THE SOUTHERLY AND EASTERLY DEVIATIONS OF FALLING BODIES FOR AN UNSYMMETRICAL GRAVITATIONAL FIELD OF FORCE*

BY

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Introduction and Statement of Results.

The adequacy with which theory may be made to account for a given class of physical phenomena depends largely upon the nature of the assumptions which are made in the mathematical formulation of the corresponding physical problem. This is well illustrated in the problem of the southerly deviation of falling bodies. In the treatment of this problem (and that of the easterly deviation) by Gauss, the assumption is made that the force in the statical field of force which is at rest with respect to the earth, i. e., the field of force in which the plumb-line is in equilibrium, is constant in magnitude and direction in the neighborhood of the path of the falling body. Under this assumption,† it follows from the laws of dynamics that the southerly deviation (denoted by S. D.) is given by the formula:‡

$$\text{(1)} \qquad \text{S. D.} = \frac{1}{3}\omega^2 \sin 2\phi \cdot \frac{h^2}{g},$$

in which ω is the angular velocity of the earth's rotation, h is the height through which the body falls, and g and ϕ are the acceleration due to weight and the astronomical latitude, respectively, at the place of observation. In a recent paper by the author § it is shown that, under the assumption of a distribution of revolution (i. e., that the potential function of the earth's gravitational (or weight) field of force is of the form $f(r, z)$, where r is the distance of a general point from the earth's axis of rotation, and z is that from a fixed plane perpendicular to the axis), the same theory yields for the southerly deviation

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† Other assumptions are also made. The body is assumed to be a particle, and the effects of air currents, air resistance and the actions of the moon and sun are disregarded.

‡ See Gauss's *Werke*, vol. 5 (1867), p. 502.

§ These Transactions, vol. 12 (1911), pp. 335–353.

the formula

$$\text{(2)} \qquad \text{S. D.} = \frac{1}{6}\left[2\omega^2 \sin 2\phi + 5\left(\frac{\partial g}{\partial x}\right)_0\right]\frac{h^2}{g_0},$$

in which $\partial g / \partial x$ is the derivative of g along the meridian measured to the north, the subscript $({}_0)$ denoting the particular value at the initial point P_0 of the falling body. It will be observed that when g is constant, $\partial g / \partial x = 0$, and therefore formula (1) is a special case of formula (2).

For the potential function for which the Besselian ellipsoid is a level surface and the formula of Helmert gives the acceleration due to weight on this ellipsoid, the following table gives a few of the corresponding values of ϕ, g, and $\partial g / \partial x$.*

TABLE I.

ϕ	g	$+10^9 \frac{\partial g}{\partial x}$
40°	980.1457	8.0399
45°	980.5966	8.1568
50°	981.0475	8.0259
55°	981.4847	7.6517
60°	981.8949	7.0463

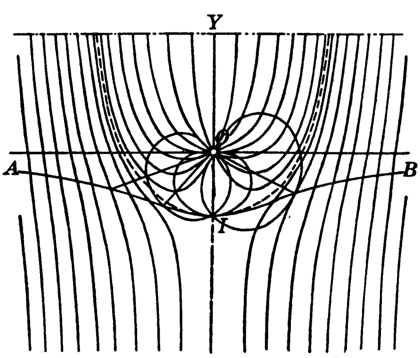
On the other hand,

$$\omega^2 = 5.3173 \cdot 10^{-9}.$$

The values given by formulas (1) and (2) are proportional to

$$2\omega^2 \sin 2\phi \qquad \text{and} \qquad 2\omega^2 \sin 2\phi + 5(\partial g / \partial x)_0.$$

Since sin 2ϕ never exceeds unity, it is evident from Table I, that, for the



potential function for which the standard ellipsoid is a level surface, formula (2) gives values for the southerly deviation which are nearly five times as great as those given by formula (1).

The expression $(\partial g / \partial x)_0 / g_0$ is the curvature, at P_0, of the line of force which passes through P_0 of the statical field of force which is at rest with respect to the earth. Thus we see that the curvature of these lines of force influences

* See 2nd reference in the second following footnote.

more strongly the southerly deviation of falling bodies than does the earth's rotation.

Simple considerations show that the introduction of an attracting mass into a field of constant force, or into one whose lines of force have slight curvature, increases considerably the curvature of the lines of force in the neighborhood of the introduced mass.*

This fact suggests that the proximity, to an experiment station, of a mountain or of a mineral deposit can change considerably the curvature of the lines of force of the statical field which is at rest with respect to the earth, and thus affect considerably the southerly deviation of falling bodies. But the proximity of such a disturbing factor would make the distribution of matter which produces the earth's gravitational field of force cease to be one of revolution. Therefore it becomes necessary to derive a formula for the southerly deviation—one for the easterly deviation will also be derived—under the assumption of *a distribution not of revolution.* This has been done in this paper. (See formulas III, where $\bar{\xi}$ = S. D. and $\bar{\eta}$ = E. D.) It turns out that the first term of the expression for the southerly deviation, under this assumption, is the same as expression (2), and that the term in $h^{5/2}$, which does not appear when the assumption of a distribution of revolution is made, is negligible in comparison with the first term.

The Hungarian physicist Baron Roland Eötvös has recently devised a method by means of which he can determine experimentally the derivative $\partial g / \partial x$, and also other second derivatives of the potential function W of the force due to weight.† The following table gives the observed values of these derivatives at a few stations which are about twenty kilometers east of Arad in Hungary and just west of some mountains.

Table II.

Station number	$10^9 \dfrac{\partial^2 W}{\partial x \partial z}$	$10^9 \dfrac{\partial^2 W}{\partial y \partial z}$	$10^9 \left(\dfrac{\partial^2 W}{\partial y^2} - \dfrac{\partial^2 W}{\partial x^2} \right)$	$10^9 \dfrac{\partial^2 W}{\partial x \partial y}$
1,018	+32.9	+88.1	+37.0	+ 3.8
1,032	+25.8	+62.8	+72.8	+ 2.5
1,035	+15.5	+69.8	+25.5	− 4.6
2,159	−37.0	+34.5	+34.3	−13.5
2,188	+77.8	+59.8	+21.5	+30.3
2,192	+82.3	+64.5	+28.2	+26.7

* The opposite figure represents the lines of force of the gravitational field which is produced by the introduction of a mass m, at 0, into a field which originally was constant (and of which the lines of force were straight lines parallel to OY). The line AIB is the locus along which the curvature has not been changed; in this case, it is the locus of the points of inflection of the lines of force of the new field of force.

† See Encyklopädie der mathematischen Wissenschaften, Band VIIB, § 23, p. 166. Also, Verhandlungen der fünfzehnten allgemeinen Conferenz der internationalen Erdmessung, 1 (1906), pp. 337–395.

Here x is measured to the north, y to the east, and z downward along the vertical. Hence

$$g = \frac{\partial W}{\partial z}, \qquad \frac{\partial g}{\partial x} = \frac{\partial^2 W}{\partial x \partial z}, \qquad \frac{\partial g}{\partial y} = \frac{\partial^2 W}{\partial y \partial z}.$$

For the latitude (46° 10′) of this region, the normal values (i. e., the values corresponding to the potential function for which the standard ellipsoid is a level surface) of the quantities given in Table II are

$$8.1, \qquad 0, \qquad 4.9, \qquad 0,$$

respectively.

At station 2,192 the value of the expression $2\omega^2 \sin 2\phi + 5\,(\partial g / \partial x)_0$ is about $422 \cdot 10^{-9}$, and hence formula (2) gives a value *forty times* as great as formula (1). On the other hand, at station 2,159, $2\omega^2 \sin 2\varphi + 5\,(\partial g / \partial x)_0$ is $-\ 174 \cdot 10^{-9}$, and hence formula (2) gives a *northerly deviation* which is more than sixteen times as great as the southerly deviation given by formula (1) The local value of the derivative $\partial g / \partial x = \partial^2 W / \partial x \partial z$ thus appears to determine the extent and the sign of the southerly deviation of falling bodies.

The easterly deviation of falling bodies is also influenced by the proximity of mountains or of mineral deposits. In fact, such disturbing factors yield the second term of the expression for the easterly deviation (denoted by E. D.). See formulas III.

$$\text{(3)} \qquad \text{E. D.} = \frac{2}{3}\sqrt{2}\,\omega \cos \phi \frac{h^{3/2}}{g_0^{1/2}} - \frac{5}{6}\frac{(\partial g / \partial y)_0}{g_0} h^2.$$

From Table II and the form of the second term in formula (3), it is evident that the effect of such disturbing factors is comparable in magnitude to the southerly deviation, but small in comparison with the first term of formula (3). The effect is also comparable in magnitude to the effect which Gauss finds is due to air resistance. Thus, for the data of Benzenberg's experiment in St. Michael's Tower, namely $\phi = 53°\ 33'$ and $h = 235$ feet, the formula of Gauss for the easterly deviation (which is equivalent to the first term of formula (3)) gives the value 3.91 lines.* When air resistance is taken into account, Gauss finds for the same data the deviation 3.86 lines. Hence the effect of air resistance is $-.05$ lines. On the other hand the value of the second term of formula (3) for the data of station 1018 (Table II) is $-\ .02$ lines. Gauss states that for the southerly deviation the effect of air resistance is negligible.

In this paper the effects of air resistance, air currents, and the actions of the moon and sun have not been taken into consideration.† While the com-

* 144 lines = 1 foot.

† Another matter which has not been taken into consideration in this paper is the weight of the string which supports the plumb-bob. In a field of force in which the lines of force are not rectilinear, this string will have some curvature. Therefore the position of a plumb-bob which is supported by a string with weight will be slightly different from that of a bob which is supported by a weightless string of the same length as the heavy string.

bined effects of these influences may be appreciable, it seems desirable to do first what has been done in this paper, namely to determine the effect of a local irregularity in the earth's gravitational field of force. *The surprising conclusion is reached that known local irregularities in the earth's gravitational field of force (caused by the presence of mountains or of mineral deposits or even by large buildings or tunnels)* (1) *influence the southerly deviation to the extent of from* -16 *times to* $+40$ *times* * *the amount which is given by the formula of Gauss for the southerly deviation (formula* (1)), (2) *affect the easterly deviation by amounts which are comparable with the effect which Gauss finds is due to air resistance.*

§ 1. *Definitions of, and formulas for, the easterly and southerly deviations of falling bodies.*

In experiments for the determination of the easterly and southerly deviations of falling bodies, a plumb-line $P_0 R$ is supported at the point P_0 from which a spherical body is later permitted to fall. On the horizontal plane which passes through the plumb-bob R, the easterly and southerly directions RE_R and RS_R are drawn (Fig. 3). The plumb-line RP_0 is the vertical at R.† The falling body, after being released from the position P_0, moves in a path which, with respect to the axes $R-E_R$, S_R, P_0, is the curve c. This curve pierces the horizontal plane $E_R R S_R$ in the point C. *The distance by which C lies to the south of RE_R is called the southerly deviation of the falling body, and that by which it lies to the east of RS_R is called the easterly deviation.*

In order to get expressions for these deviations the curve c is referred to the axes $P_0-\bar{\eta}$, $\bar{\xi}$, $\bar{\zeta}$ which pass through P_0 and are parallel to the axes $R-E_R$, S_R, P_0.‡ The equations of the curve c are (by Eqs. (17) § 7)

$$\text{I.}\quad \begin{aligned} \bar{\eta} &= -W^0_{\zeta}\left\{\frac{1}{3}\omega\cos\phi\cdot t^3+\frac{5}{24}W^0_{\eta\zeta}\cdot t^4\right. \\ &\qquad\left.+\frac{1}{60}\omega\left(\sin\phi W^0_{\xi\zeta}+\cos\phi\left(W^0_{\zeta\zeta}+W^0_{\eta\eta}-4\omega^2\right)\right)t^5\right\}, \\ \bar{\xi} &= -W^0_{\zeta}\left\{\frac{1}{24}\left(4\omega^2\sin\phi\cos\phi+5W^0_{\xi\zeta}\right)t^4\right. \\ &\qquad\left.+\frac{1}{60}\omega\left(9\sin\phi W^0_{\eta\zeta}+\cos\phi W^0_{\xi\eta}\right)t^5\right\}, \\ \bar{\zeta} &= \frac{1}{2}W^0_{\zeta}\left\{t^2+\frac{1}{12}\left(W^0_{\zeta\zeta}-4\omega^2\cos^2\phi\right)t^4-\frac{1}{3}\omega\cos\phi\cdot W^0_{\eta\zeta}\cdot t^5+\cdots\right\}, \end{aligned}$$

* In the region of the Alps this range of values would probably be much greater. See note by M. Brillouin on the curvature of the geoid in the Simplon tunnel, Comptes rendus, vol. 102 (1906), p. 916.

† It is assumed that the plumb-bob is a heavy particle and that the line is weightless and perfectly flexible.

‡ The axes $P_0-\bar{\eta}$, $\bar{\xi}$, $\bar{\zeta}$ do not coincide with the axes $P_0-\eta$, ξ, ζ. The latter axes are the easterly, southerly and vertical directions at P_0

where t represents the time measured from the instant the body begins to fall, ω represents the angular velocity of the earth's rotation, ϕ the latitude of the point P_0, W the potential function of the statical field of force which is at rest with respect to the rotating earth, and the symbols W^0_ζ, $W^0_{\eta\zeta}$, $W^0_{\xi\zeta}$, $W^0_{\zeta\zeta}$, $W^0_{\eta\eta}$, $W^0_{\xi\eta}$, stand for the values of the derivatives

$$\frac{\partial W}{\partial \zeta}, \quad \frac{\partial^2 W}{\partial \eta \partial \zeta}, \quad \frac{\partial^2 W}{\partial \xi \partial \zeta}, \quad \frac{\partial^2 W}{\partial \zeta^2}, \quad \frac{\partial^2 W}{\partial^2 \eta}, \quad \frac{\partial^2 W}{\partial \xi \partial \eta}$$

at the point P_0. If we put $-\bar{\zeta} = h = RP_0$, where h is the height through which the body falls, the corresponding values of $\bar{\eta}$ and $\bar{\xi}$ are respectively the easterly and the southerly deviations which correspond to h.

Let us now express $\bar{\eta}$ and $\bar{\xi}$ explicitly in terms of h. The last equation assumes the form

$$\frac{h}{\alpha} = t^2 + \gamma t^4 + \delta t^5 + \cdots,$$

where

$$\alpha = -\frac{1}{2} W^0_\zeta, \qquad \gamma = \frac{1}{12}(W^0_{\zeta\zeta} - 4\omega^2 \cos^2\phi), \qquad \delta = -\frac{1}{3}\omega\cos\phi \cdot W^0_{\eta\zeta}.$$

Hence

$$\pm\sqrt{\frac{h}{\alpha}} = \sqrt{t^2 + \gamma t^4 + \delta t^5 + \cdots}$$
$$= t + \frac{1}{2}\gamma t^3 + \frac{1}{2}\delta t^4 + \cdots,$$

and this when solved for t yields the relation

$$t = \left(\pm\sqrt{\frac{h}{\alpha}}\right) - \frac{1}{2}\gamma\left(\pm\sqrt{\frac{h}{\alpha}}\right)^3 - \frac{1}{2}\delta\left(\pm\sqrt{\frac{h}{\alpha}}\right)^4 + \cdots.$$

Since t is positive, the upper sign must be used. The expressions for η and $\bar{\xi}$ then assume the forms:

$$\text{II.} \qquad \begin{aligned} \bar{\eta} &= \frac{2}{3}\sqrt{2}\cdot\omega\cos\phi\cdot\frac{h^{3/2}}{\sqrt{-W^0_\zeta}} + \frac{5}{6}\frac{W^0_{\eta\xi}}{(-W^0_\zeta)}h^2 \\ &\quad + \frac{\sqrt{2}}{30}\omega\,(2\sin\phi\cdot W^0_{\xi\zeta} + 2\cos\phi\cdot W^0_{\eta\eta} - 3\cos\phi\cdot W^0_{\zeta\zeta} \\ &\qquad + 4\omega^2(5\cos^3\phi - 2\cos^2\phi))\frac{h^{5/2}}{(-W^0_\zeta)^{3/2}}, \\ \xi &= \frac{1}{6}(4\omega^2\sin\phi\cos\phi + 5W^0_{\xi\zeta})\frac{h^2}{(-W^0_\zeta)} \\ &\qquad + \frac{\sqrt{2}}{15}\omega\,(9\sin\phi\, W^0_{\eta\zeta} + \cos\phi\cdot W^0_{\xi\eta})\frac{h^{5/2}}{(-W^0_\zeta)^{3/2}}, \end{aligned}$$

and the expression for $\bar{\zeta}$ assumes the form $\bar{\zeta} = -h$, as it should. Since

W_η, W_ξ, W_ζ are the rectangular components of the acceleration g due to weight, we may write

$$-W^0_\zeta = g_0, \qquad W^0_{\eta\zeta} = -\left(\frac{\partial g}{\partial \eta}\right)_0, \qquad W^0_{\xi\zeta} = -\left(\frac{\partial g}{\partial \xi}\right)_0, \qquad W^0_{\zeta\zeta} = -\left(\frac{\partial g}{\partial \zeta}\right)_0,$$

and therefore the above formulas may be written as follows:

$$\text{III.} \qquad \begin{aligned} \bar{\eta} &= \frac{2}{3}\sqrt{2}\cdot\omega\cos\phi\cdot\frac{h^{3/2}}{g_0^{1/2}} - \frac{5}{6}\frac{(\partial g/\partial\eta)_0}{g_0}h^2 \\ &\quad + \frac{\sqrt{2}}{30}\omega\left[3\cos\phi\left(\frac{\partial g}{\partial\zeta}\right)_0 - 2\sin\phi\left(\frac{\partial g}{\partial\xi}\right)_0 + 2\cos\phi\left(\frac{\partial W_\eta}{\partial\eta}\right)_0\right. \\ &\qquad\qquad \left. + 4\omega^2(5\cos^3\phi - 2\cos^2\phi)\right]\frac{h^{5/2}}{g_0^{3/2}}, \\ \bar{\xi} &= \frac{4\omega^2\sin\phi\cos\phi - 5(\partial g/\partial\xi)_0}{6g_0}h^2 \\ &\quad + \frac{\sqrt{2}}{15}\omega\left[\cos\phi\left(\frac{\partial W_\xi}{\partial\eta}\right)_0 - 9\sin\phi\left(\frac{\partial g}{\partial\eta}\right)_0\right]\frac{h^{5/2}}{g_0^{3/2}}. \end{aligned}$$

For a distribution of revolution we have

$$\frac{\partial^2 W}{\partial\eta\,\partial\zeta} = 0, \qquad \frac{\partial^2 W}{\partial\eta\,\partial\xi} = 0,$$

and therefore the term in h^2 in the expression $(\bar{\eta})$ for the easterly deviation and the term in $h^{5/2}$ in expression $(\bar{\xi})$ for the southerly deviation drop out.

The establishment of the general equations I.

§ 2. *The curve c.* The curve c has already been defined as the path of the falling particle with respect to the rotating earth. Let us first refer it to a set of rectangular axes $O - x, y, z$, fixed in the rotating earth, and such that the origin O is a fixed point (interior to the earth) of the earth's axis a of rotation. Oz is coincident with a, and positive in the direction of the north pole, Ox is perpendicular to Oz and is so chosen that the initial point P_0 lies in the plane zOx, the positive direction being that from Oz to P_0, and finally Oy is perpendicular to Oz and Ox and is positive in the direction which Ox would have if it were revolved around the axis Oz through an angle of 90° in the direction of the earth's rotation. (See Fig. 1.)

If we represent by ω the angular velocity of the earth's rotation, and by U the function $f(x, y, z)$ which is defined by the integral:

$$U = \int_V \frac{dm}{\rho}, \qquad \rho = \sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2},$$

where dm represents that element of mass of the earth which is situated at the point whose coordinates, with respect to the axes $O - x, y, z$, are x_i, y_i, z_i, the integration being extended throughout the whole volume of the earth, then the function:

$$W = U + \frac{\omega^2}{2}(x^2 + y^2) \tag{1}$$

is the potential function of the statical field of force which is at rest with respect to the rotating earth* (i. e., the lines of force:

$$dx : dy : dz = \frac{\partial W}{\partial x} : \frac{\partial W}{\partial y} : \frac{\partial W}{\partial z}$$

of this field are at rest with respect to the rotating earth †).

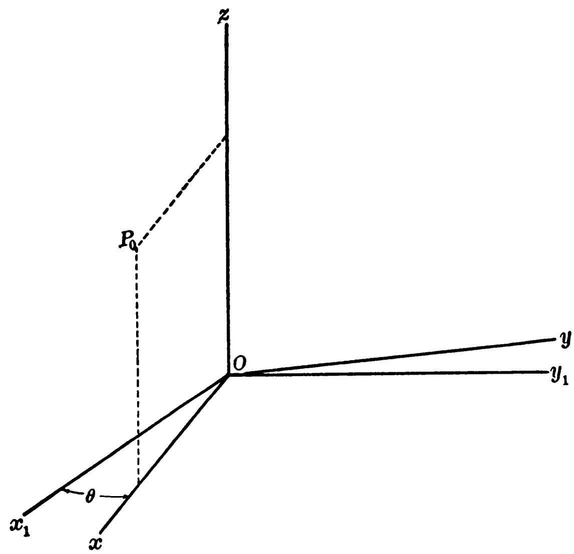


FIG. 1.

In order to find the differential equations of motion, let us denote by $O - x_1, y_1, z$ a set of rectangular axes which are fixed in space and so oriented that at the instant $t = 0$, when a falling particle leaves P_0 (at rest with respect to the earth), the set of axes $O - x, y, z$ coincides with the set $O - x_1, y_1, z$ (see Fig. 1). If we denote by θ the angle x_1Ox, positive in the direction in which the earth rotates, the relations between the coordinates are:

$$x = x_1 \cos\theta + y_1 \sin\theta, \qquad y = -x_1 \sin\theta + y_1 \cos\theta, \qquad z = z_1.$$

Then

$$U = f(x, y, z) = f_1(x_1, y_1, z, t),$$

* See PIZZETTI, *Trattato di Geodesia teoretica*, § 2 (1905).

† It should be borne in mind that the level (equipotential) surfaces, W = const., are not necessarily surfaces of revolution.

where $\theta = \omega t$ is a function of the time t. The differential equations of motion with respect to the fixed axes $O - x_1, y_1, z$, are

$$\frac{d^2x_1}{dt^2} = \frac{\partial U}{dx_1}, \quad \frac{d^2y_1}{dt^2} = \frac{\partial U}{\partial y_1}, \quad \frac{\partial^2 z}{dt^2} = \frac{\partial U}{\partial z}.$$

If we subject these differential equations to the inverse of the transformation just given, namely to the transformation

$$x_1 = x\cos\theta - y\sin\theta, \qquad y_1 = x\sin\theta + y\cos\theta, \qquad z_1 = z,$$

we obtain a set of differential equations* which is equivalent to the set:

$$\frac{d^2x}{dt^2} - 2\omega\frac{dy}{dt} - \omega^2 x = \frac{\partial U}{\partial x}, \quad \frac{d^2y}{dt^2} + 2\omega\frac{dx}{dt} - \omega^2 y = \frac{\partial U}{\partial y}, \quad \frac{d^2z}{dt^2} = \frac{\partial U}{\partial z}.$$

Since by equation (1),

$$\frac{\partial U}{\partial x} + \omega x = \frac{\partial W}{\partial x}, \quad \frac{\partial U}{\partial y} + \omega y = \frac{\partial W}{\partial y}, \quad \frac{\partial U}{\partial z} = \frac{\partial W}{\partial z},$$

these equations assume the forms:

$$(2) \qquad \frac{d^2x}{dt^2} - 2\omega\frac{dy}{dt} = \frac{\partial W}{\partial x}, \quad \frac{d^2y}{dt^2} + 2\omega\frac{dx}{dt} = \frac{\partial W}{\partial y}, \quad \frac{d^2z}{dt^2} = \frac{\partial W}{\partial z}.$$

The curve c is then that solution of the differential equations (2) which is subject to the initial conditions:

(3) When $t = 0, \quad x = r_0, \quad y = 0, \quad z = z_0, \quad \dfrac{dx}{dt} = \dfrac{dy}{dt} = \dfrac{dz}{dt} = 0.$

§ 3. *The curve c referred to the cardinal axes at P_0.* Before we can define the cardinal axes it will be necessary to define the following terms. At a general point P which is at rest with respect to the rotating earth, *the vertical* is the straight line which coincides with the plumb-line,† the plumb-bob of

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$$* \frac{d^2x}{dt^2}\cos\theta - \frac{d^2y}{dt^2}\sin\theta - 2\left(\frac{dx}{dt}\sin\theta + \frac{dy}{dt}\cos\theta\right)\omega - (x\cos\theta - y\sin\theta)\,\omega^2 = \frac{\partial U}{\partial x}\cos\theta - \frac{\partial U}{\partial y}\sin\theta,$$

$$\frac{d^2x}{dt^2}\sin\theta + \frac{d^2y}{dt^2}\cos\theta + 2\left(\frac{dx}{dt}\cos\theta - \frac{dy}{dt}\sin\theta\right)\omega - (x\sin\theta + y\cos\theta)\,\omega^2 = \frac{\partial U}{\partial x}\sin\theta + \frac{\partial U}{\partial y}\cos\theta.$$

If we multiply the first of these equations by $\cos\theta$ and the second by $\sin\theta$ and add, we obtain

$$\frac{d^2x}{dt^2} - 2\omega\frac{dy}{dt} - \omega^2 x = \frac{\partial U}{\partial x},$$

and if we multiply the first by $-\sin\theta$ and the second by $\cos\theta$ and add, we obtain

$$\frac{d^2y}{dt^2} + 2\omega\frac{dx}{dt} - \omega^2 y = \frac{\partial U}{\partial y}.$$

† It is assumed that the plumb-bob is a heavy particle and that the line is weightless and perfectly flexible.

which is situated at P (this line does not necessarily intersect the axis of rotation a of the earth), *the meridian plane* is the plane which passes through the vertical and is parallel to the axis a, *the horizontal plane* is the plane which passes through P and is perpendicular to the vertical, the *east-and-west line* is the straight line which passes through P and is perpendicular to the meridian plane, the *north-and-south line* is the line of intersection of the meridian and the horizontal planes. By *the cardinal axes at the point P* we shall mean the vertical, the east-and-west and the north-and-south lines at the point P.

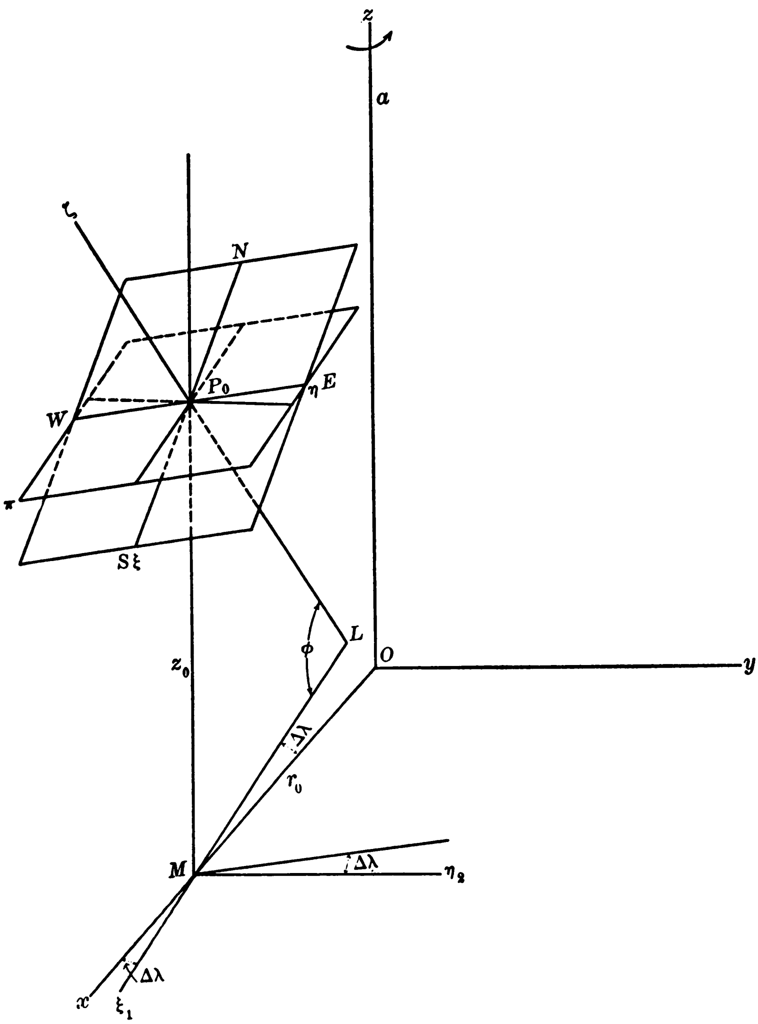

FIG. 2.

The positive directions of these axes shall be those toward the zenith, the east, and the south respectively. A few more terms will now be defined. *The* *latitude at P* is the complement of the angle which the positive direction of the vertical at P makes with the positive direction of the axis a. *The longitude at P* is the angle which the meridian plane at P makes with a fixed plane through the axis a (we shall here assume that the fixed plane is the plane zOx) measured from 0° to 360° to the east.*

Let us now take the cardinal axes at the point P_0 as the axes of the coordinates ξ, η, ζ, where ξ is measured along the north-and-south line, η along the east-and-west line and ζ along the vertical. See Fig. 2, in which $Oz \equiv a$ represents the axes of rotation of the earth, and Ox, Oy a pair of

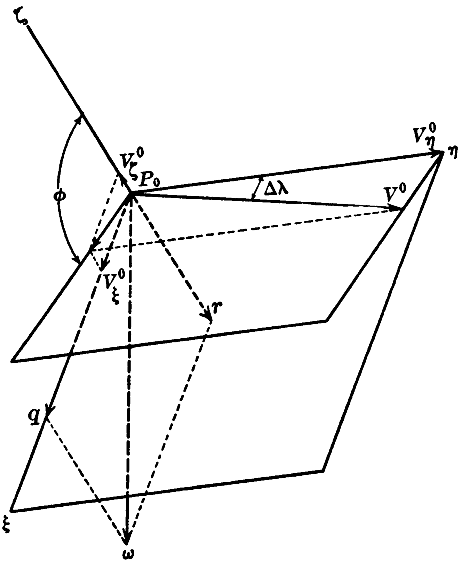

FIG. 2′.

rectangular axes which are at rest with respect to the rotating earth and are perpendicular to Oz at O. P_0 is a point of the plane xOz. $P_0 M$ is parallel to a, and intersects the plane xOy in M. We put

$$r_0 = OM, \qquad z_0 = MP_0.$$

The line $P_0 L$ is the *vertical at* P_0; it intersects the plane xOy in L. $LP_0 M$ is then the *meridian plane at* P_0. The plane $NESW$, which passes through P_0 and is perpendicular to P_0L, is the *horizontal plane at* P_0. These two planes intersect in the *north-and-south-line* NS. Let us denote by π the plane which passes through P_0 and is aparallel to the plane xOy. This plane intersects the horizontal plane in the *east-and-west line* EW.

We agreed to take the plane xOz as the plane from which to measure longitude. Therefore the *longitude of* P_0 *is the angle* $\Delta\lambda = \measuredangle OML$. *The latitude of* P_0 *is the angle* $\phi = \measuredangle MLP_0$.

* See PIZZETTI, loc. cit., § 5.

In order to get the equations of transformation from the axes $O - x, y, z$ to the axes $P_0 - \xi, \eta, \zeta$, let us introduce the auxiliary variables $\xi_1, \zeta_1, \xi_2, \eta_2$, where ξ_1 represents distance measured along ML from M in the direction LM, and ζ_1, ξ_2 and η_2 are defined by the equations

$$x = \xi_2 + r_0, \qquad y = \eta_2, \qquad z = \zeta_1 + z_0.$$

Then

$$\xi_2 = \xi_1 \cos \Delta\lambda - \eta \sin \Delta\lambda, \qquad \eta_2 = \xi_1 \sin \Delta\lambda + \eta \cos \Delta\lambda,$$

and

$$\xi_1 = \zeta \cos \phi + \xi \sin \phi, \qquad \zeta_1 = \zeta \sin \phi - \xi \cos \phi.$$

Therefore

$$\begin{aligned} x &= r_0 + (\zeta \cos \phi + \xi \sin \phi) \cos \Delta\lambda - \eta \sin \Delta\lambda, \\ y &= (\zeta \cos \phi + \xi \sin \phi) \sin \Delta\lambda + \eta \cos \Delta\lambda, \\ z &= z_0 + \zeta \sin \phi - \xi \cos \phi. \end{aligned} \tag{4}$$

The inverse of this transformation is the transformation:

$$\begin{aligned} \xi &= [(x - r_0) \cos \Delta\lambda + y \sin \Delta\lambda] \sin \phi - (z - z_0) \cos \phi, \\ \eta &= -(x - r_0) \sin \Delta\lambda + y \cos \Delta\lambda, \\ \zeta &= [(x - r_0) \cos \Delta\lambda + y \sin \Delta\lambda] \cos \phi + (z - z_0) \sin \phi. \end{aligned}$$

If now we subject the system of differential equations (2) to the transformation (4), we obtain the system of differential equations,*

* The details of this process are the following:

$$\frac{d^2 x}{dt^2} - 2\omega \frac{dy}{dt} = \frac{\partial W}{\partial x}$$

becomes

$$\cos \Delta\lambda \left(\frac{d^2 \zeta}{dt^2} \cos \phi + \frac{d^2 \xi}{dt^2} \sin \phi \right) - \sin \Delta\lambda \frac{d^2 \eta}{dt^2} - 2 \sin \Delta\lambda \left(\frac{d\zeta}{dt} \cos \phi + \frac{d\xi}{dt} \sin \phi \right) \omega - 2 \cos \Delta\lambda \cdot \frac{d\eta}{dt} \cdot \omega$$
$$= \frac{\partial W}{\partial \xi} \sin \phi \cos \Delta\lambda - \frac{\partial W}{\partial \eta} \sin \Delta\lambda + \frac{\partial W}{\partial \zeta} \cos \phi \cos \Delta\lambda$$

$$\frac{d^2 y}{dt^2} + 2\omega \frac{dx}{dt} = \frac{\partial W}{\partial y}$$

becomes

$$\sin \Delta\lambda \left(\frac{d^2 \zeta}{dt^2} \cos \phi + \frac{d^2 \xi}{dt^2} \sin \phi \right) + \cos \Delta\lambda \frac{d^2 \eta}{dt^2} + 2 \cos \Delta\lambda \left(\frac{d\zeta}{dt} \cos \phi + \frac{d\xi}{dt} \sin \phi \right) \omega - 2 \sin \Delta\lambda \frac{d\eta}{dt} \omega$$
$$= \frac{\partial W}{\partial \xi} \sin \phi \sin \Delta\lambda + \frac{\partial W}{\partial \eta} \cos \Delta\lambda + \frac{\partial W}{\partial \zeta} \cos \phi \sin \Delta\lambda .$$

$$\frac{d^2 z}{dt^2} = \frac{\partial W}{\partial z}$$

becomes

$$\frac{d^2 \zeta}{dt^2} \sin \phi - \frac{d^2 \xi}{dt^2} \cos \phi = -\frac{\partial W}{\partial \xi} \cos \phi + \frac{\partial W}{\partial \zeta} \sin \phi .$$

If we multiply these equations respectively by $\cos \Delta\lambda \sin \phi$, $\sin \Delta\lambda \sin \phi$, $-\cos \phi$ and add, we obtain the first of the equations (5); if we multiply respectively by $\cos \Delta \lambda \cos \phi$, $\sin \Delta\lambda \cos \phi$, $\sin \phi$, and add, we obtain the second of the equations (5); and if we multiply respectively by $-\sin \Delta\lambda$, $\cos \Delta\lambda$, 0, we obtain the last of the equations (5).

$$\frac{d^2\xi}{dt^2} - 2\omega\sin\phi\cdot\frac{d\eta}{dt} = \frac{\partial W}{\partial\xi},$$

(5)
$$\frac{d^2\eta}{dt^2} + 2\omega\left(\sin\phi\cdot\frac{d\xi}{dt} + \cos\phi\cdot\frac{d\zeta}{dt}\right) = \frac{\partial W}{\partial\eta},$$

$$\frac{d^2\zeta}{dt^2} - 2\omega\cos\phi\cdot\frac{d\eta}{dt} = \frac{\partial W}{\partial\zeta}.$$

The initial conditions (3) assume the form:

(6) When $\quad t = 0, \qquad \xi = \eta = \zeta = 0, \qquad \frac{d\xi}{dt} = \frac{d\eta}{dt} = \frac{d\zeta}{dt} = 0.$

Therefore the curve c is that solution of the differential equations (5) which is subject to the initial conditions (6).

§ 4. *Another derivation of equations* (5). The differential equations (5) may also be obtained as follows. The projections on the moving axes η, ξ, ζ of the absolute velocity V_a are *

$$V_{a_\eta} = \frac{d\eta}{dt} + V^0_\eta + q\zeta - r\xi,$$

$$V_{a\xi} = \frac{d\xi}{dt} + V^0_\xi + r\eta - p\zeta,$$

$$V_{a\zeta} = \frac{d\zeta}{dt} + V^0_\zeta + p\xi - q\eta,$$

where V^0_η, V^0_ξ, V^0_ζ are the projections on the axes η, ξ, ζ of the absolute velocity V^0 of the initial point P_0, and p, q, r are the projections on the same axes of the rotation ω of the earth. From Fig. 2′ it is easily seen that

$$V^0_\eta = V^0\cos\Delta\lambda,\ V^0_\xi = V^0\sin\Delta\lambda\sin\phi,\ V^0_\zeta = V^0\sin\Delta\lambda\cos\phi,$$

where

$$V^0 = \omega r_0,$$

and

$$p = 0, \qquad q = \omega\cos\phi, \qquad r = -\omega\sin\phi.$$

Therefore

$$V_{a_\eta} = \frac{d\eta}{dt} + \omega r_0\cos\Delta\lambda + \omega\cos\phi\cdot\zeta + \omega\sin\phi\cdot\xi,$$

(7)
$$V_{a\xi} = \frac{d\xi}{dt} + \omega r_0\sin\Delta\lambda\sin\phi - \omega\sin\phi\cdot\eta,$$

$$V_{a\zeta} = \frac{d\zeta}{dt} + \omega r_0\sin\Delta\lambda\cos\phi - \omega\cos\phi\cdot\eta.$$

* See Appell, *Traité de Mécanique Rationnelle*, vol. 1, (1902) §61.

The projections on the moving axes η, ξ, ζ of the absolute acceleration J_a are

$$J_{a\eta} = \frac{dV_{a\eta}}{dt} + qV_{a\zeta} - rV_{a\xi},$$

$$J_{a\xi} = \frac{dV_{a\xi}}{dt} + rV_{a\eta} - pV_{a\zeta},$$

$$J_{a\zeta} = \frac{dV_{a\zeta}}{dt} + pV_{a\xi} - qV_{a\eta}.$$

Hence by the expressions for $V_{a\eta}$, $V_{a\xi}$, $V_{a\zeta}$,

$$\begin{aligned}
&J_{a\eta} = \frac{d^2\eta}{dt^2} + 2\omega\left(\cos\phi\cdot\frac{d\zeta}{dt} + \sin\phi\,\frac{d\xi}{dt}\right) - \omega^2(\eta - r_0\sin\Delta\lambda),\\
(8)\quad &J_{a\xi} = \frac{d^2\xi}{dt^2} - 2\omega\sin\phi\cdot\frac{d\eta}{dt} - \omega^2\sin\phi\,(\cos\phi\cdot\zeta + \sin\phi\cdot\xi + r_0\cos\Delta\lambda),\\
&J_{a\zeta} = \frac{d^2\zeta}{dt^2} - 2\omega\cos\phi\cdot\frac{d\eta}{dt} - \omega^2\cos\phi(\cos\phi\cdot\zeta + \sin\phi\cdot\xi + r_0\cos\Delta\lambda).
\end{aligned}$$

Since gravitation acts instantaneously, the projections of the absolute force on the axes η, ξ, ζ are

$$\frac{\partial U}{\partial\eta},\quad \frac{\partial U}{\partial\xi},\quad \frac{\partial U}{\partial\zeta}.$$

Hence the differential equations of motion are

$$(9)\qquad J_{a\eta} = \frac{\partial U}{\partial\eta},\qquad J_{a\xi} = \frac{\partial U}{\partial\xi},\qquad J_{a\zeta} = \frac{\partial U}{\partial\zeta},$$

where the expressions for $J_{a\eta}$, $J_{a\xi}$, $J_{a\zeta}$ are given by (8). In virtue of relations (1) and (4) we find that

$$\begin{aligned}
\frac{\partial W}{\partial\eta} &= \frac{\partial U}{\partial\eta} + \omega^2(\eta - r_0\sin\Delta\lambda),\\
\frac{\partial W}{\partial\xi} &= \frac{\partial U}{\partial\xi} + \omega^2\sin\phi\,(\cos\phi\cdot\zeta + \sin\phi\cdot\xi + r_0\cos\Delta\lambda),\\
\frac{\partial W}{d\zeta} &= \frac{\partial U}{\partial\zeta} + \omega^2\cos\phi(\cos\phi\cdot\zeta + \sin\phi\cdot\xi + r_0\cos\Delta\lambda),
\end{aligned}$$

and hence equations (9) are identical with equations (5).

§ 5. *Integration of the differential equations of motion* (5). Let us write equations (5) in the following form:

$$\begin{aligned}
&\eta'' + 2\omega(\sin\phi\cdot\xi' + \cos\phi\cdot\zeta') - W_\eta = 0,\\
(\alpha)\qquad &\xi'' - 2\omega\sin\phi\cdot\eta' \qquad\qquad\qquad - W_\xi = 0,\\
&\zeta'' - 2\omega\cos\phi\cdot\eta' \qquad\qquad\qquad - W_\zeta = 0,
\end{aligned}$$

where the primes ($'$) and seconds ($''$) denote the first and second derivatives, with respect to the time t, of the functions to which they are attached, and the subscripts (η, ξ, ζ) denote the first partial derivatives of W with respect to the attached subscript. In the equations which follow thirds ($'''$), etc., and double subscript $(\eta\eta, \eta\xi \cdots)$, etc., will be used to denote the higher derivatives. For the particular values of η, ξ, ζ, and their derivatives with respect to t, which correspond to $t = 0$, we shall use the symbols η_0, ξ_0, ζ_0, η_0', ξ_0', ζ_0', η_0'', ξ_0'', etc. Hence condition (6) may be written in the following form:

$$(\beta) \qquad \eta_0 = \xi_0 = \zeta_0 = 0, \qquad \eta_0' = \xi_0' = \zeta_0' = 0.$$

For the particular values of the derivatives of W which correspond to $\eta = \eta_0$, $\xi = \xi_0$, $\zeta = \zeta_0$, we shall use the symbols W_η^0, W_ξ^0, W_ζ^0, $W_{\eta\eta}^0$, $W_{\eta\xi}^0$, $W_{\eta\zeta}^0$, $W_{\xi\xi}^0$, $W_{\xi\zeta}^0$, $W_{\zeta\zeta}^0$, etc.

For a set of functions

$$\eta = \lambda(t), \qquad \xi = \mu(t), \qquad \zeta = \nu(t),$$

which is a solution of the set of differential equations (α), the left hand member of each of the equations (α) vanishes identically. Therefore, for this solution, the derivative with respect to t of each of these left-hand members must vanish identically. Hence the identities:

$$(\alpha') \qquad \begin{aligned} &\eta''' + 2\omega(\sin\phi \cdot \xi'' + \cos\phi \cdot \zeta'') - [W_{\eta\eta} \cdot \eta' + W_{\eta\xi} \cdot \xi' + W_{\eta\zeta} \cdot \zeta'] = 0, \\ &\xi''' - 2\omega \sin\phi \cdot \eta'' \qquad\qquad\qquad - [W_{\xi\eta} \cdot \eta' + W_{\xi\xi} \cdot \xi' + W_{\xi\zeta} \cdot \zeta'] = 0, \\ &\zeta''' - 2\omega \cos\phi \cdot \eta'' \qquad\qquad\qquad - [W_{\zeta\eta} \cdot \eta' + W_{\zeta\xi} \cdot \xi' + W_{\zeta\zeta} \cdot \zeta'] = 0. \end{aligned}$$

Similarly, if we assume the existence and continuity of all the derivatives which are needed, we obtain the following sets of identities:

$$(\alpha'') \qquad \begin{aligned} &\eta^{\text{iv}} + 2\omega(\sin\phi \cdot \xi''' + \cos\phi \cdot \zeta''') \\ &\qquad - [W_{\eta\eta} \cdot \eta'' + W_{\eta\xi} \cdot \xi'' + W_{\eta\zeta} \cdot \zeta'' + \alpha_1 \eta' + \beta_1 \xi' + \gamma_1 \zeta'] = 0, \\ &\xi^{\text{iv}} - 2\omega \sin\phi \cdot \eta''' \\ &\qquad - [W_{\xi\eta} \cdot \eta'' + W_{\xi\xi} \cdot \xi'' + W_{\xi\zeta} \cdot \zeta'' + \alpha_2 \eta' + \beta_2 \xi' + \gamma_2 \zeta'] = 0, \\ &\zeta^{\text{iv}} - 2\omega \cos\phi \cdot \eta''' \\ &\qquad - [W_{\zeta\eta} \cdot \eta'' + W_{\zeta\xi} \cdot \xi'' + W_{\zeta\zeta} \cdot \zeta'' + \alpha_3 \eta' + \beta_3 \xi' + \gamma_3 \zeta'] = 0, \end{aligned}$$

where α_i, β_i, γ_i, $i = 1, 2, 3$, stand for homogeneous linear expressions in η', ξ', ζ', the coefficients of which are third derivatives of W*,

* If $u = u(x, y, z)$, $u' = u_x \cdot x' + u_y \cdot y' + u_z z'$,

$$u'' = u_x \cdot x'' + u_y \cdot y'' + u_z \cdot z'' + (u_{xx} x' + u_{xy} y' + u_{xz} z') x' + (u_{yx} x' + u_{yy} y' + u_{yz} z') y' + (u_{zx} x' + u_{zy} y' + u_{zz} z') z'.$$

$$
\begin{aligned}
&\eta^{\mathrm{v}} + 2\omega\,(\sin\phi\cdot\xi^{\mathrm{iv}} + \cos\phi\cdot\zeta^{\mathrm{iv}}) \\
&\qquad - [W_{\eta\eta}\cdot\eta''' + W_{\eta\xi}\cdot\xi''' + W_{\eta\zeta}\cdot\zeta''' + \alpha_1'\,\eta' + \beta_1'\,\xi' + \gamma_1'\,\zeta'] = 0, \\
(\alpha''')\quad &\xi^{\mathrm{v}} - 2\omega\sin\phi\cdot\eta^{\mathrm{iv}} \\
&\qquad - [W_{\xi\eta}\cdot\eta''' + W_{\xi\xi}\cdot\xi''' + W_{\xi\zeta}\cdot\zeta''' + \alpha_2'\cdot\eta' + \beta_2'\,\xi' + \gamma_2'\,\zeta'] = 0, \\
&\zeta^{\mathrm{v}} - 2\omega\cos\phi\cdot\eta^{\mathrm{iv}} \\
&\qquad - [W_{\zeta\eta}\cdot\eta''' + W_{\zeta\xi}\cdot\xi''' + W_{\zeta\zeta}\cdot\zeta''' + \alpha_3'\,\eta' + \beta_3'\,\xi' + \gamma_3'\,\zeta'] = 0,
\end{aligned}
$$

where α_i', β_i', γ_i', $i = 1, 2, 3$, stand for polynomials in the η', ξ', ζ', η'', ξ'', ζ'' and the third and fourth derivatives of W.*
By condition (β) we already know that

$$\eta_0 = \xi_0 = \zeta_0 = 0, \qquad \eta_0' = \xi_0' = \zeta_0' = 0.$$

We also know that

$$W_\eta^0 = W_\xi^0 = 0.$$

Therefore, when $t = 0$, the identities (α) yield the relations

$$\eta_0'' = 0, \qquad \xi_0'' = 0, \qquad \zeta_0'' = W_\zeta^0;$$

the identities (α'), the relations

$$\eta_0''' = -2\omega\cos\phi\cdot W_\zeta^0, \qquad \xi_0''' = 0, \quad \zeta_0''' = 0;$$

the identities (α''), the relations

$$\eta_0^{\mathrm{iv}} = W_\zeta^0\,W_{\eta\zeta}^0, \qquad \xi_0^{\mathrm{iv}} = W_\zeta^0\,(W_{\xi\zeta}^0 - 4\omega^2\sin\phi\cos\phi),$$
$$\zeta_0^{\mathrm{iv}} = W_\zeta^0\,(W_{\zeta\zeta}^0 - 4\omega^2\cos^2\phi);$$

and the identities (α'''), the relations

$$
\begin{aligned}
\eta_0^{\mathrm{v}} &= -2\omega\,W_\zeta^0\,[\sin\phi\cdot W_{\xi\zeta}^0 + \cos\phi\,(W_{\zeta\zeta}^0 + W_{\eta\eta}^0 - 4\omega^2)], \\
\xi_0^{\mathrm{v}} &= +2\omega\,W_\zeta^0[\sin\phi\cdot W_{\eta\zeta}^0 - \cos\phi\cdot W_{\xi\eta}^0], \\
\zeta_0^{\mathrm{v}} &= +2\omega\,W_\zeta^0\cos\phi\,[W_{\eta\zeta}^0 - W_{\zeta\eta}^0] = 0.
\end{aligned}
$$

---

and

$$
\begin{aligned}
u''' = u_x x''' + u_y y''' + u_z z''' + 3\,[\,(u_{xx} x'' + u_{xy} y'' + u_{xz} z'')\,x' \\
+ (u_{xy} x'' + u_{yy} y'' + u_{yz} z'')\,y' + (u_{zx} x'' + u_{zy} y'' + u_{zz} z'')\,z'\,]
\end{aligned}
$$

$$
+\left\{\begin{array}{l}
\quad (u_{xxx} x' + u_{xxy} y' + u_{xxz} z')\,x' \\
+ (u_{xyx} x' + u_{xyy} y' + u_{xyz} z')\,y' \\
+ (u_{xzx} x' + u_{xzy} y' + u_{xzz} z')\,z'
\end{array}\right\} x'
+\left\{\begin{array}{l}
\quad (u_{yxx} x' + u_{yxy} y' + u_{yxz} z')\,x' \\
+ (u_{yyx} x' + u_{yyy} y' + u_{yyz} z')\,y' \\
+ (u_{yzx} x' + u_{yzy} y' + u_{yzz} z')\,z'
\end{array}\right\} y'
$$

$$
+\left\{\begin{array}{l}
\quad (u_{zxx} x' + u_{zxy} y' + u_{zxz} z')\,x' \\
+ (u_{zyx} x' + u_{zyy} y' + u_{zyz} z')\,y' \\
+ (u_{zzx} x' + u_{zzy} y' + u_{zzz} z')\,z'
\end{array}\right\} z'.
$$

Let us now assume that the conditions are satisfied under which the set of differential equations (5) or (α) have a solution of the form

$$
\begin{aligned}
\eta &= a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + \cdots, \\
\xi &= b_0 + b_1 t + b_2 t^2 + b_3 t^3 + b_4 t^4 + b_5 t^5 + \cdots, \\
\zeta &= c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + c_5 t^5 + \cdots,
\end{aligned} \tag{10}
$$

in the neighborhood of the point P_0. It then follows, from the preceding work, that for the initial conditions (6) or (β),

$$
\begin{aligned}
&a_0 = \eta_0 = 0, \qquad a_1 = \eta_0' = 0, \qquad a_2 = \tfrac{1}{2}\,\eta_0'' = 0, \\
&b_0 = \xi_0 = 0, \qquad b_1 = \xi_0' = 0, \qquad b_2 = \tfrac{1}{2}\xi_0'' = 0, \\
&c_0 = \zeta_0 = 0, \qquad c_1 = \zeta_0' = 0, \qquad c_2 = \tfrac{1}{2}\zeta_0'' = \tfrac{1}{2}W_\zeta^0; \\
&a_3 = \tfrac{1}{6}\eta_0''' = -\tfrac{1}{3}\omega\cos\phi\cdot W_\zeta^0, \\
&b_3 = \tfrac{1}{6}\xi_0''' = 0, \\
&c_3 = \tfrac{1}{6}\zeta_0''' = 0; \\
&a_4 = \tfrac{1}{24}\eta_0^{\text{iv}} = \tfrac{1}{24}W_\zeta^0\, W_{\eta\zeta}^0, \\
&b_4 = \tfrac{1}{24}\xi_0^{\text{iv}} = \tfrac{1}{24}W_\zeta^0\,(W_{\xi\zeta}^0 - 4\omega^2\sin\phi\cos\phi), \\
&c_4 = \tfrac{1}{24}\zeta_0^{\text{iv}} = \tfrac{1}{24}W_\zeta^0\,(W_{\zeta\zeta}^0 - 4\omega^2\cos^2\phi); \\
&a_5 = \tfrac{1}{120}\eta_0^{\text{v}} = -\tfrac{1}{60}\omega W_\zeta^0\,[\sin\phi\cdot W_{\xi\zeta}^0 + \cos\phi\,(W_{\zeta\zeta}^0 + W_{\eta\eta}^0 - 4\omega^2)], \\
&b_5 = \tfrac{1}{120}\xi_0^{\text{v}} = \tfrac{1}{60}\omega W_\zeta^0\,[\sin\phi\cdot W_{\eta\zeta}^0 - \cos\phi\, W_{\xi\eta}^0], \\
&c_5 = \tfrac{1}{120}\zeta_0^{\text{v}} = 0.
\end{aligned} \tag{11}
$$

The equations (10), for which the constants are given by the relations (11), are then the equations of the curve c referred to the cardinal axes at P_0.

§ 6. *The plumb-bob locus referred to the cardinal axes of P_0.* This curve has already been defined as the locus of plumb-bobs of all plumb-lines which are supported at the initial point P_0, of the falling particle. It is therefore the locus of the feet of perpendiculars dropped from P_0 ($\eta = 0$, $\xi = 0$, $\zeta = 0$) to the level (equipotential) surfaces $W =$ const. Its equations are easily seen to be

$$
\frac{\eta}{\dfrac{\partial W}{\partial\eta}} = \frac{\xi}{\dfrac{\partial W}{\partial\xi}} = \frac{\zeta}{\dfrac{\partial W}{\partial\zeta}}. \tag{12}
$$

Now let us solve these equations in the form

$$
\eta = \alpha_1\zeta + \alpha_2\zeta^2 + \cdots, \qquad \xi = \beta_1\zeta + \beta_2\zeta^2 + \cdots.
$$

In order to do this let us write the equations (12) in the form:

$$F_i(\eta, \xi, \zeta) = 0, \qquad (i = 1, 2)$$

where

$$F_1 = \begin{vmatrix} \xi & \zeta \\ W_\xi & W_\zeta \end{vmatrix} \quad \text{and} \quad F_2 = \begin{vmatrix} \zeta & \eta \\ W_\zeta & W_\eta \end{vmatrix}.$$

The derivatives $d\eta/d\zeta$, $d\xi/d\zeta$, $d^2\eta/d\zeta^2$, $d^2\xi/d\zeta^2$ are expressible in terms of the derivatives of F_i, $i = 1, 2$,* and the derivatives of F_i are expressible in terms of the derivatives of W.† The coefficients α_1, α_2, β_1, β_2 have then the following values:

$$\alpha_1 = \left(\frac{d\eta}{d\zeta}\right)_0 = 0, \qquad \beta_1 = \left(\frac{d\xi}{d\zeta}\right)_0 = 0,$$

$$\alpha_2 = \frac{1}{2}\left(\frac{d^2\eta}{d\zeta^2}\right)_0 = \frac{W^0_{\eta\zeta}}{W^0_\zeta}, \qquad \beta_2 = \frac{W^0_{\xi\zeta}}{W^0_\zeta}.$$

Therefore the plumb-bob locus may be represented by the equations

$$\eta = \frac{W^0_{\eta\zeta}}{W^0_\zeta}\zeta^2 + \cdots, \qquad \xi = \frac{W^0_{\xi\zeta}}{W^0_\zeta}\zeta^2 + \cdots. \tag{13}$$

§ 7. *The curve c referred to axes of origin P_0 and directions which are parallel to the cardinal directions at the plumb-bob of a plumb-line which is supported at P_0.* Let us denote by R (Fig. 3) the point at which the plumb-bob of a

* The relations are the following:

$$\frac{\partial F_i}{\partial \eta}\frac{d\eta}{d\zeta} + \frac{\partial F_i}{\partial \xi}\frac{d\xi}{d\zeta} + \frac{\partial F_i}{\partial \zeta} = 0 \qquad (i = 1, 2),$$

$$\frac{\partial F_i}{\partial \eta}\frac{d^2\eta}{d\zeta^2} + \frac{d\eta}{d\zeta}\left[\frac{\partial^2 F_i}{\partial\eta\,\partial\xi}\frac{d\xi}{d\zeta} + \frac{\partial^2 F_i}{\partial\eta^2}\frac{d\eta}{d\zeta} + \frac{\partial^2 F_i}{\partial\zeta\,\partial\eta}\right] + \frac{\partial^2 F_i}{\partial\zeta\,\partial\eta}\frac{d\eta}{d\zeta} + \frac{\partial^2 F_i}{\partial\zeta^2} + \frac{\partial F_i}{\partial\xi}\frac{d^2\xi}{d\zeta^2}$$

$$+ \frac{d\xi}{d\zeta}\left[\frac{\partial^2 F_i}{\partial\xi^2}\frac{d\xi}{\partial\zeta} + \frac{\partial^2 F_i}{d\xi\,\partial\eta}\frac{d\eta}{d\zeta} + \frac{\partial^2 F_i}{\partial\zeta\,\partial\xi}\right] + \frac{\partial^2 F_i}{\partial\zeta\,\partial\xi}\frac{d\xi}{d\zeta} = 0 \quad (i = 1, 2).$$

† The relations which are needed are the following:

$$\frac{\partial F_1}{\partial\eta} = \begin{vmatrix} \xi & \zeta \\ W_{\xi\eta} & W_{\zeta\eta} \end{vmatrix}, \quad \frac{\partial F_1}{\partial\xi} = \begin{vmatrix} \xi & \zeta \\ W_{\xi\xi} & W_{\zeta\xi} \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ W_\xi & W_\zeta \end{vmatrix}, \quad \frac{\partial F_1}{\partial\zeta} = \begin{vmatrix} \xi & \zeta \\ W_{\xi\zeta} & W_{\zeta\zeta} \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ W_\xi & W_\zeta \end{vmatrix},$$

$$\frac{\partial F_2}{\partial\eta} = \begin{vmatrix} \zeta & \eta \\ W_{\zeta\eta} & W_{\eta\eta} \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ W_\zeta & W_\eta \end{vmatrix}, \quad \frac{\partial F_2}{\partial\xi} = \begin{vmatrix} \zeta & \eta \\ W_{\zeta\xi} & W_{\eta\xi} \end{vmatrix}, \quad \frac{\partial F_2}{\partial\zeta} = \begin{vmatrix} \zeta & \eta \\ W_{\zeta\zeta} & W_{\eta\zeta} \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ W_\zeta & W_\eta \end{vmatrix},$$

$$\frac{\partial^2 F_1}{\partial\zeta^2} = \begin{vmatrix} \xi & \zeta \\ W_{\xi\zeta\zeta} & W_{\zeta\zeta\zeta} \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ W_{\xi\zeta} & W_{\zeta\zeta} \end{vmatrix} - W_{\xi\zeta},$$

$$\frac{\partial^2 F_2}{\partial\zeta^2} = \begin{vmatrix} \zeta & \eta \\ W_{\zeta\zeta\zeta} & W_{\eta\zeta\zeta} \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ W_{\zeta\zeta} & W_{\eta\zeta} \end{vmatrix} + W_{\eta\zeta};$$

$$\therefore \left(\frac{\partial F_1}{\partial\eta}\right)_0 = 0, \quad \left(\frac{\partial F_1}{\partial\xi}\right)_0 = W^0_\zeta, \quad \left(\frac{\partial F_1}{\partial\zeta}\right)_0 = -W^0_\xi = 0, \quad \frac{\partial^2 F_1}{\partial\zeta^2} = -2W^0_{\xi\zeta},$$

$$\left(\frac{\partial F_2}{\partial\eta}\right)_0 = -W^0_\zeta, \quad \left(\frac{\partial F_2}{\partial\xi}\right)_0 = 0, \quad \left(\frac{\partial F_2}{\partial\zeta}\right)_0 = W^0_\eta = 0, \quad \frac{\partial^2 F_2}{\partial\zeta^2} = 2W^0_{\eta\zeta}.$$

plumb-line which is supported at P_0 is in equilibrium under the tension of the plumb-line and the force of the field whose potential function is represented by W. The line RP_0 is then the vertical at R. Let us denote by RE_R and RS_R the easterly and the southerly directions at R. Now let us draw through P_0 the axes $P_0\bar{\eta}$, $P_0\bar{\xi}$, $P_0\bar{\zeta}$ parallel to RE_R, RS_R and RP_0 respectively ($P_0\bar{\zeta}$ is the continuation of the plumb-line RP_0). Let us denote by $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$ and $(\alpha_3, \beta_3, \gamma_3)$ the directional angles of the axes $P_0 - \bar{\eta}, \bar{\xi}, \bar{\zeta}$ with respect to the cardinal axes $P_0 - \eta, \xi, \zeta$. Then

$$\begin{aligned} \bar{\eta} &= \cos\alpha_1 \cdot \eta + \cos\beta_1 \cdot \xi + \cos\gamma_1 \cdot \zeta, \\ \bar{\xi} &= \cos\alpha_2 \cdot \eta + \cos\beta_2 \cdot \xi + \cos\gamma_2 \cdot \zeta, \qquad (14) \\ \bar{\zeta} &= \cos\alpha_3 \cdot \eta + \cos\beta_3 \cdot \xi + \cos\gamma_3 \cdot \zeta. \end{aligned}$$

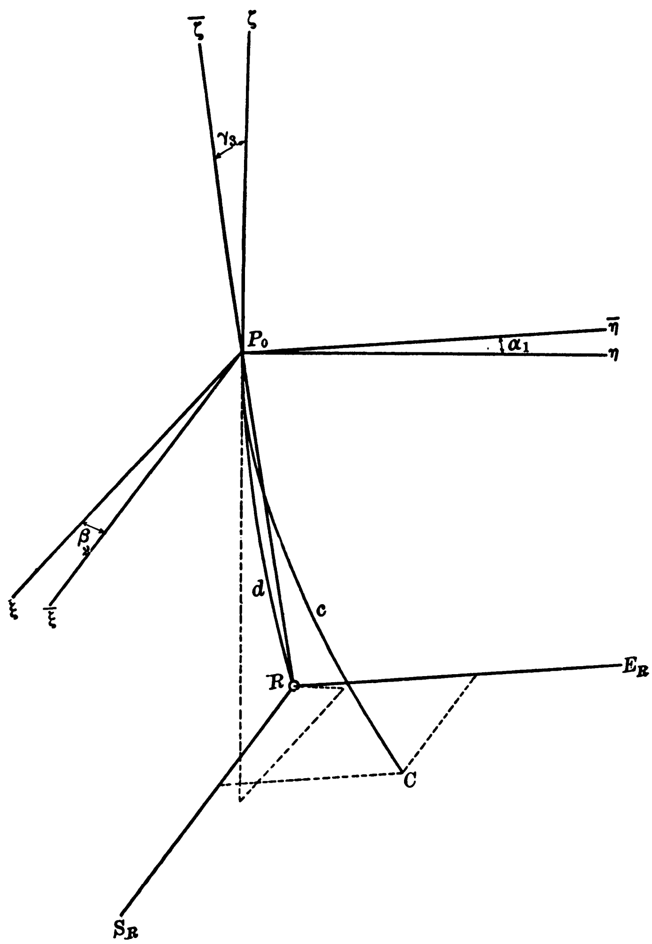

Fig. 3.

In order to obtain expressions for these directional cosines let us recall that, at a general point P, (1) the vertical is the normal to the level (equipotential) surface which passes through P, (2) the east-and-west line is the intersection of the horizontal plane and the plane ($z =$ const.) which passes through P and is perpendicular to the axis of rotation of the earth, (3) the north-and-south line is the common perpendicular to the vertical and the east-and-west line. Therefore, at a general point P, (η, ξ, ζ), (1) the directional cosines * of the vertical are proportional to

$$W_\eta, \quad W_\xi, \quad W_\zeta;$$

(2) those of the east-and-west line are proportional to

$$H, \quad \Xi, \quad Z,$$

where

$$H = \begin{vmatrix} W_\xi & W_\zeta \\ \dfrac{\partial z}{\partial \xi} & \dfrac{\partial z}{\partial \zeta} \end{vmatrix} = \sin\phi \cdot W_\xi + \cos\phi \cdot W_\zeta,$$

$$\Xi = \begin{vmatrix} W_\zeta & W_\eta \\ \dfrac{\partial z}{\partial \zeta} & \dfrac{\partial z}{\partial \eta} \end{vmatrix} = -\sin\phi \cdot W_\eta,$$

$$Z = \begin{vmatrix} W_\eta & W_\xi \\ \dfrac{\partial z}{\partial \eta} & \dfrac{\partial z}{\partial \xi} \end{vmatrix} = -\cos\phi \cdot W_\eta;$$

and (3) those of the north-and-south line are proportional to

$$\begin{vmatrix} W_\xi & W_\zeta \\ \Xi & Z \end{vmatrix}, \quad \begin{vmatrix} W_\zeta & W_\eta \\ Z & H \end{vmatrix}, \quad \begin{vmatrix} W_\eta & W_\xi \\ H & \Xi \end{vmatrix}.$$

If the point P is not too far from P_0, we may write without appreciable error,

$$\begin{aligned} W_\eta &= \qquad W^0_{\eta\eta} \cdot \eta + W^0_{\eta\xi} \cdot \xi + W^0_{\eta\zeta} \cdot \zeta, \\ W_\xi &= \qquad W^0_{\xi\eta} \cdot \eta + W^0_{\xi\xi} \cdot \xi + W^0_{\xi\zeta} \cdot \zeta, \\ W_\zeta &= W^0_\zeta + W^0_{\zeta\eta} \cdot \eta + W^0_{\zeta\xi} \cdot \xi + W^0_{\zeta\zeta} \cdot \zeta. \end{aligned}$$

Therefore, for the point R, whose coordinates are

$$\eta = \frac{W^0_{\eta\zeta}}{W^0_\zeta}\zeta^2, \qquad \xi = \frac{W^0_{\xi\zeta}}{W^0_\zeta}\zeta^2, \qquad \zeta = \zeta,$$

we may write

$$W^R_\eta = W^0_{\eta\zeta} \cdot \zeta, \qquad W^R_\xi = W^0_{\xi\zeta} \cdot \zeta, \qquad W^R_\zeta = W^0_\zeta + W^0_{\zeta\zeta} \cdot \zeta,$$

* With respect to the cardinal directions of P_0.

and hence

$$\cos\alpha_3 : \cos\beta_3 : \cos\gamma_3 = W^0_{\eta\zeta}\cdot\zeta : W^0_{\xi\zeta}\cdot\zeta : W^0_\zeta + W^0_{\zeta\zeta}\cdot\zeta,$$

$\cos\alpha_1 : \cos\beta_1 : \cos\gamma_1$

$$= \cos\phi\cdot W^0_\zeta + (\sin\phi\cdot W^0_{\xi\zeta} + \cos\phi\cdot W^0_{\zeta\zeta})\,\zeta : -\sin\phi\cdot W^0_{\eta\zeta}\cdot\zeta : -\cos\phi\cdot W^0_{\eta\zeta}\cdot\zeta,$$

$\cos\alpha_2 : \cos\beta_2 : \cos\gamma_2$

$$= \sin\phi\cdot W^0_\zeta\cdot W^0_{\eta\zeta}\cdot\zeta : \cos\phi\,(W^0_\zeta)^2$$
$$+ W^0_\zeta(\sin\phi\cdot W^0_{\xi\zeta} + 2\cos\phi\cdot W^0_{\zeta\zeta})\,\zeta : -\cos\phi\cdot W^0_\zeta\cdot W^0_{\xi\zeta}\cdot\zeta,$$

for terms of order not greater than the first. In order to get the cosines themselves, we observe that *

$$\left((W^0_{\eta\zeta}\cdot\zeta)^2 + (W^0_{\xi\zeta}\cdot\zeta)^2 + (W^0_\zeta + W^0_{\zeta\zeta}\,\zeta)^2\right)^{-\frac{1}{2}} = \frac{1}{W^0_\xi}\left(1 - \frac{W^0_{\zeta\zeta}}{W^0_\zeta}\zeta + \cdots\right),$$

$$\Big(\{\cos.\phi\cdot W^0_\zeta + (\sin\phi\cdot W^0_{\xi\zeta} + \cos\phi\cdot W^0_{\zeta\zeta})\,\zeta\}^2 + (-\sin\phi\cdot W^0_{\eta\zeta}\cdot\zeta)^2$$
$$+ (-\cos\phi\cdot W^0_{\eta\zeta}\cdot\zeta)^2\Big)^{-\frac{1}{2}} = \frac{1}{\cos\phi\cdot W^0_\zeta}\left(1 - \frac{\sin\phi\cdot W^0_{\xi\zeta} + \cos\phi\cdot W^0_{\zeta\zeta}}{\cos\phi\cdot W^0_\zeta}\zeta + \cdots\right),$$

$$\Big((\sin\phi\cdot W^0_\zeta\cdot W^0_{\eta\zeta}\cdot\zeta)^2 + \{\cos\phi\cdot(W^0_\zeta)^2 + W^0_\zeta(\sin\phi\cdot W^0_{\xi\zeta}$$
$$+ 2\cos\phi\cdot W^0_{\zeta\zeta})\,\zeta\}^2 + (-\cos\phi\cdot W^0_\zeta\cdot W^0_{\xi\zeta}\cdot\zeta)^2\Big)^{-\frac{1}{2}}$$
$$= \frac{1}{\cos\phi\cdot(W^0_\zeta)^2}\left(1 - \frac{\sin\phi\cdot W^0_{\xi\zeta} + 2\cos\phi\cdot W^0_{\zeta\zeta}}{\cos\phi\cdot W^0_\zeta}\zeta + \cdots\right).$$

Therefore

$$\cos\alpha_3 = \frac{W^0_{\eta\zeta}}{W^0_\zeta}\zeta,\qquad \cos\beta_3 = \frac{W^0_{\xi\zeta}}{W^0_\zeta}\zeta,\qquad \cos\gamma_3 = 1,$$

$$\text{(15)}\qquad \cos\alpha_1 = 1,\qquad \cos\beta_1 = -\tan\phi\,\frac{W^0_{\eta\zeta}}{W^0_\zeta}\zeta,\qquad \cos\gamma_1 = -\frac{W^0_{\eta\zeta}}{W^0_\zeta}\cdot\zeta,$$

$$\cos\alpha_2 = \tan\phi\cdot\frac{W^0_{\eta\zeta}}{W^0_\zeta}\cdot\zeta,\qquad \cos\beta_2 = 1,\qquad \cos\gamma_2 = -\frac{W^0_{\xi\zeta}}{W^0_\zeta}\cdot\zeta,$$

for terms of order not higher than the first. In virtue of the values (15), equations (14) assume the form:

$$\bar\eta = \eta - \tan\phi\,\frac{W^0_{\eta\zeta}}{W^0_\zeta}\,\xi\zeta - \frac{W^0_{\eta\zeta}}{W^0_\zeta}\zeta^2,$$

$$\text{(16)}\qquad \bar\xi = \tan\phi\cdot\frac{W^0_{\eta\zeta}}{W^0_\zeta}\,\eta\zeta + \xi - \frac{W^0_{\xi\zeta}}{W^0}\zeta^2,$$

$$\bar\zeta = \frac{W^0_{\eta\zeta}}{W^0_\zeta}\,\eta\zeta + \frac{W^0_{\xi\zeta}}{W^0_\zeta}\,\xi\zeta + \zeta,$$

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* $\dfrac{1}{\sqrt{A + B\zeta + C\zeta^2 + \cdots}} = \dfrac{1}{A^{\frac{1}{2}}}\left(1 - \dfrac{1}{2}\dfrac{B}{A}\zeta + \dfrac{3B^2 - 4AC}{8A^2}\zeta^2 + \cdots\right).$

in which we must now substitute for η, ξ, ζ the values in t which are given by equations (10). Doing this, we find *

$$\bar{\eta} = -\tfrac{1}{3}\omega\cos\phi\cdot W^0_\zeta\cdot t^3 - \tfrac{5}{24}W^0_\zeta\cdot W^0_{\eta\zeta}\cdot t^4 - \tfrac{1}{60}\,\omega\,W^0_\zeta\,(\sin\phi\cdot W^0_{\xi\zeta} + \cos\phi\,(W^0_{\zeta\zeta} + W^{0\eta}_{\eta} - 4\omega^2))\,t^5,$$

$$(17)\quad \bar{\xi} = -\left(\tfrac{1}{6}\,\omega^2\sin\phi\cos\phi + \tfrac{5}{24}W^0_{\xi\zeta}\right)W^0_\zeta\cdot t^4 - \tfrac{1}{60}\,\omega\,W^0_\zeta\,(9\sin\phi\,W^0_{\eta\zeta} + \cos\phi\,W^0_{\xi\eta})\,t^5,$$

$$\bar{\zeta} = \tfrac{1}{2}W^0_\zeta\cdot t^2 + \tfrac{1}{24}\,(W^0_{\zeta\zeta} - 4.\omega^2\cos^2\phi)\,W^0_\zeta\cdot t^4 - \tfrac{1}{6}\,\omega\cos\phi\,W^0_\zeta W^0_{\eta\zeta}\,t^5.$$

Equations (17) are the equations of the curve c referred to the axes $P_0 - \bar{\eta}, \bar{\xi}, \bar{\zeta}$.

If we put $-\bar{\zeta} = h = R\,P_0$, where h is the height through which the particle falls, then the corresponding values of $\bar{\eta}$ and $\bar{\xi}$ are the *easterly* and the *southerly deviations* which correspond to h. These, and *not* the quantities η, ξ, are the ones which are measured in experiments.

Washington University,
St. Louis, Mo., *April*, 1912.

* We can not carry these developments beyond t^5 unless we carry the developments (15) beyond ζ.