SOME THEOREMS CONCERNING GROUPS WHOSE ORDERS ARE POWERS OF A PRIME*

BY

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Professor Burnside has called attention to certain limitations on the nature of the derived groups of groups of prime power order.† One of these limitations is that such a derived group cannot be non-abelian and have a cyclic central. Theorem I of the present paper gives a slight generalization of this limitation, and an obvious modification of the argument used in the proof of this theorem establishes the important result that the second central of a group whose order is a power of an odd prime cannot be cyclic.

The results given at the end of the paper are continuations of some obtained by Professor Burnside in the article just cited.

Throughout the paper frequent use is made of the theorem that in a group whose order is a power of a prime the operations which correspond to the invariant operations of the ith cogredient are commutative with all of the ith commutators of the group.‡

We shall represent by G a group of class k and order p^m, where p is a prime; by G' the first cogredient of G; by H_i the ith central§ of G; by K_j the jth commutator subgroup; and by $K_{j,\,i}$ the subgroup of K_j that is contained in H_i.

Suppose that $K_{j,\,2}$ is cyclic and generated by the operation t of order p^a. If $k > 3$, every operation of $K_{j,\,3}$ of order p must be contained in $K_{j,\,2}$ since the operations of this latter group are invariant in K_j and its operations of order p are invariant in G. Hence $K_{j,\,3}$ has only one subgroup of order p. It follows at once that $K_{j,\,3}$ is cyclic if p is odd. If $p = 2$, $K_{j,\,2}$ contains an operation of order 2^2 that is invariant in $K_{j,\,3}$. But in the non-cyclic group of order a power of 2 with only one subgroup of order 2 no operation of

* Presented to the Society at the Cleveland meeting, January 1, 1913.

† Proceedings of the London Mathematical Society, ser. 2, vol. 11 (1912), pp. 241, 242.

‡ Fite, these Transactions, vol. 7 (1906), p. 62.

§ I follow here the notation of Burnside's *Theory of Groups*, first edition, p. 62, rather than that of the second edition, p. 120. Cf. de Séguier, *Éléments de la théorie des groupes abstraits*, p. 87.

order 2^2 is invariant. We conclude therefore that in this case also $K_{j,\,3}$ is cyclic.

Suppose then that $K_{j,\,3}$ is generated by the operation t_1 of order $p^{\alpha+\beta}$. If we denote by $K'_{j,\,i}$ the subgroup of the jth commutator subgroup of G' that is contained in the $(i-1)$th central of G', it follows from what has been proved that $K'_{j,\,4}$ must be cyclic. Hence the quotient group $K_{j,\,4}/K_{j,\,3}$ must be cyclic. If it is of order p^{γ} and t_2 is an operation of $K_{j,\,4}$ that corresponds to a generator of it, $t_2^{p^{\gamma}}$ is a power of t_1 whose exponent is relatively prime to p, and therefore $K_{j,\,4}$ is cyclic. An obvious continuation of this argument shows that K_j is cyclic. This establishes the following

Theorem I. *The jth commutator subgroup of a group of prime power order is cyclic if those of its operations that are contained in the second central of the group form a cyclic subgroup.*

Since $K_{1,\,2}$ is contained in the central of K_1, Theorem I includes as a special case Professor Burnside's theorem referred to in the introduction.

In case p is odd an argument closely analogous to the one used in the proof of Theorem I shows that if H_2 were cyclic G itself would be cyclic. But if G were cyclic there would be no H_2. This proves

Theorem II. *The second central of a group of order p^m, where p is an odd prime, cannot be cyclic.*

If $p=2$, the preceding argument does not apply, since the operations of H_2 are not all necessarily invariant in H_3 and therefore an operation of H_3 of order 2 may give a commutator of order 2^2. We can however proceed as follows.

If $k>3$, H_3 contains a commutator t_1 that is not in H_2. Now $\{H_2, t_1\}$ is an abelian group with only one subgroup of order 2, since t_1 is commutative with every one of its commutators. Hence this group is cyclic. If H_3 contained an operation s of order 2 that is not in $\{H_2, t_1\}$, s would transform t_1 into itself multiplied by an invariant commutator of G of order 2. Hence s would be commutative with t, and accordingly every operation of G would transform it into itself multiplied by an invariant commutator of G. But this is not consistent with the fact that s is not in H_2. We conclude therefore that H_3 contains only one subgroup of order 2. It contains the operation t_1 and every one of its operations transforms t_1 into itself multiplied by an operation of H_1. But this would be impossible in case H_3 were non-cyclic. We conclude therefore that H_3 is cyclic.

The continuation of the argument up to the point of showing that H_{k-1} must be cyclic is the same as in the case of an odd prime. But beyond this point the argument breaks down, since G/H_{k-1} has no commutator besides the identity. As a matter of fact, there are groups of order 2^m whose second centrals are cyclic—for example, the diedral groups of these orders when $m>3$.

Suppose now that K_1 is cyclic and generated by the commutator t of order

p^λ, where p is odd. If $k > 3$, G/H_{k-2} is metabelian* with a cyclic commutator subgroup. If p^r is the order of this subgroup, G/H_{k-3} must contain an invariant commutator of order p^r and the invariant commutators of G/H_{k-3} must form a cyclic subgroup of order p^s, where $s \geqq r$.

Since any operation of G/H_{k-3} that corresponds to an invariant operation of G/H_{k-2} is commutative with every commutator of G/H_{k-3}, no operation of H_{k-1} can transform t into itself multiplied by a commutator that is not in H_{k-3}. But some operation of G must transform t into $t^{1+xp^{s_1}}$, where x is relatively prime to p, and $r \leqq s_1 < r + s$, since otherwise t would be contained in H_{k-2} and G/H_{k-2} would be abelian. But this is impossible since G is of class k.

Suppose then that A is so selected that

$$A^{-1} tA = t^{1+xp^{s_1}},$$

where x is relatively prime to p, and s_1 is a minimum. Then

$$A^{-1} t^{p^{r-1}} A = t^{p^{r-1}+xp^{r+s_1-1}}, \qquad A^{-1} t^{p^r} A = t^{p^r+xp^{r+s_1}}.$$

Now since t^{p^r} is contained in H_{k-2}, while $t^{p^{r-1}}$ is not, it follows that

$$r + s_1 \geqq r + s, \qquad r + s_1 - 1 < r + s.$$

Hence $s_1 = s$, and therefore

$$A^{-p^r} tA^{p^r} = t^{(1+xp^s)^{p^r}}.$$

But

$$(1 + xp^s)^{p^r} \equiv 1 + xp^{r+s} \pmod{p^{2s+1}}.$$

Moreover $r + s < 2s + 1 \leqq \lambda$, since the lowest power of t that is contained in H_{k-3} is the p^{r+s}th power and since H_{k-3} must contain a commutator of order p^s. Hence

$$(1 + xp^s)^{p^r} \not\equiv 1 \pmod{p^\lambda},$$

and A^{p^r} is not commutative with t. Since however G/H_{k-2} is metabelian and contains no commutator of order greater than p^r, A^{p^r} must be contained in H_{k-1}. We conclude therefore that *if* K_1 *is cyclic and* $k > 3$, *not all of the operations of* K_1 *are invariant in* H_{k-1}.

This argument with a slight modification is valid when $p = 2$, provided that $s > 1$; but when $s = 1$ it breaks down and the conclusion does not hold, as may be seen from the groups of order 2^5.

We have assumed that $k > 3$. The existence of metabelian groups with cyclic commutator subgroups shows that the conclusion does not hold when

* I use the term *metabelian* in the sense defined by me in these Transactions, vol. 3 (1902), p. 331. Burnside uses the term in a different sense. See his *Theory of Groups*, second edition, p. 57.

$k = 2$. When $k = 3$ every operation of H_2 is commutative with t. But the following conclusion from this argument for $k > 3$ does hold when $k = 3$; namely,* *if* K_1 *is cyclic,* H_{k-1} *cannot coincide with* K_1.

The commutator formed by any two operations, t_1 and t_2, of K_1 is contained in H_{k-3}, since G/H_{k-3} is of class three and the commutator subgroup of a group of the third class is abelian. The index of $K_{1,\,k-3}$ under K_1 is at least p^2. If it is exactly p^2, we can assume that t_2 is equal to the product of some operation of H_{k-3} and t_1. Hence† the commutator formed by t_1 and t_2 is in H_{k-4} and the index of the commutator subgroup of K_1 under K_1 is at least p^3.

If t_1 and t_2 are operations of K_i they are ith commutators or products of ith commutators and correspond to invariant operations of the ith cogredient of G/H_{k-2i-1}, since they are contained in H_{k-i}. Hence the commutator formed by t_1 and t_2 is contained in H_{k-2i-1}. Now if K_i is non-abelian the index of $K_{i,\,k-2i-1}$ under K_i is at least p^{i+1}. If it is exactly p^{i+1}, we can show by an argument similar to the one used in the special case just considered that the commutator formed by t_1 and t_2 is contained in H_{k-2i-2}. Hence the index of the commutator subgroup of K_i under K_i is at least p^{i+2} if K_i is non-abelian.

If the jth derived group is contained in H_x, the $(j+1)$th derived group is contained in $H_{x-(j+1)}$, provided that $x > j + 1$. Hence the ith derived group is contained in $H_{k-i(i+1)/2}$, and therefore in a group of order p^m whose ith derived group is not the identity we must have $k > i\,(i+1)/2$. Hence $m \geqq 2 + i\,(i+1)/2$. For $i > 4$ this exceeds the lower limit for m given by Professor Burnside.

COLUMBIA UNIVERSITY,
July, 1913.

* de Séguier, *loc. cit.*, p. 127. This follows also from Theorem II.

† This result is given by Burnside in the article cited in the introduction.