mentioned.§

---

† Minkowski, Göttinger Nachrichten, 1908, pp. 53, 111. The imaginary is not really necessary.
10. We pass now to the development of certain differential formulae for three-dimensional curvilinear space, or what is the same thing, quaternions depending upon three parameters. This is equivalent to considering the three-dimensional space in a flat four-dimensional space, so that the problem becomes that of the differential geometry of curved three-spreads in four-dimensional space.\footnote{Thus, if \( q \) is to terminate on a hypersphere, we might write \( q = u_1 i + u_2 j + u_3 k + \sqrt{u_1^2 - u_2^2 - u_3^2} \).} Let us suppose, then, that

\[
q = f(u_1, u_2, u_3).
\]

This equation will limit \( q \) to a certain three spread. We shall write

\[
q_1 = \frac{\partial f}{\partial u_1}, \quad q_2 = \frac{\partial f}{\partial u_2}, \quad q_3 = \frac{\partial f}{\partial u_3}, \quad dq = q_1 du_1 + q_2 du_2 + q_3 du_3.
\]

The quaternions \( q_1, q_2, q_3 \) are independent, as the parameters are essential. Let the quaternion \( A_3 q_1 q_2 q_3 = n \). Then \( n \) is the normal at \( q \) to the three-flat tangent to the three-spread, since \( Inq_1 = 0, \ Inq_2 = 0, \ Inq_3 = 0 \). The differential \( dq \) lies in the tangent three-flat.

11. We shall indicate the quaternions

\[
\frac{-A_3 n q_2 q_3}{Inn}, \quad \frac{-A_3 n q_3 q_1}{Inn}, \quad \frac{-A_3 n q_1 q_2}{Inn}
\]

respectively by \( \bar{q}_1, \bar{q}_2, \) and \( \bar{q}_3 \). These give at once

\[
Iq_i \bar{q}_i = 1, \quad Iq_i \bar{q}_j = 0 \quad (i + j), \quad In \bar{q}_i = 0.
\]

Any quaternion in the three-dimensional space may be expanded in the forms

\[
r = q_1 \bar{q}_1 r + q_2 \bar{q}_2 r + q_3 \bar{q}_3 r = q_1 Iq_1 r + \bar{q}_2 Iq_2 r + \bar{q}_3 Iq_3 r.
\]

We now define the operator

\[
\Delta = \bar{q}_1 \frac{\partial}{\partial u_1} + \bar{q}_2 \frac{\partial}{\partial u_2} + \bar{q}_3 \frac{\partial}{\partial u_3}.
\]

We have by substitution of values the operator

\[
IdqA = du_1 \frac{\partial}{\partial u_1} + du_2 \frac{\partial}{\partial u_2} + du_3 \frac{\partial}{\partial u_3}.
\]

That is to say, the operator \( IaA \) gives the rate of variation of any operand due to a variation in the three-spread of \( q \) in the direction \( a \), where \( a \) is a unit quaternion. It is evident again that if \( Q \) is a point function defined for the three-spread, a scalar let us say at first, then it will have certain level surfaces. The quaternion \( \Delta Q \) gives us then the rate of maximum change, which will be the normal to a level surface. The operator \( \Delta \) is thus the extension to curved
space of the operator $\nabla$ for flat three-dimensional space. We proceed to show that it is invariant under transformations of the parameters. To do this let

$$u_1 = \varphi_1 (v_1, v_2, v_3), \quad u_2 = \varphi_2 (v_1, v_2, v_3), \quad u_3 = \varphi_3 (v_1, v_2, v_3),$$

where the functions $\varphi$ are differentiable and without singularities at the points considered. Then

$$A_2 q_2 q_3 = \sum A_2 q_1 v q_2 v \cdots \left( q_1 v = \frac{\partial q}{\partial \alpha}, \cdots \right).$$

Thence building up $\Delta$, we have $\Delta u = \Delta u$. Also we have

$$U_n u = UA_3 q_1 q_2 q_3 = UA_3 (q_1 v) (q_2 v) (q_3 v) = Un v.$$ 

The normal is thus an invariant under transformation of the parameters. It follows naturally that such linear operators as $I (\Delta) \cdot Un$, $I (\Delta) \cdot Ax$ and powers and combinations of these are also invariant. As an example, we see that the function $Ia \Delta \cdot Q$ gives the rate of variation of $Q$ at the point in the direction $a$. The axes of this function are the directions of extremal variation and the roots are the rates of such change, at least if the function is self-transverse. If we lay off in each direction the value of the function (when $Q$ is a scalar) the points so determined will have a hyper-quadric for their locus. The method of treatment differs little from the corresponding case for three dimensions.

Again consider the function $N = I (\Delta) \cdot Un$. This is evidently the rate of change of the unit normal at the point. From the character of $\Delta$ it is self-transverse. Hence the axes are orthogonal and are the directions of the curvatures (reciprocals of the radii) at the point. The roots are real and are the curvatures themselves for real quaternions $q_1, q_2, q_3$. One root is zero, the corresponding axis being the normal. The usual scalar invariants of $N$ are therefore the mean curvature, the second mean curvature, and the total or Kronecker-Gaussian curvature. The mean curvature (as in the similar three-dimensional case) is $I Un$. The second mean curvature and the total curvature are respectively

$$I \cdot A_3 \Delta A' A'' A_3^{-1} Un' Un'', \quad IA_3 \Delta' A'' A''' A_3 Un' Un'' Un''',$

where the accents are dropped after the expansion.

12. Since $\Delta u$, $\Delta v$, $\Delta w$, $\cdots$ are invariant, also $T\Delta u$, $T\Delta v$, $\cdots$ are invariant. Indeed it is evident that the differential parameters of the first order for the functions $u$, $v$, $\cdots$ are, in the usual notation for differential parameters,

$$\Delta_1 u = I \Delta u \Delta u, \quad \Delta_1 (u, v) = I \Delta u \Delta v.$$ 

In addition to these we have the following extensions of the Darboux $\Theta (u, v)$
for two dimensions to three dimensions:

\[ A_2 \Delta u \Delta v, \quad A_1 \Delta u \Delta v \Delta w, \quad A_3 \Delta u \Delta v \Delta w, \]

giving the respective parameters

\[ I A_2 \Delta u \Delta v A_2 \Delta u \Delta v, \quad I A_1 \Delta u \Delta v \Delta w A_1 \Delta u \Delta v \Delta w, \quad I A_3 \Delta u \Delta v \Delta w A_3 \Delta u \Delta v \Delta w, \]

\[ I \Delta u \Delta v \Delta w \Delta x, \quad I A_2 \Delta u \Delta v \Delta w \Delta x A_2 \Delta u \Delta v \Delta w \Delta x. \]

The form \( A_4 \Delta u \Delta v \Delta w \Delta x \) vanishes identically.

The formulae of part I give various syzygies connecting these, for example

\[ I \cdot \Delta u \Delta v \Delta w \Delta x = \Delta_1 (u, v) \Delta_1 (w, x) - \Delta_1 (u, w) \Delta_1 (v, x) + \Delta_1 (u, x) \Delta_1 (v, w). \]

Since \( \Delta u \) is the normal in the three-spread for the levels of the scalar function \( u \), the parameter of first order of \( u, \Delta_1 u \), is the square of the length of this normal. The differential parameter of first order of \( u \) and \( v \) is the projection of the normal to the level of either, on the other, multiplied by the length of the latter. If it vanishes the levels are orthogonal. The interpretation of the other forms follows the exactly analogous forms for \( \nabla \) in a flat three-dimensional space.

13. Any parameter of the first order is itself subject to operation and we arrive thus at compound forms such as \( I \Delta u \Delta (\Delta_1 u) \). We are led thus to consider operators such as \( \Phi = I (\ ) \Delta \cdot \Delta u \). The first scalar invariant of this linear quaternion operator is \( I \Delta \Delta u \), which is the well-known differential parameter \( \Delta_2 u \). It vanishes if \( u \) is a solution of the generalized Laplacian equation. Indeed if the space becomes flat and the position is determined by \( \rho \) instead of \( q \), the second differential parameter becomes \( -\nabla^2 \). If the lines \( u_1, u_2, u_3 \) are isothermal, we have a simplification which is not difficult to follow. The second invariant of \( \Phi \) is

\[ I A_3 n \Delta' \Delta'' A_2 n^{-1} (\Delta u)' (\Delta u)'''. \]

This can be expanded in the usual manner for the coefficients of the characteristic equation of a matrix or linear quaternion operator, in the form

\[ \frac{1}{2} (I \Delta \Delta u)^2 - I \Delta' \Delta'' u I \Delta' \Delta'' u. \]

It is the well-known parameter \( \Delta_{22} u \). Finally the third invariant is

\[ -\frac{1}{6} I A_3 \Delta' \Delta'' \Delta''' A_3 (\Delta u)' (\Delta u)'' (\Delta u)'''. \]

This might be called the parameter \( \Delta_{222} u \). The fourth invariant vanishes identically.

The function \( \Phi \) is the rate of change in any direction of the vector rate of change of the function which is itself the normal to a level of the function.
14. If we consider only the direction of the normal of the level, we have the function \( Z = I (\mathbf{A} \cdot \nabla \Delta u) \), which gives the rate of deviation of the normal direction. There are now only two invariants that do not vanish identically, and these determine what may be called geodetic curvatures. The first invariant is \( IA_3 n \Delta' A_3 n^{-1} U \Delta u' U \Delta u'' \).

These are the extensions of the usual geodetic curvature for two dimensions.

15. We notice next the fundamental forms or covariants of different orders. The first is

\[
Idqdq = \Sigma Iq_i q_j du_i du_j + 2Iq_1 q_2 du_1 du_2.
\]

The coefficients \( Iq_i q_j \) correspond to the \( E, F, G \) of the geometry of surfaces. They are expressible as parameters, for example,

\[
Iq_1 q_2 = IA_3 n \Delta u_2 \Delta u_3 A_3 n \Delta u_3 \Delta u_1.
\]

Other covariant forms are

\[
Idgdr, A_2 dqdr, A_1 dqdrds, A_3 dqdrds, Idqdrdsdt, A_2 dqdrdsdt,
\]

\[
A_2 ndq, A_1 ndqdr, A_3 ndqdrds, A_2 ndqdrds, A_4 ndqdrds = 0.
\]

Denoting differentiation as to \( u_i \) by the subscript \( i \) we have, from \( IUnUn = 1 \),

\[
IUn (Un)_i = 0,
\]

\[
IUn (Un)_ij = - I (Un)_i (Un)_j \quad (i, j = 1, 2, 3).
\]

Since \( Inq_i = 0 \), therefore

\[
Inq_{ij} = - Inq_j = - Inq_i.
\]

In this we might also have written \( Un \) instead of \( n \). We arrive thus at the extensions of \( \ldots \) of \( D, D', D'' \), of two dimensions. The Christoffel symbols are

\[
\left[ \begin{array}{c}
kl \\
\end{array} \right]_i = Iq_i q_k q_l, \\
\left[ \begin{array}{c}
kl \\
\end{array} \right]_i = Iq_k \Delta u_i = Iq_k q_l.
\]

\(
(ikrs) = IA_3 n^{-1} Un Un_k A_3 n q_s q_t.
\)

Differentiating these and remembering that also \( \Delta \) is subject to differentiation, we arrive at the symbols of higher order. This is the covariantive differentiation of Maschke.*

The second fundamental form is

\[
IdqdUn = IdqNq = \Sigma Iq_1 (Un)_i du_i du_1 + 2Iq_i (Un)_j du_i du_j.
\]

* Maschke, loc. cit. Also these Transactions, vol. 4 (1903), pp. 457.
This form vanishes for the asymptotic directions, or the principal tangent lines of the three-spread. It is worth noting that the differential equation of the lines of curvature is \( A_2 d\varphi N d\psi = 0 \). This corresponds to the similar form for surfaces \( V d\varphi d\psi d\rho = 0 \).

16. The symbolic invariants of Maschke are easily expressed in these forms by noticing the following equivalences* for his symbols

\[
\begin{align*}
fi &= I\xi q_i, \\
fi f_k &= I\xi q_i I\xi q_k = Iq_i q_k, \\
f_{ki} &= I\xi q_{ki}.
\end{align*}
\]

whence

\[
(f) = A_4 U n_1 \xi_1 \xi_2 \xi_3,
\]

\[
(f)^2 = 3! U n U n = 3!, \\
(uf) = A_4 n\Delta u_1 \xi_2.
\]

whence

\[
(fu) = A_4 U n \Delta u' \Delta u'' \xi_1,
\]

\[
(fu)^2 = 2! U n U n = 2\Delta_1 u.
\]

\[
((uf), f) = A_4 U n \Delta (A_4 U n U n_1 \xi_2) \xi_1 \xi_2 = 2U n U n I \Delta u = 2\Delta_2 u.
\]

Not to elaborate too far, we need only to remember that each ( ) is an \( A_4 \), that \( u, v, w, x, \) etc., are functions of \( q \) such as we have been using, that for every \( f \) or like symbol we substitute \( I\xi q \), for subscripts we differentiate as to the corresponding \( u \). All the Maschke symbols for four dimensions become at once interpretable in this system. The vectors of Ingold † are the quaternions appearing in these forms, his formulæ being analogues of Maschke's, not equivalents. The ordinary formulæ hold of course for these quaternions, but the Maschke forms are all scalars and commutative, and are the scalar forms with the \( t \)-pairs introduced.

---

*The bilinear function \( Q(\xi, \tau) \) is an abbreviation for \( Q(q_1, \bar{q}_1) + Q(q_2, \bar{q}_2) + Q(q_3, \bar{q}_3) \). We may always write for a bilinear function \( Q(\xi, \tau) \) the form \( Q(\Delta, q) \) or equally \( Q(q, \Delta) \). See McAulay, "Utility of Quaternions in Physics"; Shaw, Synopsis of Linear Associative Algebra.

†These Transactions, vol. 11 (1910), pp. 449-474.