ON TRANSCENDENTALLY TRANSCENDENTAL FUNCTIONS*

BY

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1. Introduction.—Following the classification made by E. H. MOORE in his fundamental paper† on transcendentally transcendental functions, we shall say that a transcendental function y of x is algebraically or transcendentally transcendental according as it does or does not satisfy a differential equation of the form

$$H(x, y, y^{(1)}, \cdots, y^{(k)}) = 0, \qquad y^{(i)} = \frac{d^i y}{dx^i} \quad (i = 1, \cdots, k),$$

where H is a polynomial in $x, y, y^{(1)}, \cdots, y^{(k)}$ with constant coefficients. The existence of transcendentally transcendental functions was first shown by HÖLDER,‡ who proved that the gamma function belongs to this class of functions. HURWITZ§ obtained important results concerning these functions and pointed out a special example. Others who have written on the subject are GRONWALL,|| BARNES,¶ TIETZE,** MORDOUCHAY-BOLTOVSKY,†† and STRIDSBERG.‡‡

The principal result of this paper is stated in section 2 as a general theorem; it is in the nature of an extension of a result due to HURWITZ. It furnishes an immediate means of writing out an unlimited number of transcendentally transcendental functions. In section 3, by means of number-theoretic considerations taken in connection with preceding results, I prove the transcendentally transcendental character of the class of functions defined in equation (5) below; and in section 4 special cases of these functions are given which satisfy functional equations of simple type.

* Presented to the Society (Chicago) December 30, 1911.

† Mathematische Annalen, vol. 48 (1896), pages 49-74.

‡ Mathematische Annalen, vol. 28 (1887), pages 1-13.

§ Annales de l'école normale supérieure, ser. 3, vol. 6 (1889), pages 327-332.

|| Öfersigt K. Svenska Vetenskap. Akad. Förh., 1898.

¶ Proceedings of the London Mathematical Society, ser. 2, vol. 2 (1904), pages 280-292. See references in this paper.

** Monatshefte für Mathematik und Physik, vol. 16 (1905), pages 329-364. See the introduction to this paper for further references.

†† Matematischiski Sbornik, vol. 27 (1909), pages 360-406.

‡‡ Arkiv för Mat., Astr. och Fys., vol. 6 (1910), numbers 15 and 18.

2. General Theorem.—Let us consider the function of x defined by the convergent infinite power series*

$$(1) \qquad y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots,$$

where $a_0, a_1, a_2, \cdots$ are rational fractions in their lowest terms. We seek properties of the coefficients a_n which are necessary if y is to satisfy an algebraic differential equation, that is, an equation of the form

$$(2) \qquad P(x, y, y^{(1)}, \cdots, y^{(k)}) = 0,$$

where P is a polynomial in $x, y, y^{(1)}, \cdots, y^{(k)}$. It is clear that if y is an integral of any such differential equation, it is an integral of one of the same form in which the coefficients are restricted to be integers. Accordingly, we suppose that (2) has this property. Let us assume further, as we may without loss of generality, that y satisfies no such equation of the form (2) and of order lower than k, and that the degree of (2) in $y^{(k)}$ is equal to or less than the degree in $y^{(k)}$ of any other such equation of order k which y satisfies.

If we differentiate (2) with respect to x an appropriate number of times, and in the result put $x = 0$ and then clear of fractions, we have finally (see HURWITZ, l. c., p. 329, equation (8)) an equation of the form

$$(3) \qquad y_0^{(n)}(b_0 + b_1 n + \cdots + b_\gamma n^\gamma) = G_n(y_0, y_0^{(1)}, \cdots, y_0^{(n-1)})$$

where $b_0, b_1, \cdots, b_\gamma, \gamma$ are integers independent of n, and G_n is a polynomial in $y_0, y_0^{(1)}, \cdots, y_0^{(n-1)}$ with integral coefficients; the equation being valid and sufficing for the determination of $y_0^{(n)}$ for every value of n greater than some particular integer N. Here $y_0, y_0^{(1)}, \cdots, y_0^{(n)}$ are what $y, y^{(1)}, \cdots, y^{(n)}$ become when $x = 0$.

An upper limit to the degree of G_n in its arguments is readily found. Suppose that equation (2) is of degree t in $y, y^{(1)}, \cdots, y^{(k)}$. Then, after any number of differentiations, the resulting equation is of degree t (at most) in y and its derivatives. If in this we put $x = 0$, the equation so obtained is at most of degree t in $y_0, y_0^{(1)}, y_0^{(2)}, \cdots$. Hence the degree of G_n in its arguments $y_0, y_0^{(1)}, \cdots, y_0^{(n-1)}$ is at most t.

Now let m be any integer such that, for all values of n equal to or greater than m, equation (3) is valid and the polynomial

$$h(n) = b_0 + b_1 n + \cdots + b_\gamma n^\gamma$$

is different from zero. Let D be the least common multiple of the denominators of $y_0, y_0^{(1)}, \cdots, y_0^{(m-1)}$. Then form the value of $y_0^{(m+s)}$, $s \geqq 0$, by

* By an "infinite power series" I mean a power series with an infinite number of coefficients different from zero. A zero coefficient in (1) we shall suppose to be written in the form $0/1$.

means of a successive use of (3); we shall show by induction that the result may be written in the form

$$(4)\quad y_0^{(m+s)} = \frac{\bar{A}_{m+s}}{D^{t^{s+1}}[h(m)]^{t^s}[h(m+1)]^{t^{s-1}}\cdots[h(m+s-1)]^t h(m+s)},$$

where $\bar{A}_{m+s}$ is an integer. For $s = 0$, (4) takes the form

$$y_0^{(m)} = \frac{\bar{A}_m}{D^t h(m)}.$$

That this relation is true follows from (3) by putting $n = m$. Consequently the proof of (4) may be completed by showing that if (4) is true for all values of s equal to or less than a given value σ, it is also true for $s = \sigma + 1$. From (3) we have

$$y_0^{(m+\sigma+1)} = \frac{1}{h(m+\sigma+1)} G_{m+\sigma+1}(y_0, y_0^{(1)}, \cdots, y_0^{(m+\sigma)}),$$

where $G_{m+\sigma+1}$ is a polynomial of degree t (at most) in its arguments. The rational numbers $y_0, y_0^{(1)}, \cdots, y_0^{(m-1)}$ have a common denominator D. Hence if we assume (4) for $s = 0, 1, 2, \cdots, \sigma$, we see that the rational numbers $y_0, y_0^{(1)}, \cdots, y_0^{(m+\sigma)}$ have as a common denominator the denominator in the second member of (4) for $s = \sigma$. Writing each of these numbers as a fraction having this common denominator, and substituting in the last equation above (remembering that G is of degree t at most in its arguments), we are led at once to an equation of the form (4) in which s is replaced by $\sigma + 1$. Hence the proof of (4) is complete.

Now if we put

$$g(x) = (m-1)!\,D^t x(b_0 + b_1 x + \cdots + b_\gamma x^\gamma) \equiv c_0 + c_1 x + \cdots + c_\alpha x^\alpha$$

and remember that $a_n = y_0^{(n)}/n!$, we see that (4) leads at once to the following theorem:*

THEOREM: *If the function y of x defined by the convergent series*

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots,$$

where $a_0, a_1, a_2, \cdots$ are rational fractions (which we take to be in their lowest terms), is an integral of an algebraic differential equation whose coefficients are rational functions of x, then there exist positive integers m and t, and a polynomial

$$g(x) = c_0 + c_1 x + \cdots + c_\alpha x^\alpha$$

whose coefficients $c_0, c_1, \cdots, c_\alpha$ are integers independent of x, such that a_n,

* This theorem is in the nature of an extension of the result stated by HURWITZ in the footnote on p. 330 of the paper already referred to, the extension consisting essentially in the determination of Hurwitz's constants $\alpha_0, \alpha_1, \cdots$ and $\beta_0, \beta_1, \cdots$.

$n > m$, has the value

$$a_n = \frac{A_n}{[g(m)]^{t^{n-m}}[g(m+1)]^{t^{n-m-1}}\cdots[g(n-1)]^t g(n)},$$

where A_n is an integer and no one of the numbers $g(m)$, $g(m+1)$, $\cdots$ is zero.

The integer t is the degree in $y, y^{(1)}, \cdots, y^{(k)}$ of the polynomial P in equation (2) defined as in the first paragraph of this section.

COROLLARY I: *For such a function y there exists a positive integer τ such that the denominator [numerator] of a_n, for $n > 1$, is in absolute value less than n^{τ^n}.*

The part of this corollary which refers to the denominators is an immediate consequence of the theorem and the fact that

$$\lim_{n=\infty} n^{-\tau^n}[g(m)]^{t^{n-m}}\cdots[g(n-1)]^t g(n) = 0$$

when $\tau > t$. The part referring to the numerators follows from the other part and the fact that the series for y is convergent for some x different from zero. (It may be necessary to take the value of τ greater in the latter case than in the former.)

COROLLARY II: *If $b_0, b_1, b_2, \cdots$ is any set of rational fractions in their lowest terms, an infinite number of which are different from zero, and if there exists an integer τ such that the numerator of b_n, for increasing n, is ultimately less than n^{τ^n}, then each of the following series is permanently convergent and represents a transcendentally transcendental entire function*:

$$b_0 + b_1 x + \frac{b_2}{16}x^2 + \cdots + \frac{b_n}{n^{n^n}}x^n + \cdots,$$

$$b_0 + b_1 x + \frac{b_2}{24}x^2 + \cdots + \frac{b_n}{(n^n)!}x^n + \cdots,$$

$$b_0 + \frac{b_1}{a}x + \frac{b_2}{a^a}x^2 + \frac{b_3}{a^{a^a}}x^3 + \cdots.$$

In the last example, a is a positive integer equal to or greater than 2, and the denominator in the $(n+1)$th term is $a^{a^{\cdots}}$, a being repeated n times.

The proof follows at once from the fact that $n^{(t+\tau)^n}$ (t = any positive integer), which, for increasing n, is ultimately greater than n^{t^n} times the numerator of b_n, is at the same time ultimately less than n^{n^n} or $(n^n)!$ or $a^{a^{\cdots}}$. For when the coefficients of the powers of x in any one of the above series are reduced to their lowest terms their denominators fail to satisfy the condition of Corollary I, which is necessary if the function is an integral of an algebraic differential equation.

From the method of proof of this corollary it follows that any of these series will continue to represent a transcendentally transcendental function if any of its denominators are increased (in absolute value) in any way, provided that the resulting number is still rational. Various other forms of examples suggest themselves at once.

From Corollary I it follows readily that for the numbers $b_0, b_1, \cdots$ in Corollary II we may take any set of rational numbers (an infinite number of which are different from zero) such that $y = b_0 + b_1 x + b_2 x^2 + \cdots$ is an integral of an algebraic differential equation.

3. Examples of transcendentally transcendental functions.—In Corollary II above, and the remarks following it, we have suggested an unlimited number of examples of transcendentally transcendental functions. In the present section we obtain others of a different character.

From equation (3) the following result is readily deduced: *If y is defined as in the theorem above, then there exists an integer m and a polynomial $h(x) = b_0 + b_1 x + \cdots + b_\gamma x^\gamma$ whose coefficients are integers independent of x, such that every divisor d of the denominator of a_n which is prime to $n\,!$ and to the denominator of every a_i, $i < n$, is a factor of $h(n)$.* This theorem is practically proved by Hurwitz (l. c., pp. 329, 330) though not explicitlv stated. We have the following immediate consequence of this result:

LEMMA. *If y (as defined in the theorem above) is an integral of an algebraic differential equation, and if the denominator of a_n is written in the form $r_n s_n$ where s_n is prime to $n\,!$ and to the denominator of every a_i, $i < n$, then there exists an integer ϵ such that, for increasing n, s_n ultimately becomes and remains less than n^ϵ.*

This lemma we shall now employ to prove the following:

Let $b_0, b_1, b_2, \cdots$ be a set of rational fractions in their lowest terms, an infinite number of which are different from zero. Let them have the property that a number N exists such that, for $n > N$, neither the numerator nor the denominator of b_n (when $b_n \neq 0$) contains a prime factor greater than n. Let α and β be any relatively prime integers one of which is in absolute value greater than 1. Then the series

$$(5)\quad y = b_0 + \frac{b_1}{(\alpha-\beta)^{k_{11}}} x + \cdots + \frac{b_n}{(\alpha-\beta)^{k_{1n}}(\alpha^2-\beta^2)^{k_{2n}} \cdots (\alpha^n-\beta^n)^{k_{nn}}} x^n + \cdots,$$

if it converges, defines a function which is transcendentally transcendental. Here the numbers k_{ij} are integers (positive, negative, or zero) and k_{nn} is equal to or greater than 1 for an infinite number of values of n for which $b_n \neq 0$.

Let $n_1, n_2, n_3, \cdots$ be an infinite sequence of numbers greater than N and such that $k_{n_i n_i} \geqq 1$ while $b_{n_i} \neq 0$. Denote by β_n the coefficient of x^n in series (5), when reduced to its lowest terms, and let s_{n_i} be the greatest factor of the denominator of β_{n_i} which is prime to $n_i!$ and to the denominator of every b_k, $k < n_i$. We shall show first that s_{n_i}, for every i, contains the greatest factor F_i of $\alpha^{n_i} - \beta^{n_i}$ which is prime to every $\alpha^s - \beta^s$, $s < n_i$. It is well known* that every prime factor q of $\alpha^{n_i} - \beta^{n_i}$ which is not contained in any $\alpha^s - \beta^s$, $s < n_i$, is of the form $\lambda n_i + 1$. It is therefore greater than any factor of either the numerator or the denominator of any b_k, $k \gtreqqless n_i$, and hence it is not contained in any of these. Similarly, q is not contained in $n_i!$. From these results and the fact that q is prime to every $\alpha^s - \beta^s$, $s < n_i$, it follows that F_i is a factor of s_{n_i}.

In view of our lemma, the proof of the transcendentally transcendental character of y as defined in (5) will be completed if we show that for every integer ϵ we have

$$\lim_{n_i=\infty} \frac{n_i^\epsilon}{F_i} = 0.$$

To effect the proof of this we shall need certain properties of the number F_i, and these we shall now obtain.

Let $Q_n(x) = 0$ be the equation whose roots are the primitive nth roots of unity without repetition, the coefficient of the leading term in $Q_n(x)$ being unity. The degree of $Q_n(x)$ is $\varphi(n)$, where φ denotes Euler's φ-function. Form the quantity $Q_n(\alpha, \beta)$ defined by the equation

$$Q_n(\alpha, \beta) = \beta^{\phi(n)} Q_n\left(\frac{\alpha}{\beta}\right).$$

It is evident that $Q_n(\alpha, \beta)$ is a homogeneous form of degree $\varphi(n)$ in α, β. I have shown† that the greatest common divisor of $Q_n(\alpha, \beta)$ and $\alpha^{n/p} - \beta^{n/p}$ is 1 or p, p being any prime factor of n. From this it follows readily that $Q_n(\alpha, \beta)$ has no factor in common with the numbers $\alpha^s - \beta^s$, $s < n$, except possibly prime factors of n and their products. Hence it follows that

$$F_i \gtreqless \frac{|Q_{n_i}(\alpha, \beta)|}{n_i}.$$

Now suppose that ρ lies between the two quantities $|\alpha|$ and $|\beta|$ and is greater than 1. By using the product form for $Q_n(\alpha, \beta)$ obtained from the product form‡ for $Q_n(x)$, one may show without difficulty that we have

$$\lim_{n_i=\infty} \frac{\rho^{\phi(n_i)}}{|Q_{n_i}(\alpha, \beta)|} = \lim_{n_i=\infty} \frac{\Pi(\alpha_1^{n/p} - \beta_1^{n/p}) \cdot \Pi(\alpha_1^{n/pqr} - \beta_1^{n/pqr}) \cdots}{(\alpha_1^n - \beta_1^n)\, \Pi(\alpha_1^{n/pq} - \beta_1^{n/pq}) \cdots} = 0,$$

* Lucas, American Journal of Mathematics, vol. 1 (1878), p. 291.
† American Mathematical Monthly, vol. 16 (1909), pages 153–159.
‡ See Bachmann's *Kreisteilung*, p. 16.

where

$$\alpha_1 = \frac{\alpha}{\rho}, \qquad \beta_1 = \frac{\beta}{\rho},$$

and $p, q, r, \cdots$ are the different prime factors of n. Hence one sees readily that the proof of our theorem will be completed if we show that

$$\lim_{n_i=\infty} \frac{n_i^{\epsilon+1}}{\rho^{\phi(n_i)}} = 0$$

for every integer ϵ; and the existence and value of the last limit follow at once from the obvious fact that for increasing n_i

$$\rho^{\frac{\phi(n_i)}{\epsilon+2}}$$

ultimately becomes and remains greater than n_i. Thus we have completed the demonstration of the transcendentally transcendental character of y as defined in equation (5).

4. Transcendentally transcendental functions satisfying functional equations.—In the preceding sections we have established the transcendentally transcendental character of functions which may be written down in unlimited number. Usually, however, there will be nothing to ensure that a function so defined shall possess valuable or interesting special properties. In the present section we give several special cases of functions of the general form in equation (5) so chosen that they satisfy functional equations of simple type and of such form as to make certain that the functions are essentially simple in character and possess a set of important special properties which are easily found. In no case, however, will the detailed discussion of the properties of the functions be given.

Throughout this section we shall denote by α and β two relatively prime integers one at least of which is in absolute value different from unity; and we shall write $\alpha / \beta = q$.

EXAMPLE 1. If $b_1 x + b_2 x^2 + b_3 x^3 + \cdots$ is a convergent infinite power series whose coefficients are rational fractions in their lowest terms such that for every n greater than some preassigned N neither the numerator nor the denominator of b_n (when $b_n \neq 0$) contains a prime factor greater than n, then the function

$$g(x) = \frac{b_1 \beta}{\alpha - \beta} x + \frac{b_2 \beta^2}{\alpha^2 - \beta^2} x^2 + \frac{b_3 \beta^3}{\alpha^3 - \beta^3} x^3 + \cdots$$

is transcendentally transcendental and satisfies the functional equation

$$g(qx) - g(x) = b_1 x + b_2 x^2 + b_3 x^3 + \cdots .$$

Special cases are afforded by taking as the second member of this equation the developments of the functions

$$\frac{x}{1-x}, \quad e^x - 1, \quad \log(1+x), \quad \sin x, \quad \cos x - 1, \quad \cdots.$$

In particular we have a transcendentally transcendental function

$$g(x) = \frac{x}{a-1} + \frac{x^2}{a^2-1} + \frac{x^3}{a^3-1} + \cdots, \quad g(ax) - g(x) = \frac{x}{1-x},$$

a being any integer greater than 1.

EXAMPLE 2:

$$g(x) = 1 + \frac{x}{(q-1)^n} + \cdots + \frac{x^m}{(q-1)^n (q^2-1)^n \cdots (q^m-1)^n} + \cdots;$$

$$g(q^n x) - ng(q^{n-1}x) + \frac{n(n-1)}{2!} g(q^{n-2}x) - \cdots + (-1)^n g(x) = xg(x).$$

For the special case when $n = 1$ we have

$$g(x) = 1 + \frac{x}{(q-1)} + \frac{x^2}{(q-1)(q^2-1)^2} + \cdots; \quad g(qx) = (1+x)g(x).$$

The transcendentally transcendental character of the last function has been previously established by Stridsberg (see the first paper referred to above).

EXAMPLE 3:

$$g(x) = \frac{x}{\alpha - \beta} + \frac{\gamma x^n}{(\alpha-\beta)(\alpha^n - \beta^n)} + \frac{\gamma^{1+n} x^{n^2}}{(\alpha-\beta)(\alpha^n-\beta^n)(\alpha^{n^2}-\beta^{n^2})}$$
$$+ \frac{\gamma^{1+n+n^2} x^{n^3}}{(\alpha-\beta)(\alpha^n-\beta^n)\cdots(\alpha^{n^3}-\beta^{n^3})} + \cdots,$$

$$g(\alpha x) - g(\beta x) - g(\gamma x^n) = x,$$

where γ is any rational number and n is an integer greater than 1.

EXAMPLE 4:

$$g(x) = 1 + \frac{x}{\alpha-\beta} + \frac{x^n}{\alpha^n - \beta^n} + \frac{x^{n^2}}{\alpha^{n^2}-\beta^{n^2}} + \frac{x^{n^3}}{\alpha^{n^3}-\beta^{n^3}} + \cdots,$$

$$g(\alpha x) - g(\beta x) - g(\alpha x^n) + g(\beta x^n) = x,$$

where n is an integer greater than 1.

EXAMPLE 5:

$$g(x) = 1 + \frac{x}{\alpha-\beta} + \frac{x^2}{(\alpha-\beta)(\alpha^2-\beta^2)} + \frac{x^3}{(\alpha-\beta)(\alpha^2-\beta^2)(\alpha^3-\beta^3)} + \cdots,$$

$$g(\alpha x) - g(\beta x) = xg(x).$$

EXAMPLE 6: The functional equation

$$g(\alpha x) - g(\beta x) - g(x^2) - g(x^3) = x$$

possesses a unique solution which is analytic at the point zero. This function is transcendentally transcendental.

It is obvious that such special examples might be multiplied indefinitely.

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