ON THE PSEUDO-RESOLVENT TO THE KERNEL OF AN INTEGRAL EQUATION*

BY

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The method developed by FREDHOLM† for the solution of a non-homogeneous linear integral equation is comparatively simple in case the corresponding homogeneous equation admits no solutions not identically zero; in terms of the determinant and the first minor for the kernel of the given equation a resolvent function is defined, by means of which the solution is readily found. If, on the other hand, the homogeneous equation admits non-zero solutions, the development of necessary and sufficient conditions for the existence of solutions of the non-homogeneous equation by FREDHOLM'S method, while similar in character to the simpler case, involves the introduction of minors of higher order, thereby necessitating somewhat tedious and complicated algebraic manipulation. Other methods of approach have been devised in this case—for example, SCHMIDT'S detour through the easier theories of equations with kernels which are symmetric, which remain small in absolute value, or which consist of the sum of a finite number of terms of the form $\varphi(x)\psi(y)$, to that of the equation with general unsymmetric kernel.‡ Such methods, although they present many points of interest in themselves, involve a considerable amount of work before arriving at the few essential facts regarding the solution of the integral equation. It is the purpose of the present paper to furnish a scheme for obtaining these essential facts rapidly and directly, without the introduction of any new concepts beyond those involved in the simpler case; by an easy artifice, the solution of a given integral equation is reduced to the solution of another, for whose kernel a resolvent function exists;

* Read before the American Mathematical Society, September 12, 1911.

† Acta Mathematica, vol. 27 (1903), p. 365.

‡ Mathematische Annalen, vol. 63 (1907), p. 459; vol. 64 (1907), p. 161. A recent paper by T. LALESCO, Bulletin de la Société Mathématique de France, vol. 39 (1911), p. 85, is also concerned with the problem here presented; the method is more complicated than that here used and the steps of the work, as well as the final result, are quite different. On the other hand, LALESCO obtains valuable information regarding the behavior of the resolvent in the neighborhood of a pole, which is not obtainable by the procedure here suggested.

a *pseudo-resolvent* is thus defined for the kernel of the given equation, and the theory is readily established. It should be noted that the facts are given completely by Theorems I, II, and III; the remainder of the paper is concerned with various subsidiary considerations.

Throughout, the known functions of one variable are supposed to be given, and the unknown to be sought, in the interval $a \leqq x \leqq b$; the kernel $K(x, y)$ is supposed given in the region $a \leqq x \leqq b$, $a \leqq y \leqq b$; these premises will not be repeated in the statement of the theorems. The treatment is restricted to the case that all functions involved are continuous.

From the elementary part of the FREDHOLM theory* we assume the following

LEMMA I. *If the kernel $K(x, y)$ is continuous, and if either of the equations*

$$\varphi(x) = \int_a^b K(x, s)\, \varphi(s)\, ds, \tag{1}$$

$$\psi(x) = \int_a^b \psi(s)\, K(s, x)\, ds, \tag{1'}$$

has no continuous solutions not identically zero, then the other has no continuous solutions not identically zero; there exists a continuous function $k(x, y)$, the resolvent to the given kernel $K(x, y)$, such that

$$k(x, y) = K(x, y) + \int_a^b K(x, s)\, k(s, y)\, ds,$$

$$k(x, y) = K(x, y) + \int_a^b k(x, s)\, K(s, y)\, ds;$$

if $f(x)$ is continuous, the equation

$$u(x) = f(x) + \int_a^b K(x, s)\, u(s)\, ds, \tag{2}$$

has one and only one continuous solution, given by the formula:

$$u(x) = f(x) + \int_a^b k(x, s) f(s)\, ds.$$

We assume also the properties of normal orthogonal sets of continuous functions,† and in particular the following:

LEMMA II. *If $u_1(x), u_2(x), \cdots, u_n(x)$ are linearly independent continuous functions, then there exists a normal orthogonal set of continuous functions $v_1(x), v_2(x), \cdots, v_n(x)$, such that any function of either set is linearly dependent on the functions of the other set.*

We proceed to the proof of the theorems giving the desired facts.

* Cf., for instance, BÔCHER, *An Introduction to the Study of Integral Equations*, p. 29.

† BÔCHER, loc. cit., p. 52.

THEOREM I. *If the kernel $K(x, y)$ is continuous, the equations* (1), (1′) *have each a finite number of linearly independent continuous solutions, on which every other continuous solution is linearly dependent; and this number is the same for the two equations.*

The proof of the first part of the theorem by direct methods is well known,* and will not be repeated here; it is only the direct proof of the equality of the numbers of solutions which is new. Let the sets of solutions of (1), (1′) be replaced, according to Lemma II, by equivalent normal orthogonal sets $\varphi_1(x), \varphi_2(x), \cdots \varphi_n(x); \psi_1(x), \psi_2(x), \cdots, \psi_m(x)$. If their numbers are unequal, let $n < m$. Define

$$L(x, y) = K(x, y) - \sum_{i=1}^{n} \psi_i(x)\, \varphi_i(y). \tag{3}$$

It will be shown that of the two homogeneous equations

$$u(x) = \int_a^b L(x, s)\, u(s)\, ds, \tag{4}$$

$$v(x) = \int_a^b v(s)\, L(s, x)\, ds, \tag{4'}$$

one possesses a continuous solution not identically zero, while the other does not; so that Lemma I is contradicted.

On replacing $L(x, s)$ by its defining formula, we see that any continuous solution of (4) must satisfy the equation

$$u(x) = -\sum_{i=1}^{n} \psi_i(x) \int_a^b \varphi_i(s)\, u(s)\, ds + \int_a^b K(x, s)\, u(s)\, ds; \tag{5}$$

if we multiply this equation by $\psi_j(x)$ $[j = 1, 2, \cdots, n]$, and integrate with respect to x between the limits a and b, we find, on account of the normal orthogonality of the set $\psi_1(x), \psi_2(x), \cdots, \psi_n(x)$,

$$\int_a^b \varphi_j(s)\, u(s)\, ds = 0 \qquad [j = 1, 2, \cdots, n]; \tag{6}$$

hence $u(x)$ must satisfy

$$u(x) = \int_a^b K(x, s)\, u(s)\, ds, \tag{7}$$

so that

$$u(x) = \sum_{i=1}^{n} c_i\, \varphi_i(x); \tag{8}$$

but again in view of (6),

$$c_i = 0 \qquad [i = 1, 2, \cdots, n],$$

* BÔCHER, loc. cit., p. 56.

and therefore

$$u(x) = 0,$$

as was asserted.

On the other hand, at least one solution of (4′) not identically zero can be found, namely $v(x) = \psi_{n+1}(x)$; for

$$\psi_{n+1}(x) - \int_a^b \psi_{n+1}(s) L(s, x)\, ds = \psi_{n+1}(x) - \int_a^b \psi_{n+1}(s) K(s, x)\, ds$$
$$+ \sum_{i=1}^{n} \left[\int_a^b \psi_{n+1}(s) \psi_i(s)\, ds \right] \varphi_i(x) = 0.$$

Thus the assumption that $n < m$ leads to a contradiction of Lemma I, and is therefore untenable. A similar proof holds for the hypothesis $n > m$, so that the theorem is proved.

THEOREM II. *If the complete sets of normal orthogonal solutions of* (1), (1′), *are respectively* $\varphi_1(x), \varphi_2(x), \cdots, \varphi_n(x)$; $\psi_1(x), \psi_2(x), \cdots, \psi_n(x)$, *there exists a* continuous function* $k(x, y)$, *the pseudo-resolvent to the given kernel* $K(x, y)$, *such that*

$$(9) \quad k(x, y) = K(x, y) + \int_a^b K(x, s) k(s, y)\, ds - \sum_{i=1}^{n} \psi_i(x) \psi_i(y),$$

$$(9') \quad k(x, y) = K(x, y) + \int_a^b k(x, s) K(s, y)\, ds - \sum_{i=1}^{n} \varphi_i(x) \varphi_i(y).$$

To prove this, we carry over from the proof of Theorem I the definition (3) of $L(x, y)$ and the property that the equation (4) has no continuous solution not identically zero. Let us construct the *resolvent* $k(x, y)$ to the kernel $L(x, y)$; its existence is assured by Lemma I, which also furnishes the identities:

$$(10) \qquad k(x, y) = L(x, y) + \int_a^b L(x, s) k(s, y)\, ds,$$

$$(10') \qquad k(x, y) = L(x, y) + \int_a^b k(x, s) L(s, y)\, ds.$$

The first of these identities may be rewritten in the form:

$$(11) \quad \begin{aligned} k(x, y) = K(x, y) - \sum_{i=1}^{n} \psi_i(x) \varphi_i(y) + \int_a^b K(x, s) k(s, y)\, ds \\ - \sum_{i=1}^{n} \psi_i(x) \int_a^b \varphi_i(s) k(s, y)\, ds; \end{aligned}$$

* In fact, an infinite number of such functions; their relations to one another are mentioned later.

on multiplying by $\psi_j(x)$ $[j = 1, 2, \cdots, n]$, and integrating with respect to x from a to b, we obtain, after easy simplifications,

$$\int_a^b \varphi_j(s)\, k(s, y)\, ds = \psi_j(y) - \varphi_j(y) \quad [j = 1, 2, \cdots, n];$$

and the substitution of this result in (11) yields at once (9), the first of the formulæ to be proved. The second formula may be verified in like manner.

THEOREM III. *A necessary and sufficient condition that the equation*

$$u(x) = f(x) + \int_a^b K(x, s)\, u(s)\, ds \tag{2}$$

(where $f(x)$ is continuous) have a continuous solution, is that

$$\int_a^b \psi_i(x) f(x)\, dx = 0 \qquad [i = 1, 2, \cdots, n]. \tag{12}$$

If this condition is fulfilled, any continuous solution may be written in the form

$$u(x) = f(x) + \int_a^b k(x, s) f(s)\, ds + \sum_{i=1}^n c_i\, \varphi_i(x), \tag{13}$$

where $c_1, c_2, \cdots, c_n$ are constants.

The theorem may be paraphrased as follows: if (2) has a continuous solution, it must be given by (13); the continuous function defined by (13) is a solution of (2) when and only when conditions (12) are fulfilled. The proof of these two assertions is conducted by the use of the pseudo-resolvent and the identities (9), (9′), in exactly the same fashion as the proof of the corresponding part of Lemma I is ordinarily given by use of the resolvent and its characteristic properties. If we multiply (2) by $k(z, x)$ and integrate with respect to x, we infer, by use of (9′), that

$$\int_a^b K(z, s)\, u(s)\, ds = \int_a^b k(z, x) f(x)\, dx + \sum_{i=1}^n \varphi_i(z) \int_a^b \varphi_i(s)\, u(s)\, ds;$$

the direct substitution of this result in (2) leads to:

$$u(x) = f(x) + \int_a^b k(x, s) f(s)\, ds + \sum_{i=1}^n \varphi_i(x) \int_a^b \varphi_i(s)\, u(s)\, ds,$$

which is of the form (13), with

$$c_i = \int_a^b \varphi_i(s)\, u(s)\, ds \qquad [i = 1, 2, \cdots, n].$$

On the other hand, suppose that (13) is satisfied; a similar process of multiplication by $K(z, x)$, integration with respect to x, use of (9), and substitution in (13), leads to the result:

$$u(x) = f(x) + \int_a^b K(x, s)\, u(s)\, ds - \sum_{i=1}^{n} \psi_i(x) \int_a^b \psi_i(s) f(s)\, ds.$$

Thus the function $u(x)$ defined by (13) will actually be a solution of (2) when and, on account of the linear independence of $\psi_1(x), \psi_2(x), \cdots, \psi_n(x)$, only when conditions (12) are satisfied. Our theorem is thus proved.

It is evident that the function $k(x, y)$ defined in the course of the proof of Theorem II is not uniquely defined by the relations (9), (9′); these are satisfied by every function of the form

$$k(x, y) + \sum_{i=1, j=1}^{n, n} c_{i,j}\, \varphi_i(x)\, \psi_j(y),$$

(where $c_{i,j}$ are constants) and by no other continuous functions. The particular function which arose in the proof may be completely specified, if this is desired, by demanding, in addition to (9) and (9′) the validity of one of the relations

$$\int_a^b \varphi_i(s)\, k(s, x)\, ds = \psi_i(x) - \varphi_i(x) \quad [i = 1, 2, \cdots, n].$$

$$\int_a^b k(x, s)\, \psi_i(s)\, ds = \varphi_i(x) - \psi_i(x) \quad [i = 1, 2, \cdots, n].$$

It is natural to ask whether functions of still more general character could serve the same purpose—functions which do not necessarily satisfy (9), (9′), but which nevertheless suffice to give the solutions of (2) in the form (13) under conditions (12). *Any continuous function $l(x, y)$ will be called a pseudo-resolvent to the kernel $K(x, y)$ if for every continuous function $f(x)$ satisfying conditions* (12), *every continuous solution of* (2) *is expressible in the form*:

$$u(x) = f(x) + \int_a^b l(x, s) f(s)\, ds + \sum_{i=1}^{n} c_i\, \varphi_i(x) \tag{14}$$

and every function of the form (14) *is a solution of* (2). We then have

THEOREM IV. *In order that the continuous function $l(x, y)$ be a pseudo-resolvent, it is necessary and sufficient that*

$$(15)\quad l(x, y) = K(x, y) + \int_a^b K(x, s)\, l(s, y)\, ds - \sum_{i=1}^{n} \Psi_i(x)\psi_i(y);$$

$$(15')\quad l(x, y) = K(x, y) + \int_a^b l(x, s)\, K(s, y)\, ds - \sum_{i=1}^{n} \varphi_i(x)\Phi_i(y);$$

where $\Phi_1(x), \Phi_2(x), \cdots, \Phi_n(x)$; $\Psi_1(x), \Psi_2(x), \cdots, \Psi_n(x)$ are continuous functions such that

$$(16)\qquad \int_a^b \varphi_i(x)\Phi_j(x)\, dx = \begin{Bmatrix} 0, & i \neq j \\ 1, & i = j \end{Bmatrix};$$

$$(16')\qquad \int_a^b \Psi_i(x)\psi_j(x)\, dx = \begin{Bmatrix} 0, & i \neq j \\ 1, & i = j \end{Bmatrix} \quad [i, j = 1, 2, \cdots, n].$$

By way of proof, we assume that $l(x, y)$ is a pseudo-resolvent and write

$$Q(x, y) = l(x, y) - K(x, y) - \int_a^b K(x, s)\, l(s, y)\, ds.$$

It is clear that

$$\int_a^b \psi_i(s)\, Q(s, x)\, ds = -\psi_i(x) \qquad [i = 1, 2, \cdots, n],$$

and that if we define

$$\Psi_i(x) = -\int_a^b Q(x, s)\, \psi_i(s)\, ds \qquad [i = 1, 2, \cdots, n],$$

the relations (16′) are satisfied. Now by our definition of a pseudo-resolvent the function

$$u(x) = f(x) + \int_a^b l(x, s) f(s)\, ds$$

must verify the equation

$$u(x) = f(x) + \int_a^b K(x, s)\, u(s)\, ds$$

for every $f(x)$ satisfying (12). Carrying out the process of substitution in the equation, we obtain, after trifling reductions, the condition

$$\int_a^b Q(x, s) f(s)\, ds = 0,$$

valid for every such function $f(x)$. But such a function is given by the formula

$$f(s) = Q(x, s) + \sum_{i=1}^{n} \Psi_i(x)\psi_i(s)$$

for any value of x; hence we have the identity

$$\int_a^b Q(x, s)\left[Q(x, s) + \sum_{i=1}^{n} \Psi_i(x)\psi_i(s)\right] ds = 0.$$

As however evidently

$$\int_a^b \psi_j(s)\left[Q(x, s) + \sum_{i=1}^{n} \Psi_i(x)\psi_i(s)\right] ds = 0 \quad [j = 1, 2, \cdots, n],$$

we have on multiplying the latter equations by $\Psi_j(x)$ and adding to the preceding

$$\int_a^b \left[Q(x, s) + \sum_{i=1}^{n} \Psi_i(x)\psi_i(s)\right]^2 ds = 0,$$

so that

$$Q(x, s) = -\sum_{i=1}^{n} \Psi_i(x)\psi_i(s),$$

and (15) is demonstrated.

The same plan cannot be used in the proof of (15′);* we shall deduce this formula from (15), making use of the properties of the simpler pseudo-resolvent $k(x, y)$. Operating with $k(x, y)$ in the usual manner on (15) as if it were an equation to be solved for an unknown function $l(x, y)$, we deduce that

$$l(x, y) = K(x, y) - \sum_{i=1}^{n} \Psi_i(x)\psi_i(y) + \int_a^b k(x, s)K(s, y)\,ds$$

$$- \sum_{i=1}^{n}\left[\int_a^b k(x, s)\Psi_i(s)\,ds\right]\psi_i(y) + \sum_{i=1}^{n} \varphi_i(x)\int_a^b \varphi_i(s)l(s, y)ds,$$

or, by (9′),

$$l(x, y) = k(x, y) + \sum_{i=1}^{n} \varphi_i(x)\left[\varphi_i(y) + \int_a^b \varphi_i(s)\,l(s, y)\,ds\right]$$

$$- \sum_{i=1}^{n}\left[\Psi_i(x) + \int_a^b k(x, s)\Psi_i(s)\,ds\right]\psi_i(y).$$

* The same proof *would* of course apply if we assumed as part of the definition of a pseudo-resolvent that its connection with the kernel remains unaltered under interchange of the two variables. From the standpoint of the logic of mathematics some interest attaches to the fact that it is unnecessary to make this assumption.

If we multiply this equation by $K(y, z)$ and integrate with respect to y between the limits a and b, making use again of (9′), we find that

$$\int_a^b l(x, y) K(y, z)\, dy = k(x, z) - K(x, z) + \sum_{i=1}^{n} \varphi_i(x) \left[\varphi_i(z) + \int_a^b \varphi_i(y) K(y, z)\, dy + \int_a^b \int_a^b \varphi_i(s) l(s, y) K(y, z)\, dy\, ds \right] - \sum_{i=1}^{n} \left[\Psi_i(y) + \int_a^b k(y, s) \Psi_i(s)\, ds \right] \psi_i(z),$$

so that

$$l(x, z) - K(x, z) - \int_a^b l(x, y) K(y, z)\, dy = \sum_{i=1}^{n} \varphi_i(x) \left[\int_a^b \varphi_i(s) l(s, z)\, ds - \int_a^b \varphi_i(s) K(s, z)\, ds - \int_a^b \int_a^b \varphi_i(s) l(s, y) K(y, z)\, dy\, ds \right].$$

This is of the form (15′), provided we set

$$-\Phi_i(z) = \int_a^b \varphi_i(s) l(s, z)\, ds - \int_a^b \varphi_i(s) K(s, z)\, ds - \int_a^b \int_a^b \varphi_i(s) l(s, y) K(y, z)\, dy\, ds \qquad [i = 1, 2, \cdots, n]$$

and it is evident that with this choice of $\Phi_i(z)$, the restrictions (16′) are also satisfied.

We have thus shown the necessity of the conditions stated for a pseudo-resolvent; that they are sufficient may be verified at once by following out exactly the lines of the proof of Theorem III.

To complete the theory of these more general pseudo-resolvents, we add a sort of partial converse of Theorem IV:

THEOREM V. *For any set of continuous functions* $\Phi_1(x), \Phi_2(x), \cdots, \Phi_n(x); \Psi_1(x), \Psi_2(x), \cdots, \Psi_n(x)$, *satisfying* (16), (16′), *there exist pseudo-resolvents* $l(x, y)$ *satisfying* (15), (15′).

To construct them we have only to solve (15), (15′) for $l(x, y)$ by the use of any special pseudo-resolvent (such as $k(x, y)$ of Theorems II and III),

and to pick out a solution common to the two. Such a solution is

$$l(x, y) = k(x, y) - \sum_{i=1}^{n} \varphi_i(x) \left[\Phi_i(y) + \int_a^b \Phi_i(s) k(s, y) ds \right]$$
$$- \sum_{i=1}^{n} \left[\Psi_i(x) + \int_a^b k(x, s) \Psi_i(s) ds \right] \psi_i(y);$$

the general solution is obtained by adding to this special one:

$$\sum_{i=1, j=1}^{n, n} c_{i, j} \varphi_i(x) \psi_j(y),$$

where $c_{i, j}$ are constants.

The question as to the most general possible pseudo-resolvent is thus completely answered by Theorems IV and V. The pseudo-resolvent $l(x, y)$ reduces to the simpler one $k(x, y)$ in case we choose

$$\Phi_i(x) = \varphi_i(x), \tag{17}$$

$$\Psi_i(x) = \psi_i(x) \qquad [i = 1, 2, \cdots, n]. \tag{17'}$$

A case in which the second kind of pseudo-resolvent differs from the first is obtained on replacing (17) by

$$\Phi_i(x) = \int_a^b \varphi_i(s) K(s, x) ds \qquad [i = 1, 2, \cdots, n]$$

or (17′) by

$$\Psi_i(x) = \int_a^b K(x, s) \psi_i(s) ds \qquad [i = 1, 2, \cdots, n]$$

or both. Out of the large number of possible pseudo-resolvents, that of theorem II especially recommends itself both by the symmetry of its characteristic formulæ (9), (9′), and by the ease with which its existence can be proved directly.

It is of interest to investigate from the present standpoint the pseudo-resolvent which presents itself in Fredholm's treatment;* it must of course fall under the case of Theorem IV for some choice of $\Phi_1, \Phi_2, \cdots, \Phi_n$; $\Psi_1, \Psi_2, \cdots, \Psi_n$. If we remove from a statement of the properties of the Fredholm pseudo-resolvent all reference to the successive minors, in terms of which it is derived, we obtain the following essential facts:† *There exist n points*

* FREDHOLM, loc. cit., p. 374. See also the very clear and succinct presentation by HORN, Einführung in die Theorie der partiellen Differentialgleichungen p. 199.

† HORN, loc. cit., p. 209.

$x_1, x_2, \cdots, x_n$,* *and* n *continuous solutions* $\bar{\varphi}_1(x), \bar{\varphi}_2(x), \cdots, \bar{\varphi}_n(x)$ *of* (1), *such that*

$$\bar{\varphi}_i(x_j) = \begin{Bmatrix} 0, & i \neq j \\ 1, & i = j \end{Bmatrix} \qquad [i, j = 1, 2, \cdots, n]; \tag{18}$$

and there exist n *points* $y_1, y_2, \cdots, y_n$,* and n *continuous solutions* $\bar{\psi}_1(x)$, $\bar{\psi}_2(x), \cdots, \bar{\psi}_n(x)$ *of* (1′), *such that*

$$\psi_i(y_j) = \begin{Bmatrix} 0, & i \neq j \\ 1, & i = j \end{Bmatrix} \qquad [i, j = 1, 2, \cdots, n]. \tag{18'}$$

The Fredholm pseudo-resolvent $l(x, y)$ *satisfies the relations*:

$$l(x, y) = K(x, y) + \int_a^b K(x, s) l(s, y)\, ds - \sum_{i=1}^{n} K(x, y_i) \bar{\psi}_i(y), \tag{19}$$

$$l(x, y) = K(x, y) + \int_a^b l(x, s) K(s, y)\, ds - \sum_{i=1}^{n} \bar{\varphi}_i(x) K(x_i, y). \tag{19'}$$

In order to bring this into connection with our former work, we make first the following observation: the necessary and sufficient condition on the points $x_1, x_2, \cdots, x_n$ for the existence of the solutions $\bar{\varphi}_1(x), \bar{\varphi}_2(x), \cdots, \bar{\varphi}_n(x)$ is that

$$\Delta = \begin{vmatrix} \varphi_1(x_1) & \varphi_2(x_1) & \cdots & \varphi_n(x_1) \\ \varphi_1(x_2) & \varphi_2(x_2) & \cdots & \varphi_n(x_2) \\ \cdot & \cdot & \cdot & \cdot \\ \varphi_1(x_n) & \varphi_2(x_n) & \cdots & \varphi_n(x_n) \end{vmatrix} \neq 0. \tag{20}$$

That this condition is necessary appears from the fact that if $\bar{\varphi}_1(x), \bar{\varphi}_2(x), \cdots, \bar{\varphi}_n(x)$ have the properties (18), they must then be connected with $\varphi_1(x), \varphi_2(x), \cdots, \varphi_n(x)$ by the relations

$$\begin{aligned} \varphi_1(x) &= \varphi_1(x_1)\bar{\varphi}_1(x) + \varphi_1(x_2)\varphi_2(x) + \cdots + \varphi_1(x_n)\bar{\varphi}_n(x), \\ \varphi_2(x) &= \varphi_2(x_1)\bar{\varphi}_1(x) + \varphi_2(x_2)\bar{\varphi}_2(x) + \cdots + \varphi_2(x_n)\bar{\varphi}_n(x), \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \varphi_n(x) &= \varphi_n(x_1)\bar{\varphi}_1(x) + \varphi_n(x_2)\bar{\varphi}_2(x) + \cdots + \varphi_n(x_n)\bar{\varphi}_n(x), \end{aligned} \tag{21}$$

* The sets of points $x_1, x_2, \cdots, x_n$; $y_1, y_2, \cdots, y_n$ may be chosen in a great variety of ways; indeed they are subject only to the condition of yielding a value different from zero on being substituted for the $2n$ arguments of the earliest minor which does not vanish identically; there is thus a correspondingly great freedom of choice in the selection of the FREDHOLM pseudo-resolvent.

and the vanishing of Δ would contradict the linear independence of $\varphi_1(x)$, $\varphi_2(x), \cdots, \varphi_n(x)$. On the other hand, the condition is sufficient, the values of $\bar{\varphi}_1(x), \bar{\varphi}_2(x), \cdots, \bar{\varphi}_n(x)$ being given by the solution of the equations (21).

In the next place, there will certainly exist some set of points $x_1, x_2, \cdots, x_n$ rendering $\Delta \neq 0$; for the identical vanishing of Δ for all choices of $x_1, x_2, \cdots, x_n$ would involve the linear dependence of $\varphi_1(x), \varphi_2(x), \cdots, \varphi_n(x)$.*

If now (19′) and (15′) are to be in agreement, we must have

$$\sum_{i=1}^{n} \varphi_i(x)\Phi_i(y) = \sum_{i=1}^{n} \bar{\varphi}_i(x) K(x_i, y); \tag{22}$$

substituting here for $\varphi_1(x), \varphi_2(x), \cdots, \varphi_n(x)$ the values given by (21), we may write this in the form

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \varphi_i(x_j)\bar{\varphi}_j(x)\Phi_i(y) = \sum_{i=1}^{n} \bar{\varphi}_i(x) K(x_i, y),$$

or, giving to x the special value x_p $[p = 1, 2, \cdots, n]$,

$$\sum_{i=1}^{n} \varphi_i(x_p)\Phi_i(y) = K(x_p, y) \qquad [p = 1, 2, \cdots, n].$$

These equations suffice to determine $\Phi_i(y)$ as being equal to $1/\Delta$ multiplied into the determinant obtained on replacing the ith column of Δ by $K(x_1, y)$, $K(x_2, y), \cdots, K(x_n, y)$. The functions $\Phi_i(y)$ thus obtained do actually satisfy (22), thus identifying (19′) with (15′); they give these relations the form:†

$$l(x, y) = K(x, y) + \int_a^b l(x, s) K(s, y)\, ds$$

$$+ \frac{1}{\Delta}\begin{vmatrix} 0 & \varphi_1(x) & \varphi_2(x) & \cdots & \varphi_n(x) \\ K(x_1, y) & \varphi_1(x_1) & \varphi_2(x_1) & \cdots & \varphi_n(x_1) \\ K(x_2, y) & \varphi_1(x_2) & \varphi_2(x_2) & \cdots & \varphi_n(x_2) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ K(x_n, y) & \varphi_1(x_n) & \varphi_2(x_n) & \cdots & \varphi_n(x_n) \end{vmatrix};$$

finally, without any further restrictions, they satisfy (16). To exemplify

* This theorem is stated incidentally in a note by Richardson and Hurwitz, Bulletin of the American Mathematical Society, vol. 16 (1909–10), p. 18. It is stated explicitly and proved by Curtiss, Bulletin of the American Mathematical Society, vol. 17 (1910–11), p. 463.

† In the final statement of the theorem, this form is further condensed by the absorption of the term $K(x, y)$ into the determinant.

the latter statement, let us write out the value, say of $\Phi_1(y)$:

$$\Phi_1(y) = \frac{1}{\Delta}\begin{vmatrix} K(x_1, y) & \varphi_2(x_1) & \cdots & \varphi_n(x_1) \\ K(x_2, y) & \varphi_2(x_2) & \cdots & \varphi_n(x_2) \\ \cdot & \cdot & \cdot & \cdot \\ K(x_n, y) & \varphi_2(x_n) & \cdots & \varphi_n(x_n) \end{vmatrix},$$

multiply by $\varphi_i(y)$ $[i = 1, 2, \cdots, n]$, and integrate. Since

$$\int_a^b K(x_j, y)\,\varphi_i(y)\,dy = \varphi_i(x_j) \quad [i, j = 1, 2, \cdots, n],$$

this process shows at once that

$$\int_a^b \varphi_i(y)\,\Phi_1(y)\,dy = 0 \qquad [i = 2, 3, \cdots, n],$$

$$\int_a^b \varphi_1(y)\,\Phi_1(y)\,dy = 1;$$

a similar proof applies to $\Phi_2(y), \cdots, \Phi_n(y)$.

Of course, the same reasoning could be used for the identification of the formulæ (16), (19). We state the result as follows:

THEOREM VI. *There exist sets of points* $x_1, x_2, \cdots, x_n$; $y_1, y_2, \cdots, y_n$, *such that*

$$\Delta = \begin{vmatrix} \varphi_1(x_1) & \varphi_2(x_1) & \cdots & \varphi_n(x_1) \\ \varphi_1(x_2) & \varphi_2(x_2) & \cdots & \varphi_n(x_2) \\ \cdot & \cdot & \cdot & \cdot \\ \varphi_1(x_n) & \varphi_2(x_n) & \cdots & \varphi_n(x_n) \end{vmatrix} \neq 0,$$

$$\Delta_1 = \begin{vmatrix} \psi_1(y_1) & \psi_2(y_1) & \cdots & \psi_n(y_1) \\ \psi_1(y_2) & \psi_2(y_2) & \cdots & \psi_n(y_2) \\ \cdot & \cdot & \cdot & \cdot \\ \psi_1(y_n) & \psi_2(y_n) & \cdots & \psi_n(y_n) \end{vmatrix} \neq 0.$$

A Fredholm pseudo-resolvent is obtained by selecting any such set, and choosing for $\Phi_i(y)$ *the constant* $1/\Delta$ *multiplied into the determinant obtained on replacing the ith column of* Δ *by* $K(x_1, y), K(x_2, y), \cdots, K(x_n, y)$; *and for* $\Psi_i(x)$ *the constant* $1/\Delta_1$ *multiplied into the determinant obtained on replacing the ith*

column of Δ_1 *by* $K(x, y_1), K(x, y_2), \cdots, K(x, y_n)$. *Such a pseudo-resolvent satisfies the relations*:

$$l(x, y) = \int_a^b K(x, s)\, l(s, y)\, ds$$

$$+\frac{1}{\Delta_1}\begin{vmatrix} K(x, y) & \psi_1(y) & \psi_2(y) & \cdots & \psi_n(y) \\ K(x, y_1) & \psi_1(y_1) & \psi_2(y_1) & \cdots & \psi_n(y_1) \\ K(x, y_2) & \psi_1(y_2) & \psi_2(y_2) & \cdots & \psi_n(y_2) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ K(x, y_n) & \psi_1(y_n) & \psi_2(y_n) & \cdots & \psi_n(y_n) \end{vmatrix},$$

$$l(x, y) = \int_a^b l(x, s)\, K(s, y)\, ds$$

$$+\frac{1}{\Delta}\begin{vmatrix} K(x, y) & \varphi_1(x) & \varphi_2(x) & \cdots & \varphi_n(x) \\ K(x_1, y) & \varphi_1(x_1) & \varphi_2(x_1) & \cdots & \varphi_n(x_1) \\ K(x_2, y) & \varphi_1(x_2) & \varphi_2(x_2) & \cdots & \varphi_n(x_2) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ K(x_n, y) & \varphi_1(x_n) & \varphi_2(x_n) & \cdots & \varphi_n(x_n) \end{vmatrix}.$$

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