ON THE CHARACTER OF A TRANSFORMATION IN THE NEIGHBORHOOD OF A POINT WHERE ITS JACOBIAN VANISHES*

BY

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If we have a transformation of n variables

$$T: \qquad y_i = \varphi_i(x_1, \cdots, x_n) \qquad (i = 1, \cdots, n),$$

where in the x domain under consideration the functions φ are single-valued, and either (A) analytic functions of complex variables, or else (B) real functions of real variables with continuous partial derivatives of the first order, the character of the transformation is well known for the neighborhood of an interior point $a = (a_1, \cdots, a_n)$ in the x domain for which $J(a_1, \cdots, a_n) \neq 0$, where J is the Jacobian of the transformation,

$$J(x_1, \cdots, x_n) \equiv \frac{D(\varphi_1, \cdots, \varphi_n)}{D(x_1, \cdots, x_n)}.$$

Thus we know that in a sufficiently small neighborhood of a the transformation T possesses a single-valued inverse, having the property (A) when T has it, and the property (B) when T has it. We also know that if the x's are connected by one or more functional relations, the values at a of the resulting ordinary or partial derivatives of the x's with regard to one another or to parameters are transformed linearly by T into similar derivatives of the y's, if we make the provision in case (B) that the φ's shall have continuous derivatives of as high an order as those to be transformed. This fact as applied to first derivatives is commonly stated in the form that in a small neighborhood of the point a the transformation T is approximately projective. When $n = 2$ or 3 it leads to the geometric theorem that the order of contact of two curves or surfaces through a is unchanged by the transformation T.†

* Presented to the Society, October 26, 1912. This paper deals with a generalization of a theorem for two real variables included in the thesis presented by the author for the doctor's degree at Harvard University in 1909. This generalization was made at the suggestion of Professor Bouton, under whose direction the thesis was written. See also articles by G. R. CLEMENTS, Bulletin of the American Mathematical Society, vol. 18 (June, 1912), p. 455, Theorem X, and by S. E. URNER, these Transactions, vol. 13 (April, 1912), p. 232.

† Any of these statements which are not made explicitly in the standard treatises may be easily verified by computation. It should be noted that the proofs ordinarily given assume that the point a is an interior point of the region considered. It is easy, however, to modify the proofs so that they hold when a is a boundary point, provided the boundary locus has no singularity at a.

If on the other hand $J(a_1, \cdots, a_n) = 0$, none of these statements is necessarily true. The point a will then be called a *singular* point of the transformation; and a transformation will be said to be *regular* in a region in which it is free from singular points. The purpose of the present paper is to indicate the general character of the transformation in the neighborhood of a singular point by showing that, in general, it is essentially similar to the transformation

$$(1) \qquad \begin{aligned} y_1 &= x_1^2, \\ y_i &= x_i \qquad (i = 2, \cdots, n), \end{aligned}$$

in the neighborhood of a point where $x_1 = 0$. By "essentially similar" here is meant, differing only in properties which are altered even by a regular transformation. That is to say, the transformation of the neighborhood of a singular point by T can be obtained, in general, as the result of three transformations, of which the first and third are regular and the second has the form (1). This statement, however, is subject to the restriction of the non-vanishing at a of at least one of the n functional determinants which may be obtained from J by replacing one of the functions φ by J itself. As a matter of notation this function is assumed to be φ_1 in the explicit statement of the theorems which follow. For brevity the notation x will be used for $(x_1, \cdots, x_n)$, $y = \varphi(x)$ for the transformation T, and so on.

THEOREM 1: *If in the transformation T the functions φ are single-valued and analytic in the neighborhood of a point a, and at this point $J = 0$ but the determinant*

$$J_1(x) \equiv \frac{D(J, \varphi_2, \cdots, \varphi_n)}{D(x_1, \cdots, x_n)} \neq 0,$$

then there is a neighborhood of a in which the tranformation T can be broken up into three successive transformations, of which the first and third are regular and analytic and the second is

$$T_1: \qquad \begin{aligned} y_1' &= x_1'^2, \\ y_i' &= x_i' \qquad (i = 2, \cdots, n). \end{aligned}$$

To prove this, consider first the transformation

$$R_1: \qquad \begin{aligned} x_1'' &= J(x), \\ x_i'' &= \varphi_i(x) \qquad (i = 2, \cdots, n). \end{aligned}$$

This is analytic near a and its Jacobian is $J_1(x)$, which does not vanish at a. There will therefore be a neighborhood D of this point in which the transformation R_1 will have a single-valued analytic inverse. Let this be

$$R_1^{-1}: \qquad x = \psi(x'').$$

Its Jacobian is $1/J_1$. We then have

$$\varphi_1(x) = \varphi_1(\psi(x'')) = \omega(x''),$$

a single-valued analytic function of x'' throughout the x'' region which is the image of D. Hence y is connected with x'' by the transformation

$$R_1^{-1}T: \quad \begin{aligned} y_1 &= \omega(x''), \\ y_i &= x_i'' \qquad (i = 2, \cdots, n). \end{aligned}$$

Of this transformation the Jacobian is

$$\frac{\partial \omega}{\partial x_1''} = \frac{J}{J_1} = \frac{x_1''}{J_1},$$

the second form being derived from the theorem that the Jacobian of the product of two transformations is equal to the product of their Jacobians. Therefore

$$\frac{\partial^2 \omega}{\partial x_1''^2} = \frac{J_1 - x_1'' \dfrac{\partial J_1}{\partial x_1''}}{J_1^2}.$$

When $x_1'' = 0$ this becomes $1/J_1 \neq 0$. Hence, regarding y_1 as a function of x_1'', we may write

$$\omega(x_1'', \cdots, x_n'') - \omega(0, x_2'', \cdots, x_n'') = x_1''^2 \chi(x_1'', \cdots, x_n'') = x_1''^2 \chi(x''),$$

where χ is a single-valued analytic function which remains different from zero in some neighborhood D_1 of a'', where a'' is the x'' image of a. In D_1 consider the transformation

$$R_2: \quad \begin{aligned} x_1' &= x_1'' \sqrt{\chi(x'')}, \\ x_i' &= x_i'', \qquad (i = 2, \cdots, n). \end{aligned}$$

where the radical denotes *one* of the two distinct functions whose square is χ and which are not connected by any branch point in D_1. The Jacobian of R_2 is

$$\frac{\partial x_1'}{\partial x_1''} = \sqrt{\chi} + \frac{x_1''}{2\sqrt{\chi}} \frac{\partial \chi}{\partial x_1''},$$

which is different from zero when $x_1'' = 0$. The transformation R_2 is therefore a regular analytic transformation in the neighborhood of a''. Let R denote the product of the two transformations R_1 and R_2, i. e., the result of applying them in succession. Then there is a neighborhood of a in which R is a regular analytic transformation because R_1 and R_2 are such. We shall show that R may be taken as the first of the three transformations mentioned in the theorem.

To do this it is only necessary to prove that if y' is defined by the trans-

formation T_1 then y is connected with y' by a regular transformation in the neighborhood of b', where b' is the y' image of a. Now since

$$y_1' = x_1'^2 = x_1''^2 \chi(x'') = \omega(x_1'', \cdots, x_n'') - \omega(0, x_2'', \cdots, x_n'')$$
$$= y_1 - \omega(0, y_2', \cdots, y_n'),$$

it follows that y and y' are connected by the transformation

$$S: \quad \begin{aligned} y_1 &= y_1' + \omega(0, y_2', \cdots, y_n'), \\ y_i &= y_i' \qquad (i = 2, \cdots, n). \end{aligned}$$

In the neighborhood of b' this is clearly single-valued and has a Jacobian equal to unity. We have therefore effected the resolution

$$T = RT_1 S,$$

where R, T_1, and S fulfil the requirements of the theorem.

THEOREM 2: *If in the transformation T the functions φ are real and single-valued and have continuous partial derivatives of the fourth order in the neighborhood of the point a, and at this point $J = 0$ but $J_1 \neq 0$, then there is a neighborhood of a in which the transformation T can be broken up into three successive transformations of which the first and third are regular and the second is T_1.*

The proof of this is the same as that of Theorem 1, except for a few changes, of which all are obvious except perhaps one change in the transformation R_2. Here if $\chi(a'')$ is negative we must set $x_1' = x_1''\sqrt{-\chi(x'')}$ in order that x_1' may be real.

From these two theorems we may make several inferences about the character of the transformation T at any singular point where $J_1 \neq 0$. In the first place if a is such a singular point and $b = \varphi(a)$ is its image, then all the points in the neighborhood of a which are not themselves singular points are grouped in pairs such that the two points of a pair are transformed into the same point. In the case of reals the two points of a pair lie on opposite sides of the Jacobian locus, i. e., the locus $J(x) = 0$, and all points of the neighborhood are transformed into points on the same side of the image of this locus which we shall call $\bar{J}(y) = 0$.*

Moreover, in the neighborhood of b, the inverse transformation T^{-1} is in the complex case everywhere two-valued except on $\bar{J}(y) = 0$, where the two values coincide. In the real case, the inverse exists only on one side of $\bar{J}(y) = 0$ and on this locus itself, and is two-valued except on this locus. These properties hold at any singular point of T where $J_1 \neq 0$ because they

* This fact for the transformation of two real variables was stated (without proof) by G. A. BLISS at the Princeton Colloquium in September, 1909. It had also been communicated previously by Professor Bliss to the author of this paper.

hold at any singular point of T_1 and are unaltered by a regular transformation. In the same way we can infer that all the curves through a except those having one particular direction are transformed by T into curves which are tangent to the locus $\bar{J}(y) = 0$; also that the neighborhood of $J(x) = 0$ is flattened down by the transformation into a much narrower neighborhood of $\bar{J}(y) = 0$, the degree of flattening increasing as the original neighborhood becomes narrower; so that whereas the transformation of the neighborhood of an ordinary point approximates to an ordinary projective transformation, that of a singular point where $J_1 \neq 0$ approximates to a degenerate projective transformation.

Although the proof of Theorem 2 requires the existence of partial derivatives of the fourth order for the functions φ, the properties that we have just derived for the neighborhood of a general singular point hold also if we assume only the continuity of the second order partial derivatives of the functions φ, as may be derived from the following theorem.

THEOREM 3: *If in the transformation T the functions φ are real and single-valued and have continuous partial derivatives of the second order in the neighborhood of the point a, and at this point $J = 0$ but $J_1 \neq 0$, then there is a neighborhood of a in which the transformation T can be resolved into three successive transformations, of which the first is regular, the second is T_1, and the third has a non-vanishing Jacobian but is in general two-valued.*

Let the first transformation be

$$P: \qquad \begin{aligned} x_1' &= J(x), \\ x_i' &= \varphi_i(x) \qquad (i = 2, \cdots, n). \end{aligned}$$

This has the Jacobian J_1 which does not vanish near a. Let its inverse be

$$P^{-1}: \qquad x = \psi(x'),$$

a regular transformation with the Jacobian $1/J_1$. Let the second transformation be

$$T_1: \qquad \begin{aligned} y_1' &= x_1'^2, \\ y_i' &= x_i' \qquad (i = 2, \cdots, n), \end{aligned}$$

of which the Jacobian is $2x_1'$. Here the Jacobian locus $x_1' = 0$ is the image of $J(x) = 0$. The inverse of this is

$$T_1^{-1}: \qquad \begin{aligned} x_1' &= \pm\sqrt{y_1'}, \\ x_i' &= y_i' \qquad (i = 2, \cdots, n), \end{aligned}$$

which is of course two-valued and has the Jacobian $1/2x_1'$, a two-valued

function of y'. The transformation from y' to y must then be $T_1^{-1} P^{-1} T$, or

$$Q: \quad \begin{aligned} y_1 &= \varphi_1 (\psi (\pm \sqrt{y_1'}, y_2', \cdots, y_n')), \\ y_i &= y_i' \qquad\qquad (i = 2, \cdots, n). \end{aligned}$$

This has the Jacobian

$$\frac{1}{2x_1'} \cdot \frac{1}{J_1} \cdot J = \frac{x_1'}{2x_1' J_1} = \frac{\partial y_1}{\partial y_1'},$$

which is equal to $1 / 2J_1$ except perhaps when $y_1' = 0$. We can see, however, that y_1 is a continuous function of y_1' when $y_1' = 0$, and since its derivative approaches a limit at this point, the derivative also exists at this point and is equal to the limit.* Therefore $\partial y_1 / \partial y_1'$, which is the Jacobian of Q, is continuous and different from zero when $y_1' = 0$.

Hence in Q each sign of the radical gives a single-valued *regular* transformation up to *and including* $y_1' = 0$.† Moreover, on the locus $y_1' = 0$ these two transformations coincide as regards the values of the variables and of all the first partial derivatives. One of them gives the transformation of points in the y' domain which is derived from their correspondence with points on one side of $x_1' = 0$, the other that derived from their correspondence with points on the other side. Hence one of them is to be associated with one inverse of T_1, the other with the other. Thus any point y which corresponds to a point in the y' domain where y_1' is positive, is the image of two points in the x domain, one on each side of $J = 0$. In like manner we can make all the inferences from Theorem 3 that we did from Theorem 2.

Thus the properties that we derived for a singular point of T where $J_1 \neq 0$ hold if we know that the functions in T have continuous partial derivatives of the second order. The author has not been able, however, to find *single-valued* transformations for reducing T to T_1 without assuming the continuity of the fourth order derivatives in T.

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* DINI, *Funzioni di variabili reali*, § 75; or E. W. HOBSON, *Theory of functions of a real variable*, § 220.

† Since this is a boundary point of the region of definition for Q we need the modified proofs referred to in the second footnote of this article.

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