THE SYMBOLICAL THEORY OF FINITE EXPANSIONS*

BY

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Introduction.

The first two sections of this paper are the result of an attempt to identify, and state in general terms, certain fundamental common properties of a large number of known types of finite expansions of polynomials in p variables. The number of such expansions which are known and available for such a synthesis is of course large, and only the most salient properties can be comprised under one general theory. The principles proved in these preliminary sections are subjected to verification when it is shown that they hold for a number of new expansions. These are derived in sections 3 to 6 inclusive. They are of considerable interest in themselves, apart from the general theory, to the fundamental laws of which they conform.

The Aronhold symbolical notation† for algebraic forms subject to linear transformations constitutes the basis of the general methods of the paper.

To state briefly our initial point of view; if a function f can be given an expression of the form

$$f = B_0 A_1^m + mB_1 A_1^{m-1} A_2 + \binom{m}{2} B_2 A_1^{m-2} A_2^2 + \cdots + B_m A_2^m,$$

it can be represented symbolically as an mth power, say

$$f = \Xi_A^m = (\Xi_1 A_1 + \Xi_2 A_2)^m = \Xi_1^m A_1^m + m \Xi_1^{m-1} \Xi_2 A_1^{m-1} A_2 + \cdots.$$

But with $\nabla_1 = \Xi_2 (\partial / \partial \Xi_1)$ this may be written

$$f = \Xi_1^m A_1^m + \frac{\nabla_1 \Xi_1^m}{\lfloor 1} A_1^{m-1} A_2 + \frac{\nabla_1^2 \Xi_1^m}{\lfloor 2} A_1^{m-2} A_2^2 + \cdots + \frac{\nabla_1^m \Xi_1^m}{\lfloor m} A_2^m.$$

Now the symbol Ξ_1^m equals B_0. If it prove possible to find a non-symbolical operator ∇ equivalent (§ 1) to ∇_1, then we shall have expressed f in the funda-

* Presented to the Society, April, 1911, and March, 1913.

† Aronhold, Journal für Mathematik, vol. 55 (1858), p. 97.

mental, non-symbolical form

$$f = B_0 A_1^m + \frac{\nabla B_0}{\lfloor\underline{1}} A_1^{m-1} A_2 + \frac{\nabla^2 B_0}{\lfloor\underline{2}} A_1^{m-2} A_2^2 + \cdots + \frac{\nabla^m B_0}{\lfloor\underline{m}} A_2^m .$$

This is what is actually done in what follows.

§ 1. Fundamental Lemma.

Definitions. The general term *function* will be used to indicate an algebraic form in p variables.

By an *expansion* of a function we shall mean a representation of it as a finite series proceeding by a law in powers of one or more arguments, with coefficients dependent upon the coefficients, and possibly upon the variables also (§3), of the original function.

Equivalent operators. In connection with the familiar symbolical representation of a binary form

$$f = a_x^n = (\alpha_1 x_1 + \alpha_2 x_2)^n = \alpha_1^n x_1^n + n\alpha_1^{n-1} \alpha_2 x_1^{n-1} x_2 + \cdots$$
$$= a_0 x_1^n + n a_1 x_1^{n-1} x_2 + \cdots ,$$

we note that the expressions $\alpha_1^n \,(= a_0)$, $\alpha_1^{n-1}\alpha_2 \,(= a_1)$, $\cdots$ and linear combinations of these, are the only expressions in the symbolical α_i, which are *defined* in terms of the actual coefficients. We call an expression, as $I = \alpha_1^n + p\alpha_1^{n-2}\alpha_2^2$, the *symbolical equivalent* of the corresponding expression in the a's, $J = a_0 + pa_2$.

The differential operator $\alpha_2 \,(\partial / \partial\alpha_1)$ is such that when

$$\text{(i)} \qquad \alpha_2 \frac{\partial}{\partial \alpha_1} I = K,$$

and I is a defined expression, then K is a defined expression. Let the non-symbolical equivalent of K be L, that of I being J as illustrated above. Then the non-symbolical operator

$$O = n a_1 \frac{\partial}{\partial a_0} + (n-1)\, a_2 \frac{\partial}{\partial a_1} + \cdots + a_n \frac{\partial}{\partial a_{n-1}}$$

has the property that*

* For example when I is the particu'ar expression given above, equations (i), (ii) are respectively

$$\alpha_2 \frac{\partial}{\partial \alpha_1} (\alpha_1^n + p\alpha_1^{n-2}\alpha_1^n) = n\alpha_1^{n-1}\alpha_2 + p\,(n-2)\,\alpha_1^{n-3}\alpha_2^3 ,$$

$$O\,(a_0 + pa_2) = na_1 + p\,(n-2)\,a_3 .$$

The operands on the left are equivalent, and the right-hand members are also equivalent expressions.

(ii) $$OJ = L.$$

Two operators, one symbolical and the other non-symbolical, related like $\alpha_2(\partial/\partial\alpha_1)$ and O will be called equivalent or isomorphic operators. We shall refer to this important property of equivalence of two operators as the property (A), and we now state it in such general terms as will completely define it for all the symbolical representations of functions that are employed in the sequel.

(A) Let there be given a definite symbolical representation of a function f. A symbolical derivative operator on an expression in the defined combinations of the symbols, which produces a result containing the symbols in defined combinations only, has corresponding to it a non-symbolical derivative operator which carries the non-symbolic equivalent of the first expression into that of the second.

Finite Expansions. Any finite expansion of a function f can be made the basis of a symbolical representation of that function. For, if there exists an expansion of f in powers of a given argument A,

$$f = B_0 + \binom{m}{1} B_1 A + \binom{m}{2} B_2 A^2 + \cdots + B_m A^m,$$

then f may be represented as the mth power of a purely symbolical binomial expression

(1) $$\Xi_1 + \Xi_2 A.$$

That is, if we write $\Xi_{(A)}$ as an abbreviation of this non-homogeneous (as to A) expression, then

(1′) $$f = \Xi_{(A)}^m = (\Xi_1 + \Xi_2 A)^m = \Xi_1^m + m\Xi_1^{m-1}\Xi_2 A + \cdots = B_0 + mB_1 A + \cdots.$$

Then the only defined expressions in the symbols are

$$\Xi_1^{m-k}\Xi_2^k \ (= B_k) \qquad (k = 0, 1, \cdots, m),$$

and linear combinations of these, where, according to our definition of an expansion, the B_k are non-symbolical, determinate expressions in the coefficients and variables of f.

Symbolical basis. In the case of every finite expansion with which we shall deal, (i) the property (A) holds for both operators

$$\nabla_1 = \Xi_2 \frac{\partial}{\partial \Xi_1}, \qquad \nabla_2 = \Xi_1 \frac{\partial}{\partial \Xi_2},$$

and (ii) the non-symbolical equivalents of ∇_i can actually be found, expressed, not in terms of the unknown B_k, but in terms of the coefficients of the known function f, and the known argument A.

When (i), (ii) have actually been verified we say that the existence of a symbolical basis of the expansion, viz.,

$$\Xi_{(A)} = \Xi_1 + \Xi_2 A,$$

has been verified.

Fundamental lemma. For an expansion for which a symbolical basis exists we now prove a lemma. Evidently

$$\nabla_1 \Xi_1^{m-k} \Xi_2^k = (m-k) \Xi_1^{m-k-1} \Xi_2^{k+1} \qquad (k = 0, 1, \cdots, m).$$

Hence (1′) gives

$$f = \Xi_1^m + \nabla_1 \Xi_1^m A + \frac{\nabla_1^2 \Xi_1^m}{\underline{|2}} A^2 + \cdots. \tag{2'}$$

Write ∇ for the non-symbolical equivalent of ∇_1, according to the property (A), and set $\Phi = B_0 \,(= \Xi_1^m)$. Then from (2′) we have

Lemma. *There exists a function Φ and a differential operator ∇, capable of being represented symbolically by the linear operator*

$$\nabla_1 = \Xi_2 \frac{\partial}{\partial \Xi_1},$$

such that the explicit expansion of f is

$$f = \Phi + \frac{\nabla\Phi}{\underline{|1}} A + \frac{\nabla^2 \Phi}{\underline{|2}} A^2 + \cdots + \frac{\nabla^m \Phi}{\underline{|m}} A^m. \tag{2}$$

If a symbolical basis be assumed in homogeneous form, or, what amounts to the same thing, if f be assumed to have a homogeneous expansion in two arguments A_1, A_2 for which a symbolical basis exists in the form

$$\Xi_A = \Xi_1 A_1 + \Xi_2 A_2,$$

then

$$f = \Xi_A^m = \Xi_1^m A_1^m + m \Xi_1^{m-1} \Xi_2 A_1^{m-1} A_2 + \cdots = \Phi_1 A_1^m + \cdots + \Phi_2 A_2^m. \tag{3'}$$

This is the same as (2) when $A_1 = 1$. But, in general, we have

$$\frac{\nabla_1^r \Xi_1^m}{\underline{|r}} = \frac{\nabla_2^{m-r} \Xi_2^m}{\underline{|m-r}} = \frac{1}{2}\left(\frac{\nabla_1^r \Xi_1^m}{\underline{|r}} + \frac{\nabla_2^{m-r} \Xi_2^m}{\underline{|m-r}}\right) = \binom{m}{r} \Xi_1^{m-r} \Xi_2^r.$$

Thus the third expression can be used in place of the general coefficient in (3′). Let ∇_i' be the non-symbolical equivalent of $\nabla_i \, (i = 1, 2)$. Then we get from (3′) an expansion formula more general, and more symmetrical than (2), viz.,

$$f = \Phi_1 A_1^m + \frac{1}{2}\left(\frac{\nabla_1' \Phi_1}{\underline{|1}} + \frac{\nabla_2'^{m-1} \Phi_2}{\underline{|m-1}}\right) A_1^{m-1} A_2 + \cdots$$
$$+ \frac{1}{2}\left(\frac{\nabla_1'^{m-1} \Phi_1}{\underline{|m-1}} + \frac{\nabla_2' \Phi_2}{\underline{|1}}\right) A_1 A_2^{m-1} + \Phi_2 A_2^m. \tag{3}$$

Such an expansion will be completely given by two non-symbolical operators ∇_i' and two pre-determined functions Φ_i $(i = 1, 2)$.

The importance of a general formula like (3) evidently depends upon the number of expansions which can be found to conform to its laws. We proceed to point out the fact that many of the known finite expansions are special cases of (2) or (3).

(*a*) Any binary form $f(x_1, x_2) = a_0 x_1^m + ma_1 x_1^{m-1} x_2 + \cdots$, when transformed by

$$x_1 = \lambda_1 x_1' + \mu_1 x_2', \qquad x_2 = \lambda_2 x_1' + \mu_2 x_2',$$

gives a result $f_1(x_1', x_2')$ which is an expansion of f in the arguments x_i'. It is then a case of (3) where now $A_i = x_i'$, and

$$\Phi_1 = f(\lambda_1, \lambda_2), \qquad \Phi_2 = f(\mu_1, \mu_2),$$

$$\nabla_1' = \mu_1 \frac{\partial}{\partial \lambda_1} + \mu_2 \frac{\partial}{\partial \lambda_2}, \qquad \nabla_2' = \lambda_1 \frac{\partial}{\partial \mu_1} + \lambda_2 \frac{\partial}{\partial \mu_2}.$$

(*b*) Any covariant of $f(x_1, x_2)$, say $C = C_0 x_1^w + \cdots$, considered as an expansion in x_1, x_2 is a case of (3). Here $\Phi_1 = C_0$, the seminvariant leading coefficient; $\Phi_2 = C_w$. Or $x_1^{-w} C$ is a case of (2), where $A = x_1 / x_2$. In either case

$$\nabla_1' = ma_1 \frac{\partial}{\partial a_0} + (m-1) a_2 \frac{\partial}{\partial a_1} + \cdots + a_m \frac{\partial}{\partial a_{m-1}},$$

$$\nabla_2' = a_0 \frac{\partial}{\partial a_1} + 2a_1 \frac{\partial}{\partial a_2} + \cdots + ma_{m-1} \frac{\partial}{\partial a_m}.$$

(*c*) If $\varphi = (a_0, a_1, \cdots)(x_1, x_2)^m$ goes into $(a_0', a_1', \cdots)(x_1', x_2')^m$ under the special transformation $x_1 = x_1' + Ax_2'$, $x_2 = x_2'$, and $I(a_0, a_1, \cdots)$ is any rational integral function* of $a_0, a_1, \cdots$, then the expansion of $I(a_0', a_1', \cdots)$ in powers of A is a case of (2), where $f = I(a_0', \cdots)$, $\Phi = I(a_0, a_1, \cdots)$, and ∇ is the preceding ∇_2'.

(*d*) Taylor's expansion of a non-homogeneous polynomial $f(x)$ in powers of $x - a$ is a case of (2), where

$$\Phi = f(a), \qquad \nabla = \frac{\partial}{\partial a}.$$

(*e*) As a proposition inverse to the fundamental lemma (2), or (3), any formal identity containing three terms may be employed as the symbolical basis of finite expansions. In fact, if the identity is

$$\Xi_A = \Xi_1 A_1 + \Xi_2 A_2,$$

* Elliott, *Algebra of Quantics*, p. 115.

then functions $f = \Xi_A^m$ and corresponding expansions in the arguments A_i exist for all values of m. Gordan's* series may be developed (see § 6) from this point of view, from the formal identity

$$a_x b_y - a_y b_x = (ab)(xy).$$

This is also the principle upon which Hermite's theory of typical representation† and associated forms is based.

§ 2. Extension of the fundamental lemma.

The extension of (2) to the case of the general homogeneous expansion in p arguments can be readily made. We set

$$(4') \quad f = \Xi_A^m = (\Xi_1 A_1 + \Xi_2 A_2 + \cdots + \Xi_p A_p)^m = \Xi_1^m A_1^m + \cdots = \Phi_1 A_1^m + \cdots,$$

and assume that the property (A) is satisfied by all of the symbolical operators of the set

$$\nabla_i = \Xi_{i+1} \frac{\partial}{\partial \Xi_i} \qquad (i = 1, 2, \cdots, p-1).$$

Then it is easy to verify, by actually performing the indicated differentiations, that (4′) may be written

$$f = \sum \frac{\nabla_1^{i_1} \nabla_2^{i_2} \cdots \nabla_{p-1}^{i_{p-1}} \Xi_1^m}{\underline{|i_1} \, \underline{|i_2} \cdots \underline{|i_{p-1}}} A_1^{m-i_1} A_2^{i_1-i_2} \cdots A_p^{i_{p-1}},$$

where the set $(m - i_1, i_1 - i_2, \cdots, i_{p-2} - i_{p-1}, i_{p-1})$ ranges over the partitions of m into p parts, assuming each partition once and once only. Hence if, according to the property (A), ∇_i' is the non-symbolical equivalent of ∇_i, then there exists a function Φ and a set of $p - 1$ operators ∇_i' such that the explicit expansion of f is (§ 5)

$$(4) \qquad f = \sum \frac{\nabla_1'^{i_1} \nabla_2'^{i_2} \cdots \nabla_{p-1}'^{i_{p-1}} \Phi}{\underline{|i_1} \, \underline{|i_2} \cdots \underline{|i_{p-1}}} A_1^{m-i_1} A_2^{i_1-i_2} \cdots A_p^{i_{p-1}}.$$

§ 3. On expressing one polynomial in terms of another.

Non-homogeneous variables will be used throughout this section. We let

$$f(x) = a_x^m = a_0 x^m + a_1 x^{m-1} + \cdots + a_m \quad (a_0 = 1, m = \mu\nu),$$

$$\varphi(x) = \xi_x^\nu = \xi_0 x^\nu + \xi_1 x^{\nu-1} + \cdots + \xi_\nu \equiv \xi_0 (x - s_1)(x - s_2) \cdots (x - s_\nu).$$

* Gordan, *Vorlesungen über Invariantentheorie*, vol. 2, § 117.

† Hermite, Journal für Mathematik, vol. 52 (1855), p. 23.

It may be shown by a form of the Euclidean process of successive division that there exists an expansion of f in powers of ξ_x^ν,

$$(5)\qquad f(x) = \Phi_1 + \Phi_2\,\xi_x^\nu + \Phi_3\,(\xi_x^\nu)^2 + \cdots + \Phi_\mu\,(\xi_x^\nu)^{\mu-1} + \Phi_{\mu+1}\,(\xi_x^\nu)^\mu ,$$

where Φ_i is a polynomial in x of order $\nu - 1$, and in particular, say

$$\Phi_1 = \gamma_0\,x^{\nu-1} + \gamma_1\,x^{\nu-2} + \cdots + \gamma_{\nu-1}.$$

We proceed to the determination of (the coefficients of) Φ_1 and the operator ∇ in (2). These will then give the whole expansion, and it will appear that (5) has a symbolical basis of the form

$$(6)\qquad \Xi_{(\phi)} = \Xi_{1f(x)} + \Xi_{2f(x)}\,\varphi .$$

Substitution of the roots of $\varphi(x)$ for x in (5) gives at once

$$(7)\qquad f(s_i) = \gamma_0\,s_i^{\nu-1} + \gamma_1\,s_i^{\nu-2} + \cdots + \gamma_{\nu-1} \quad (i = 1, 2, \cdots, \nu).$$

These are linear equations in the γ_j $(j = 0, 1, \cdots, \nu - 1)$. Their solution is possible provided $D \neq 0$, where D is the discriminant of $\varphi(x)$. This solution yields

$$(8)\qquad \gamma_j = \frac{\begin{vmatrix} s_1^{\nu-1} & s_1^{\nu-2} & \cdots & s_1^{\nu-j} & f(s_1) & s_1^{\nu-j-2} & \cdots & s_1 & 1 \\ s_2^{\nu-1} & s_2^{\nu-2} & \cdots & s_2^{\nu-j} & f(s_2) & s_2^{\nu-j-2} & \cdots & s_2 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ s_\nu^{\nu-1} & s_\nu^{\nu-2} & \cdots & s_\nu^{\nu-j} & f(s_\nu) & s_\nu^{\nu-j-2} & \cdots & s_\nu & 1 \end{vmatrix}}{\sqrt{D}} \qquad (j = 0, 1, 2, \cdots, \nu - 1).$$

Thus γ_j is symmetric in the roots s_i, rational in the coefficients ξ_k $(k = 0, 1, \cdots, \nu)$ and linear in $a_0\,(= 1), a_1, a_2, \cdots, a_m$. It is necessary to evaluate the symmetric functions γ_j of the roots s_i in order to determine the operator ∇ required by the theory of the symbolical basis, and (2). I did this originally by means of the Euclidean process mentioned above, i. e., $f(x)$ was divided by φ^μ, the remainder by $\varphi^{\mu-1}$, and so forth.* The results for the general case are given below, without details of derivation. In these formulas, Ω indicates the following operator:

$$\Omega = a_0\frac{\partial}{\partial a_1} + a_1\frac{\partial}{\partial a_2} + \cdots + a_{m-1}\frac{\partial}{\partial a_m}.$$

* This work was subsequently checked by inductive steps based upon ordinary computation by symmetric functions.

I obtain

$$
\begin{aligned}
F = \gamma_0 = {} & \Omega^{\nu-1} f(-\xi_1) + \sum_{i=2}^{k_{11}} \xi_i \, \Omega^{\nu+i-2} f'(-\xi_1) \\
& + \sum_{i=2}^{k_{21}} \sum_{j=2}^{k_{22}} \frac{\xi_i \, \xi_j}{\underline{|2}} \Omega^{\nu+i+j-3} f''(-\xi_1) \\
& + \sum_{i=2}^{k_{31}} \sum_{j=2}^{k_{32}} \sum_{k=2}^{k_{33}} \frac{\xi_i \, \xi_j \, \xi_k}{\underline{|3}} \Omega^{\nu+i+j+k-4} f'''(-\xi_1) \qquad (9) \\
& + \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \\
& + \sum_{i=2}^{k_{t1}} \sum_{j=2}^{k_{t2}} \cdots \sum_{n=2}^{k_{tt}} \frac{\xi_i \, \xi_j \cdots \xi_n}{\underline{|t}} \Omega^{\nu+i+j+\cdots+n-t-1} f^{(t)}(-\xi_1)
\end{aligned}
$$

$$
(k_{rs} \leqq \nu, \; k_{s1} + k_{s2} + \cdots + k_{ss} \leqq (\mu - 1)\nu + 1).
$$

Then there results

$$
\begin{aligned}
\Phi = \Phi_1 = {} & F x^{\nu-1} + [-(\xi_2 \, \Omega + \xi_3 \, \Omega^2 + \cdots + \xi_\nu \, \Omega^{\nu-1}) F + a_{m-\nu+2}] \, x^{\nu-2} \\
& + [-(\xi_3 \, \Omega + \xi_4 \, \Omega^2 + \cdots + \xi_\nu \, \Omega^{\nu-2}) F + a_{m-\nu+3}] \, x^{\nu-3} \\
& + \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \qquad (10) \\
& + [-(\xi_{\nu-1} \, \Omega + \xi_\nu \, \Omega^2) F + a_{m-1}] \, x \\
& + [-\xi_\nu \, \Omega F + a_m].
\end{aligned}
$$

Furthermore, we find during the course of the derivation of these results by the division process that an operator ∇ in (2) exists and is precisely [see (6)]

$$
\nabla = \Xi_{2f(x)} \frac{\partial}{\partial \, \Xi_{1f(x)}} = - \frac{\partial}{\partial \xi_\nu}. \qquad (11)
$$

Hence expansion (5) takes the form

$$
\begin{aligned}
f(x) = {} & \Phi - \frac{\partial \Phi}{\partial \xi_\nu} \xi_x^\nu + \frac{\partial^2 \Phi}{\partial (\xi_\nu)^2} \frac{(\xi_x^\nu)^2}{\underline{|2}} - \cdots + (-1)^{\mu-1} \frac{\partial^{\mu-1} \Phi}{\partial (\xi_\nu)^{\mu-1}} \frac{(\xi_x^\nu)^{\mu-1}}{\underline{|\mu - 1}} \qquad (12) \\
& + (-1)^\mu \frac{\partial^\mu \Phi}{\partial (\xi_\nu)^\mu} \frac{(\xi_x^\nu)^\mu}{\underline{|\mu}}.
\end{aligned}
$$

It is thus analogous to the Taylor expansion, to which it reduces when $\nu = 1$.

We note in passing the following interesting property of the function F:

$$
-(\xi_1 \, \Omega + \xi_2 \, \Omega^2 + \cdots + \xi_\nu \, \Omega^\nu) F + a_{m-\nu+1} \equiv F.
$$

§ 4. Homogeneous expansions.

As an illustration of formula (3), we consider the expansion of a homogeneous binary form in terms of two other binary forms of lower order. Let

$$f_M = a_x^M = \prod_{i=1}^{M} (r_2^{(i)} x_1 - r_1^{(i)} x_2) = a_0 x_1^M + \cdots$$

be the form to be expanded. Let the arguments of the expansion be

$$A_1 = f_{1n} = \alpha_x^n = \prod_{j=1}^{n} (\alpha_2^{(j)} x_1 - \alpha_1^{(j)} x_2) = \alpha_0 x_1^n + \cdots,$$

$$A_2 = f_{2n} = \beta_x^n = \prod_{j=1}^{n} (\beta_2^{(j)} x_1 - \beta_1^{(j)} x_2) = \beta_0 x_1^n + \cdots.$$

Then the expansion sought will be of the type

$$f_M = \varphi_{0p} f_{1n}^m + \varphi_{1p} f_{1n}^{m-1} f_{2n} + \cdots + \varphi_{mp} f_{2n}^m, \tag{13}$$

where φ_{ip} is a homogeneous binary form of order p.

It is assumed at the outset that f_M, f_{in} are perfectly general forms, i. e., that their coefficients are independent variables. These forms are also understood to be given in the sense that the coefficient forms φ_{ip} are to be determined from them. Hence the totality of coefficients of the φ_{ip} ($i = 0, 1, \cdots, m$) must be equal in number to those of f_M,

$$M + 1 = (m + 1)(p + 1). \tag{14}$$

Also

$$M = mn + p.$$

Thus

$$p = n - 1, \qquad M = n(m + 1) - 1.$$

When M and p have these values the solution of $n(m + 1)$ linear equations will determine the coefficients of φ_{in-1} ($i = 0, 1, \cdots, m$), and consequently the whole expansion. Hence the expansion exists provided these linear equations are consistent. It will be shown that the determinant representing the condition for consistency is a power of the resultant of f_{1n}, f_{2n}, multiplied by a constant.

For this determinant D is of order $n(m + 1)$, and is of the form of the eliminant of the set of $m + 1$ forms of order mn,

$$f_{1n}^m, \quad f_{1n}^{m-1} f_{2n}, \quad \cdots, \quad f_{2n}^m.$$

If $D = 0$, then there exists a linear relation of the type

$$\lambda_1 f_{1n}^m + \lambda_2 f_{1n}^{m-1} f_{2n} + \cdots + \lambda_{m+1} f_{2n}^m = 0, \tag{15}$$

where λ_{m+1} is of order $n-1$ in (x), and so f_{1n}, f_{2n} have at least one common root. Conversely if f_{1n}, f_{2n} have a common root then $D=0$, for otherwise $f_{n(m+1)-1}$ would have a root of multiplicity m.

An expansion (13) *of* $f_{n(m+1)-1}$ *exists provided the resultant* $R(\alpha, \beta)$ *of* f_{1n}, f_{2n} *does not vanish.*

To determine our expansion (13), let

$$(16)\quad \varphi_{in-1} = p_{i0}\, x_1^{n-1} + p_{i1}\, x_1^{n-2} x_2 + \cdots + p_{in-1}\, x_2^{n-1} = p_{ix}^{n-1} \quad (i=0,1,\cdots m).$$

Substituting $\beta^{(k)}$ for x in (13) we get

$$(17)\qquad p_{0\beta^{(k)}}^{n-1}\,(\alpha_{\beta^{(k)}}^{n})^{m} = a_{\beta^{(k)}}^{n(m+1)-1} \qquad (k=1,2,\cdots n).$$

Also owing to the symmetry of (13) in (α, β),

$$(18)\qquad p_{m\alpha^{(k)}}^{n-1}\,(\beta_{\alpha^{(k)}}^{n})^{m} = a_{\alpha^{(k)}}^{n(m+1)-1} \qquad (k=1,2,\cdots,n).$$

Equations (17) form a set of n linear equations for the determination of the n coefficients of p_{0x}^{n-1}. In order to exhibit the solutions as briefly as may be let Δ_β be the discriminant of β_x^n and Δ_α that of α_x^n. Then

$$\pm \Delta_\beta = \left| \beta_1^{(1)n-1},\ \beta_1^{(2)n-2}\beta_2^{(2)},\ \beta_1^{(3)n-3}\beta_2^{(3)2},\ \cdots,\ \beta_2^{(n)n-1} \right|^2 \div \prod_{k=1}^{n} \beta_2^{(k)2(n-1)}.$$

Also let

$$\Sigma_j(\alpha,\beta,a) = \left| \alpha_{\beta^{(1)}}^{mn}\beta_1^{(1)n-1},\ \alpha_{\beta^{(2)}}^{mn}\beta_1^{(2)n-2}\beta_2^{(2)},\ \cdots,\ \alpha_{\beta^{(j)}}^{mn}\beta_1^{(j)n-j}\beta_2^{(j)j-1}, \right.$$
$$\left. a_{\beta^{(j+1)}}^{n(m+1)-1},\ \alpha_{\beta^{(j+2)}}^{mn}\beta_1^{(j+2)n-j-2}\beta_2^{(j+2)j+2},\ \cdots,\ \alpha_{\beta^{(n)}}^{mn}\beta_2^{(n)n-1} \right|,$$

then we get

$$(19)\qquad p_{0j} = \frac{1}{R(\alpha,\beta)^m}\cdot\frac{\Sigma_j(\alpha,\beta,a)}{\prod_{k=1}^{n}\beta_2^{(k)n-1}\sqrt{\Delta_\beta}} \quad (j=0,1,\cdots,n-1).$$

From symmetry, or directly from (18), we have also

$$(20)\qquad p_{mj} = \frac{1}{R(\beta,\alpha)^m}\cdot\frac{\Sigma_j(\beta,\alpha,a)}{\prod_{k=1}^{n}\alpha^{(k)n-1}\sqrt{\Delta_\alpha}} \quad (j=0,1,\cdots,n-1).$$

The coefficients of $R(\alpha,\beta)^{-m}$ in (19), (20) are rational, *integral* and symmetric in the roots of f_{1n}, f_{2n} and hence rational in the quantities $\alpha_0, \alpha_1, \cdots;$ $\beta_0, \beta_1, \cdots$

Referring now to expansion formula (3) we note that the functions Φ_i $(i=1,2)$ are determined in the present case by (19), (20).

The operators ∇_1', ∇_2' required by (3) to complete the determination of (13) are the Aronhold operators obtained from α_x^n, β_x^n; i. e.,

$$(21)\qquad \begin{aligned} -\nabla_1' &= \alpha_0 \frac{\partial}{\partial\beta_0} + \alpha_1 \frac{\partial}{\partial\beta_1} + \cdots + \alpha_n \frac{\partial}{\partial\beta_n}, \\ -\nabla_2' &= \beta_0 \frac{\partial}{\partial\alpha_0} + \beta_1 \frac{\partial}{\partial\alpha_1} + \cdots + \beta_n \frac{\partial}{\partial\alpha_n}. \end{aligned}$$

These with $\Phi_1 = \varphi_{0n-1}$, $\Phi_2 = \varphi_{mn-1}$, $A_i = f_{in}$, give in explicit form the desired expansion of $f_{n(m+1)-1}$; that is, it takes the form (3):

$$(22)\quad f_{n(m+1)-1} = \Phi_1 f_{1n}^m + \frac{1}{2}\left(\frac{\nabla_1' \Phi_1}{\underline{|1}} + \frac{\nabla_2'^{\,m-1} \Phi_2}{\underline{|m-1}}\right) f_{1n}^{m-1} f_{2n} + \cdots + \Phi_2 f_{2n}^m .$$

It is of some interest to enumerate the conditions, necessary and sufficient, in order that series (22), taken in non-homogeneous form with $x_2 = 1$, say, should have constant coefficients. We get a minimum set of such conditions by equating to zero all of the coefficients of φ_{in-1} $(i = 0, 1, \cdots, m)$ except the last.

Let (19) be written in the form

$$p_{0j} = \frac{Q_j(\alpha, \beta)}{R(\alpha, \beta)^m} \qquad (j = 0, 1, \cdots, n-1).$$

Then the aforesaid conditions are given by

$$(23)\qquad C_{ij} \equiv \nabla_1'^{\,i} Q_j = 0 \qquad \begin{pmatrix} i = 0, 1, \cdots, m \\ j = 0, 1, \cdots, n-2 \end{pmatrix}.$$

Their number is

$$N = (n-1)(m+1).$$

It follows that a series free from conditions $(N = 0)$, having constant coefficients, exists only when the arguments f_{in} are linear $(n = 1)$.

By constructing equations analogous to (14) it is easy to show that a ternary form f_m can be expressed as a ternary expansion in three ternary arguments f_{in} $(i = 1, 2, 3)$,

$$f_m = \varphi_{m00} f_{1n}^m + \cdots,$$

with the coefficient forms φ all of order $n - 1$, and with the expansion free from conditions, only when the arguments are linear.

§ 5. Expansions with linear arguments.

Consider next the problem of expanding a p-ary form a_x^m in terms of powers of p linear p-ary forms α_{ix} $(i = 1, 2, \cdots, p)$. Expansion formula (4) applies, and

$$(24)\qquad a_{x_\iota}^m = \sum_{i_\nu=1}^{m} \frac{\nabla_1'^{\,i_1} \nabla_2'^{\,i_2} \cdots \nabla_{p-1}'^{\,i_{p-1}} \Phi_1}{\underline{|i_1} \, \underline{|i_2} \cdots \underline{|i_{p-1}}} \alpha_{1x}^{m-i_1} \alpha_{2x}^{i_1 - i_2} \cdots \alpha_{px}^{i_{p-1}}.$$

In this the coefficient Φ_j is given by the set

$$i_1 = i_2 = \cdots = i_{j-1} = m; \qquad i_j = i_{j+1} = \cdots = i_{p-1} = 0 \quad (j = 1, 2, \cdots, p).$$

To get Φ_1 we substitute for (x) in (24) the coördinates of the point of intersection of the $p-1$ hyperplanes

$$\alpha_{ix} = \alpha_{i1} x_1 + \alpha_{i2} x_2 + \cdots + \alpha_{ip} x_p = 0 \quad (i = 2, 3, \cdots, p)$$

This gives

$$x_j = -\frac{(\alpha_2 \alpha_3 \cdots \alpha_{j-1} \alpha_p \alpha_{j+1} \cdots \alpha_{p-1}) x_p}{(\alpha_2 \alpha_3 \cdots \alpha_p)} \quad (j = 1, 2, \cdots, p),$$

and by substitution

$$\Phi_1 = \frac{(a\alpha_2 \alpha_3 \cdots \alpha_p)^m}{(\alpha_1 \alpha_2 \cdots \alpha_p)^m}.$$

For this expansion there exists a symbolical basis in the form of (4′), as may readily be verified by considering (24) to be the transformed of a_x^m by the inverse of

$$X_i = \alpha_{ix} \qquad (i = 1, 2, \cdots, p).$$

The operators for this case are the Aronhold operators

$$-\nabla_i' = \alpha_{i1} \frac{\partial}{\partial \alpha_{i+11}} + \alpha_{i2} \frac{\partial}{\partial \alpha_{i+12}} + \cdots + \alpha_{ip} \frac{\partial}{\partial \alpha_{i+1p}} \quad (i = 1, 2, \cdots, p-1),$$

the whole expansion (24) being given by Φ_1 and ∇_i'.

It can be shown that corresponding results hold for forms whose coefficients belong to a field or reduced residue system* [modd p, $P(x)$], where p is a prime number and $P(x)$ an irreducible quantic modulo p of order n. Thus when $m = p = 2$, we have, provided $(\alpha\beta) \not\equiv 0$ [modd p, $P(x)$], an expansion of the form

$$f \equiv \Phi_1 (\alpha_1 x_1 - \alpha_2 x_2)^2 + I (\alpha_1 x_1 - \alpha_2 x_2)(\beta_1 x_1 - \beta_2 x_2) + \Phi_2 (\beta_1 x_1 - \beta_2 x_2)^2 \,[\text{modd } p, P(x)],$$

where the linear arguments have coefficients belonging to the residue system. Then

$$\Phi_1 \equiv (\alpha_1 \beta_2 - \alpha_2 \beta_1)^{p^n-3} (a_0 \beta_2^2 + a_1 \beta_2 \beta_1 + a_2 \beta_1^2),$$

$$\Phi_2 \equiv (\alpha_1 \beta_2 - \alpha_2 \beta_1)^{p^n-3} (a_0 \alpha_2^2 + a_1 \alpha_2 \alpha_1 + a_2 \alpha_1^2)$$

$$[\text{modd } p, P(x)],$$

* Dickson, *Linear Groups*, p. 7.

and with

$$\nabla_1' = -\left(\alpha_1\frac{\partial}{\partial\beta_1} + \alpha_2\frac{\partial}{\partial\beta_2}\right), \qquad \nabla_2' = -\left(\beta_1\frac{\partial}{\partial\alpha_1} + \beta_2\frac{\partial}{\partial\alpha_2}\right),$$

we have

$$I = \tfrac{1}{2}(\Delta_1'\Phi_1 + \nabla_2'\Phi_2) \equiv -(\alpha\beta)^{p^n-3}[2a_0\alpha_2\beta_2 + a_1(\alpha_1\beta_2 + \alpha_2\beta_1) + 2a_2\alpha_1\beta_1] \qquad [\text{modd } p, P(x)].$$

The form $f = a_x b_x$ is apolar to $\varphi = (\alpha_1 x_1 - \alpha_2 x_2)(\beta_1 x_1 - \beta_2 x_2) = \alpha_x\beta_x$ provided $I \equiv 0$. In fact the latter congruence is a necessary and sufficient condition that the two forms f, φ respectively be the following linear combinations of two squares in the field:

$$f \equiv s\alpha_x^2 + t\beta_x^2; \qquad \varphi \equiv \sigma a_x^2 + \tau b_x^2.$$

§ 6. The Clebsch-Gordan Expansion.

Gordan's series* is a case of expansion (2). It may be of interest to give a sketch of the derivation of this series from the present point of view, for two reasons: first to show how the preceding general theory applies, and secondly because our methods give a very compact treatment of the details of the derivation, which is believed to be novel.

We employ as symbolical basis the formal identity

$$(E) \qquad a_x b_y - a_y b_x = (ab)(xy).$$

From it we have

$$a_x^m b_y^n \equiv [a_y b_x + (ab)(xy)]^m b_y^{n-m} \qquad (n \geqq m),$$

or

$$(25) \qquad a_x^m b_y^n = a_y^m b_x^m b_y^{n-m} + \binom{m}{1} a_y^{m-1} b_x^{m-1} b_y^{n-m}(ab)(xy) + \cdots + \binom{m}{k} a_y^{m-k} b_x^{m-k} b_y^{n-m}(ab)^k(xy)^k + \cdots + (ab)^m(xy)^m b_y^{n-m}.$$

This is one form of Gordan's series. Here we have (§ 1)

$$\Phi = a_y^m b_x^m b_y^{n-m}, \qquad \nabla = \frac{\partial^2}{\partial x_1\partial y_2} - \frac{\partial^2}{\partial x_2\partial y_1}.$$

But to reduce the series to the ordinary form we use the following polar formula:

$$(a_x^m, b_x^n)_{y^{n-k}}^k$$

$$(26) \qquad = \frac{(ab)^k}{\binom{m+n-2k}{n-k}} \sum_{h=0}^{m-k} \binom{m-k}{m-k-h}\binom{n-k}{n-m+h} a_y^{m-k-h} a_x^h b_x^{m-k-h} b_y^{n-m+h}.$$

* Gordan, *Ueber Combinanten*, Mathematische Annalen, vol. 5 (1872), p. 95.

Since the sum of the numerical coefficients in the polar of a product is unity, (26) can be written

$$(27)\qquad \begin{aligned}(a_x^m, b_x^n)^k_{y^{n-k}} &= (ab)^k a_y^{m-k} b_x^{m-k} b_y^{n-m} \\ &+ \frac{(ab)^k}{\binom{m+n-2k}{n-k}} \sum_{h=1}^{m-k} \binom{m-k}{m-k-h}\binom{n-k}{n-m+h} a_y^{m-k-h} b_x^{m-k-h} b_y^{n-m} \\ &\qquad \times (a_x^h b_y^h - a_y^h b_x^h).\end{aligned}$$

The first term of this is, aside from the factor $\binom{m}{k}$, the coefficient of $(xy)^k$ in (25). Hence this coefficient is the $(n-k)$th transvectant of a_x^m and b_x^n, minus terms which contain the factor $(ab)^{k+1}(xy)$. Hence (27) may be used as a recursion formula; and, starting with $k = m$, we get

$$(28)\qquad \begin{aligned}(ab)^m b_y^{n-m} &= (a_x^m, b_x^n)^m_{y^{n-m}}, \\ (ab)^{m-1} a_y b_x b_y^{n-m} &= (a_x^m, b_x^n)^{m-1}_{y^{n-m+1}} - \frac{1}{n-m+2}(a_x^m, b_x^n)^m_{y^{n-m}}(xy), \\ (ab)^{m-2} a_y^2 b_x^2 b_y^{n-m} &= (a_x^m, b_x^n)^{m-2}_{y^{n-m+2}} - \frac{4}{n-m+4}(a_x^m, b_x^n)^{m-1}_{y^{n-m+1}}(xy) \\ &\qquad + \frac{2}{(n-m+2)(n-m+3)}(a_x^m, b_x^n)^m_{y^{n-m}}(xy)^2\end{aligned}$$

And, in general, by induction

$$(29)\qquad \begin{aligned}(ab)^{m-k} a_y^k b_x^k b_y^{n-m} &= \alpha_0 (a_x^m, b_x^n)^{m-k}_{y^{n-m+k}} + \alpha_1 (a_x^m, b_x^n)^{m-k+1}_{y^{n-m+k-1}}(xy) + \cdots \\ &+ \alpha_j (a_x^m, b_x^n)^{m-k+j}_{y^{n-m+k-j}}(xy)^j + \cdots + \alpha_k (a_x^m, b_x^n)^m_{y^{n-m}}(xy)^k.\end{aligned}$$

Consequently it follows at once from (25) that

$$(30)\qquad \begin{aligned}a_x^m b_y^n &= C_0 (a_x^m, b_x^n)^0_{y^n} + C_1 (a_x^m, b_x^n)^1_{y^{n-1}}(xy) \\ &+ \cdots + C_j (a_x^m, b_x^n)^j_{y^{n-j}}(xy)^j + \cdots + C_m (a_x^m, b_x^n)^m_{y^{n-m}}(xy)^m\end{aligned}$$

The constants C_i may now be determined in the usual manner by operating on both sides of (30) with the proper power of Δ, and noting that when $y \backsim x$ in any polar it goes back into the original polarized form. The following known formula* may be used for this purpose:

$$(31)\qquad \Delta^i (xy)^j \alpha_x^{n-j} \alpha_y^{m-j} = \frac{\underline{|j}\ \ \underline{|m+n-j+1}}{\underline{|j-i}\ \ \underline{|m+n-j-i+1}} (xy)^{j-i} \alpha_x^{n-j} \alpha_y^{m-j} \qquad (i \leqq j).$$

* Grace and Young, *Algebra of Invariants*, p. 54.

The coefficients may also be derived by multiplying the respective members of the column (28) by

$$\binom{m}{m}(xy)^m,\ \binom{m}{m-1}(xy)^{m-1},\ \cdots,\ \binom{m}{2}(xy)^2,\ \binom{m}{1}(xy),\ 1;$$

and adding the results. Either method gives the well-known result

(32) $$a_x^m b_y^n = \sum_{i=0}^{m} \frac{\binom{m}{i}\binom{n}{i}}{\binom{m+n-i+1}{i}} (xy)^i (a_x^m, b_x^n)^i_{y^{n-i}}.$$

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