CONGRUENCES AND COMPLEXES OF CIRCLES*

BY

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The differential geometry of circle systems has received a large amount of attention of late. The subject has been approached from two quite distinct sides. On the one hand we have Koenigs, Cosserat, Moore, Bompiani and others who fix their attention on what we may call the descriptive differential properties of circles. The totality of circles in three-dimensional space may be represented by an S_6^5, lying in an S_9, and this variety may be studied by the now familiar methods of projective differential geometry. The theorems so reached are invariant under the twenty-four parameter group of sphere transformations. The other class of writers, wherein we may include Bianchi, Tzitzeica, Guichard and Eisenhart, have confined themselves largely to congruences (two-parameter systems) of circles, frequently to normal congruences. The methods employed have been the general ones of differential geometry and the center of interest has been rather more in the axes of the circles than the circles themselves.

It has seemed for some time to the present writer that the last word on these subjects had not by any means been written, and that by a different method of approach not only might we obtain simpler proofs of known theorems but discover a number of new theorems as well. The most interesting properties of circles are those which are invariant under inversion, or under the ten-parameter group of conformal transformations of space; the best approach to this group is through the use of pentaspherical coördinates. It is true that some of the writers mentioned above have made use of these coördinates, and still more has been done by Darboux in his *Théorie des Surfaces*, yet the possibilities of these coördinates have been by no means exhausted, and in the present paper they are more systematically applied to problems in differential circle geometry than has been the case in the past.

A circle may be regarded in two different aspects, either as a locus of points, or an envelop of spheres. The two points of view are, of course, closely related, but the change of emphasis leads naturally to rather different sets of theorems. The first section of the present article is devoted to preliminary formulæ for points and spheres in pentaspherical coördinates, and certain

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fundamental theorems concerning cospherical and orthogonal circles. In section 2 a parametric representation is given of the points of a circle, and it is shown how this leads to direct and elegant proofs of the usual theorems about circle congruences, in so far as they are invariant under the conformal group. In the third section the circle is considered in its other aspect, and the classic methods of the Kummer line-geometry are applied to circle congruences, thus giving a goodly number of new theorems. The fourth section is devoted to three-parameter families of circles.

§ 1. Preliminary Formulæ.

In the present work we shall confine ourselves to points in the finite region of space, locating each by a system of homogeneous pentaspherical coördinates, whose relation to the homogeneous rectangular Cartesian system is exhibited by the following equations:

$$\begin{aligned} \rho x_0 &= i\,(x^2+y^2+z^2+t^2), & \sigma x &= x_2, \\ \rho x_1 &= x^2+y^2+z^2-t^2, & \sigma y &= x_3, \\ \rho x_2 &= 2xt, & \sigma z &= x_4, \\ \rho x_3 &= 2yt, & \sigma t &= -\,(ix_0+x_1), \\ \rho x_4 &= 2zt, \end{aligned} \tag{1}$$

$$\sum_{i=0}^{i=4} x_i^2 \equiv (xx) = 0. \tag{2}$$

A sphere will be given by an equation of the form

$$\sum_{i=0}^{i=4} y_i\,x_i \equiv (yx) = 0. \tag{3}$$

The sphere will be null if (y) satisfy the identity (2) in which case y_i will be the coördinates of the center in our present system. The angle of two not null spheres (y) and (z) will be

$$\theta = \cos^{-1}\frac{(yz)}{\sqrt{(yy)}\,\sqrt{(zz)}}. \tag{4}$$

It is convenient when a sphere is not null to assume that its coördinates (coefficients) satisfy the identity

$$(yy) = 1. \tag{5}$$

If (x) and (x') be two points, then every sphere (y), where

$$y_i = \lambda x_i + \mu x'_i,$$

will be a sphere of inversion to interchange the two. We see, in fact, that every sphere orthogonal to (x) and (x') is orthogonal to (y) also. By our identity,

$$(xx) = (x'x') = 0.$$

If then

$$(xx') = 0$$

the sphere (y) will be null. The circle common to the null spheres whose centers are (x) and (x') will contain none but null spheres, i. e., (x) and (x') lie on an isotropic line. Suppose that we have two infinitely near points (x) and $(x + dx)$,

$$(xdx) = -\tfrac{1}{2}(dxdx).$$

Since this is a differential expression of the second order, we may take (dx) as the coördinates of the sphere through (x) orthogonal to the direction of advance to $(x) + (dx)$. The condition that two directions of advance should be mutually perpendicular is thus

$$(dx\delta x) = 0. \tag{6}$$

If a direction of advance be isotropic we have

$$(dxdx) = 0. \tag{7}$$

Through each circle of finite radius will pass two null spheres whose centers are called the *foci* of the circle. The necessary and sufficient condition that a circle should be null is that the foci should fall together. If, thus, the circle be given by the spheres (y) and (z) it will be null if we have equal roots in the quadratic

$$\lambda^2 (yy) + 2\lambda\mu (yz) + \mu^2 (zz) = 0,$$

that is,

$$(yy)(zz) - (yz)^2 = 0.$$

The following well-known theorems are given for reference:

A. A necessary and sufficient condition that two circles of finite radius should be cospherical is that their foci should be concyclic.

B. A necessary and sufficient condition that two circles of finite radius should touch one another is that their foci should lie on two intersecting isotropic lines.

C. If two circles be cospherical they will be cospherical and orthogonal to an infinite number of circles; if they be not cospherical and their foci do not lie in pairs on two isotropics, there will be two circles cospherical and orthogonal to both. The last two circles are in bi-involution, i. e., every sphere through the one is orthogonal to every sphere through the other.

There remains the case where the foci of the two circles lie in pairs on two skew isotropics. These two will be generators of one system of a sphere.

There is a one-parameter group of conformal transformations leaving invariant these two generators as well as all generators of the other system. For instance, if the fixed sphere were

$$x_0 = 0$$

we might consider transformations of the quaternion type

$$(x_1' + ix_2' + jx_3' + kx_4') = (a + bi + cj + dk)(x_1 + ix_2 + jx_3 + kx_4),$$
$$i^2 = j^2 = k^2 = ijk = -1.$$

It appears thus that circles cospherical and orthogonal to our given circles will be continuously transformed by this group, so that there are an infinite number of such circles. Our original circles shall be said in this case to be paratactic.*

D. A necessary and sufficient condition that two non-cospherical circles should be cospherical and orthogonal to more than two circles is that they should be paratactic.

§ 2. The Parametric Method. The Classical Theorems of Ribaucour and Darboux.

Let a circle be determined by a sphere (β) to which it is orthogonal, and the two points of intersection therewith (α) and (γ). The coördinates of every point thereon may be exhibited in the parametric form

$$x_i = \rho^2\alpha_i + \rho\beta_i + \gamma_i,$$
$$(8) \qquad (\alpha\alpha) = (\alpha\beta) = (\beta\gamma) = (\gamma\gamma) = 0,$$
$$(\beta\beta) = -2(\alpha\gamma) = 1.$$

This form is especially suitable to the study of congruences of circles. We therefore assume that (α), (β), (γ) are functions of two independent variables u and v, the various ratios being naturally supposed not to be all constant or functions of a single variable. We have the additional equations

$$\left(\alpha\frac{\partial\alpha}{\partial u}\right) = \left(\alpha\frac{\partial\alpha}{\partial v}\right) = \left(\beta\frac{\partial\beta}{\partial u}\right) = \left(\beta\frac{\partial\beta}{\partial v}\right) = \left(\gamma\frac{\partial\gamma}{\partial u}\right) = \left(\gamma\frac{\partial\gamma}{\partial v}\right) = 0,$$

$$(9) \quad \left(\alpha\frac{\partial\beta}{\partial u}\right) + \left(\beta\frac{\partial\alpha}{\partial u}\right) = \left(\alpha\frac{\partial\beta}{\partial v}\right) + \left(\beta\frac{\partial\alpha}{\partial v}\right) = \left(\beta\frac{\partial\gamma}{\partial u}\right) + \left(\gamma\frac{\partial\beta}{\partial u}\right) =$$

$$\left(\beta\frac{\partial\gamma}{\partial v}\right) + \left(\gamma\frac{\partial\beta}{\partial v}\right) = \left(\alpha\frac{\partial\gamma}{\partial u}\right) + \left(\gamma\frac{\partial\alpha}{\partial u}\right) = \left(\alpha\frac{\partial\gamma}{\partial v}\right) + \left(\gamma\frac{\partial\alpha}{\partial v}\right) = 0.$$

Our groups of equations (8) and (9) lead to easy proofs of all the classical

* The concept of paratactic circles seems first to have been introduced in an article by the Author, *A Study of the Circle Cross*, These Transactions, vol. 14 (1913), pp. 149–174. Proofs of our theorems *A–D* will there be found.

theorems about circle congruences, in so far, that is, as these theorems are invariant for inversion. We state a number of these theorems which are particularly important. They will be found for the most part in the second volume of Darboux's *Theorie generale des surfaces*, but mostly trace their origin to Ribaucour.*

THEOREM 1. *The circles of a congruence are, in general, tangent to four surfaces.*

In practice it is more convenient to say that a circle in general position intersects four adjacent circles. When this intersection arises from the fact that it is cospherical with two adjacent circles, i. e., strictly speaking, when each circle is a generator of two annular surfaces of the congruence, the latter is said to be *focal*. The corresponding generating spheres through the circle are called the *focal spheres*. The points where the circle touches the four surfaces are the *focal points*, and must not be confused with the *foci* mentioned above. If the circles of a congruence be orthogonal to an infinite number of surfaces, the congruence is said to be *normal*. In counting the number of orthogonal trajectories attached to a given congruence, we count the number of points on each circle where it meets some surface orthogonally. Thus, the circles orthogonal to a given sphere are said to have (at least) *two* orthogonal trajectories.

THEOREM 2. *If the circles of a congruence be orthogonal to more than two surfaces, the congruence is normal.*

THEOREM 3. *Any four orthogonal trajectories of the circles of a normal congruence will meet those circles in sets of points having a fixed cross ratio.*

THEOREM 4. *The foci of the circles of a focal congruence generate the two nappes of the envelope of a congruence of spheres.*

THEOREM 5. *If the foci of the circles of a congruence be the pairs of points of contact of the spheres of a non-parabolic congruence with their envelope, the congruence of circles is focal.*

THEOREM 6. *A necessary and sufficient condition that a focal congruence should be normal is that the focal spheres through a circle in general position should be mutually orthogonal, or that it should consist in circles through two points.*

THEOREM 7. *Every normal congruence is focal.*

THEOREM 8. *In a normal congruence the lines of curvature correspond to one another in all orthogonal trajectories, and give the annular surfaces of the congruence.*

THEOREM 9. *If a congruence of spheres have an envelope of two nappes whereon the lines of curvature correspond to one another, the circles orthogonal to the various spheres at their points of contact will generate a normal congruence.*

* See his two notes *Sur les systemès cyclicques* and *Sur les faisceaux de cercles*, Paris, Comptes Rendus, vol. 76 (1873); also an earlier note *Sur la déformation des surfaces*, ibid., vol. 70 (1870).

THEOREM 10. *If the circles of a congruence have two orthogonal trajectories the congruence will be focal when, and only when it is normal.*

THEOREM 11. *If the circles of a congruence be normal to two surfaces which do not consist in a sphere counted twice, and can be assembled into a one-parameter family of annular surfaces, the congruence is normal.*

As examples of the method of applying our equations (8), (9) to the proof of these theorems, we select theorems 2, 3, 6, and 8.

2. We wish to make t in (8) such a function of u and v that if dx_i be the differential along this surface, and δx_i the differential along the circle, then

$$(dx\,\delta x) = 0.$$

This involves

$$-\frac{\partial t}{\partial u} = \left(\alpha\frac{\partial\beta}{\partial u}\right)t^2 + 2\left(\alpha\frac{\partial\gamma}{\partial u}\right)t + \left(\beta\frac{\partial\gamma}{\partial u}\right),$$

$$-\frac{\partial t}{\partial v} = \left(\alpha\frac{\partial\beta}{\partial v}\right)t^2 + 2\left(\alpha\frac{\partial\gamma}{\partial v}\right)t + \left(\beta\frac{\partial\gamma}{\partial v}\right).$$

The condition for compatibility becomes, in the light of (9),

$$\begin{aligned}&\left\{\left(\frac{\partial\alpha}{\partial v}\frac{\partial\beta}{\partial u}\right) - \left(\frac{\partial\alpha}{\partial u}\frac{\partial\beta}{\partial v}\right) - 2\left[\left(\alpha\frac{\partial\beta}{\partial u}\right)\left(\alpha\frac{\partial\gamma}{\partial v}\right) - \left(\alpha\frac{\partial\beta}{\partial v}\right)\left(\alpha\frac{\partial\gamma}{\partial u}\right)\right]\right\}t^2\\ &+ 2\left\{\left(\alpha\frac{\partial\beta}{\partial v}\right)\left(\beta\frac{\partial\gamma}{\partial u}\right) - \left(\alpha\frac{\partial\beta}{\partial u}\right)\left(\beta\frac{\partial\gamma}{\partial v}\right) + \left(\frac{\partial\alpha}{\partial v}\frac{\partial\gamma}{\partial u}\right) - \left(\frac{\partial\alpha}{\partial u}\frac{\partial\gamma}{\partial v}\right)\right\}t\\ &+ \left\{\left(\frac{\partial\beta}{\partial v}\frac{\partial\gamma}{\partial u}\right) - \left(\frac{\partial\beta}{\partial u}\frac{\partial\gamma}{\partial v}\right) + 2\left[\left(\alpha\frac{\partial\gamma}{\partial v}\right)\left(\beta\frac{\partial\gamma}{\partial u}\right) - \left(\alpha\frac{\partial\gamma}{\partial u}\right)\left(\beta\frac{\partial\gamma}{\partial v}\right)\right]\right\} = 0.\end{aligned}\tag{10}$$

Since this equation is quadratic, if it have more than two solutions it is an identity.

3. Let (α) and (γ) be so chosen as to trace two orthogonal trajectories

$$(\alpha d\beta) = (\beta d\alpha) = (\gamma d\beta) = (\beta d\gamma) = 0.\tag{11}$$

For another orthogonal trajectory we must have

$$-\frac{\partial t}{\partial u} = 2\left(\alpha\frac{\partial\gamma}{\partial u}\right)t, \qquad -\frac{\partial t}{\partial v} = 2\left(\alpha\frac{\partial\gamma}{\partial v}\right)t.$$

The condition of compatibility is

$$\left(\frac{\partial\alpha}{\partial u}\frac{\partial\gamma}{\partial v}\right) = \left(\frac{\partial\alpha}{\partial v}\frac{\partial\gamma}{\partial u}\right).\tag{12}$$

This equation will be of great importance later. If t be one solution of our system of differential equations above, rt is another, if r be constant. But this constant is the cross ratio of the four points corresponding to the parameter values ∞, 0, t, rt.

6. Our congruence being focal, we may take u and v as focal parameters. We take for (β) the sphere whose points of contact with its envelop will, by theorem 4, be the foci of our circles; these foci shall be (s) and (t)

$$(ss) = (s\alpha) = (s\beta) = (s\gamma) = (sd\beta) = 0,$$
$$(tt) = (t\alpha) = (t\beta) = (t\gamma) = (td\beta) = 0.$$

From the first equations we get

$$A\alpha_i + B\beta_i + C\gamma_i + D\frac{\partial\beta_i}{\partial u} + Es_i = 0,$$

$$A'\alpha_i + B'\beta_i + C'\gamma_i + D'\frac{\partial\beta_i}{\partial v} + E's_i = 0.$$

Multiplying through by t_i and summing, then by β_i and summing, we have

$$E = B = 0, \qquad D \neq 0,$$

$$\frac{\partial\beta_i}{\partial u} = a\alpha_i + c\gamma_i, \qquad \frac{\partial\beta_i}{\partial v} = a'\alpha_i + c'\gamma_i.$$

Since all points of the circle $(\alpha) + (\partial\alpha/\partial u)\,du$, $(\gamma) + (\partial\gamma/\partial u)\,du$, $(\beta) + (\partial\beta/\partial u)\,du$ lie on a sphere through our circle, a focal sphere,

$$\frac{\partial\gamma_i}{\partial u} = l\alpha_i + m\beta_i + n\gamma_i + r\frac{\partial\alpha_i}{\partial u},$$

$$\frac{\partial\gamma_i}{\partial v} = \lambda\alpha_i + \mu\beta_i + \nu\gamma_i + \rho\frac{\partial\alpha_i}{\partial v}.$$

If the congruence be normal, (10) must be satisfied identically. On substituting, the coefficient of t^2 and the constant vanish. The condition is thus

$$\left[cc' - 4\left(\frac{\partial\alpha}{\partial u}\frac{\partial\alpha}{\partial v}\right)\right](r - \rho) = 0.$$

If $r = \rho$, every solution of

$$(x\alpha) = (x\beta) = (x\gamma) = (xd\alpha) = 0$$

is also a solution of

$$(xd\beta) = (xd\gamma) = 0.$$

Each two adjacent circles are co-spherical, and all of our circles pass through two points or lie on one sphere. The latter would not however yield a normal congruence.

If $r \neq \rho$ we may take as focal spheres

$$\rho y_i = \left| \alpha_j \, \beta_k \, \gamma_l \frac{\partial \alpha_m}{\partial u} \right|, \qquad \sigma z_i = \left| \alpha_j \, \beta_k \, \gamma_l \frac{\partial \alpha_m}{\partial v} \right|.$$

The condition that these should be mutually orthogonal is

$$cc' - 4\left(\frac{\partial \alpha}{\partial u} \frac{\partial \alpha}{\partial v}\right) = 0.$$

8. We have a normal congruence, so that we have the use of equations (11). We may in two ways (usually distinct) solve the following equations for s_i, du, dv

$$(s\alpha) = (s\beta) = (s\gamma) = (sd\alpha) = (sd\gamma).$$

By a change of parameters we may write

$$\frac{\partial \gamma_i}{\partial u} = l\alpha_i + b\beta_i + m\gamma_i + n\frac{\partial \alpha_i}{\partial u},$$

$$\frac{\partial \gamma_i}{\partial v} = \lambda\alpha_i + b'\beta_i + \mu\gamma_i + \rho\frac{\partial \alpha_i}{\partial u}.$$

Multiplying through by β_i and summing, we get

$$b = b' = 0.$$

The condition for a normal congruence (12) gives

$$n\left(\frac{\partial \alpha}{\partial u} \frac{\partial \alpha}{\partial v}\right) = \rho\left(\frac{\partial \alpha}{\partial u} \frac{\partial \alpha}{\partial v}\right).$$

If $n = \rho$, then

$$d\gamma_i = P\alpha_i + Q\gamma_i + Rd\alpha_i.$$

The points (α), (γ), $(\alpha) + (d\alpha)$, $(\gamma) + (d\gamma)$ are always concyclic. Hence the circles with these as foci pass through two points or lie on a sphere, since any two adjacent ones are cospherical. The first case is inadmissible as (α) and (γ) would trace a circle. In the other case they trace surfaces mutually inverse in this sphere and the lines of curvature correspond.

If $n \neq \rho$, then

$$\left(\frac{\partial \alpha}{\partial u} \frac{\partial \alpha}{\partial v}\right) = -\left(\alpha \frac{\partial^2 \alpha}{\partial u \, \partial v}\right) = 0,$$

$$\left(\frac{\partial \gamma}{\partial u} \frac{\partial \gamma}{\partial v}\right) = -\left(\gamma \frac{\partial^2 \gamma}{\partial u \, \partial v}\right) = 0.$$

The condition of compatibility of the two partial differential equations for (γ) is

$$(R - P)\frac{\partial^2 \alpha_i}{\partial u\,\partial v} + \beta\frac{\partial \alpha_i}{\partial u} + C\frac{\partial \alpha_i}{\partial v} + D\alpha_i + E\gamma_i = 0.$$

Multiplying through by α_i and summing, we find $E = 0$ and

$$\frac{\partial^2 \alpha_i}{\partial u\,\partial v} = a\alpha_i + b\frac{\partial \alpha_i}{\partial u} + c\frac{\partial \alpha_i}{\partial v},$$

$$\frac{\partial^2 \gamma_i}{\partial u\,\partial v} = a'\alpha_i + b'\frac{\partial \alpha_i}{\partial u} + c'\frac{\partial \alpha_i}{\partial v}.$$

These equations are characteristic for the lines of curvature, for they show that the Gauss coefficient D' must vanish for both surfaces, while we saw above

$$\left(\frac{\partial \alpha}{\partial u}\,\frac{\partial \alpha}{\partial v}\right) = \left(\frac{\partial \gamma}{\partial u}\,\frac{\partial \gamma}{\partial v}\right) = 0,$$

showing that the u and v curves are mutually orthogonal.*

§ 3. The Kummer Method.

Since the totality of spheres can be put into one-to-one correspondence with that of all points of a four-dimensional space (if planes be counted as spheres) so there is a one-to-one correspondence between the circles of our space and the lines of a four-dimensional space. Moreover our formula (4) for the angle of two spheres is identical with the non-euclidean distance formula, so that the angles of spheres will correspond to the non-euclidean distances of points. We are thus led to the idea of approaching our circle systems by the classical methods of Kummer for line geometry, with the modifications necessary for a four-dimensional universe, and non-euclidean measurement.† We shall fix a circle by two mutually orthogonal spheres (y) and (z) whose coördinates are supposed to be functions of u and v

$$y_i = y_i(u, v), \qquad z_i = z_i(u, v), \qquad (yy) = (zz) = 1, \qquad (yz) = 0.$$

We have the three fundamental equations

$$\begin{aligned} &(dy\,dy) - (zdy)^2 = Edu^2 + 2Fdu\,dv + Gdv^2, \\ (13) \qquad &(dz\,dz) - (ydz)^2 = E'\,du^2 + 2F'\,du\,dv + G'\,dv^2, \\ &(dy\,dz) = e\,du^2 + (f + f')\,du\,dv + ydv^2. \end{aligned}$$

More specifically

* Conf. Darboux, *Surfaces*, vol. I, p. 221.

† The Kummer formulæ for non-euclidean space were first worked out by Fibbi, *I sistemi doppiamente infiniti di raggi negli spazii di curvatura costante*, Annali della R. Scuola Normale Superiore, Pisa, 1891. See also the Author's *Elements of Non-Euclidean Geometry*, Oxford, 1909, Chap. 16.

$$\left(\frac{\partial y}{\partial u}\frac{\partial y}{\partial u}\right)-\left(z\frac{\partial y}{\partial u}\right)^2=E,\quad \left(\frac{\partial y}{\partial u}\frac{\partial y}{\partial v}\right)-\left(z\frac{\partial y}{\partial u}\right)\left(z\frac{\partial y}{\partial v}\right)=F,$$

$$\left(\frac{\partial y}{\partial v}\frac{\partial y}{\partial v}\right)-\left(z\frac{\partial y}{\partial v}\right)^2=G,$$

$$(14)\quad \left(\frac{\partial z}{\partial u}\frac{\partial z}{\partial u}\right)-\left(y\frac{\partial z}{\partial u}\right)^2=E',\quad \left(\frac{\partial z}{\partial u}\frac{\partial z}{\partial v}\right)-\left(y\frac{\partial z}{\partial u}\right)\left(y\frac{\partial z}{\partial v}\right)=F',$$

$$\left(\frac{\partial z}{\partial v}\frac{\partial z}{\partial v}\right)-\left(y\frac{\partial z}{\partial v}\right)^2=G',$$

$$\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial u}\right)=e,\quad \left(\frac{\partial y}{\partial v}\frac{\partial z}{\partial u}\right)=f,\quad \left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v}\right)=f',\quad \left(\frac{\partial y}{\partial v}\frac{\partial z}{\partial v}\right)=g.$$

These various coefficients are connected by the following syzygy:

$$(15)\quad \begin{vmatrix} y_0 & \cdots & y_4 & 0 \\ z_0 & \cdots & z_4 & 0 \\ \frac{\partial y_0}{\partial u} & \cdots & \frac{\partial y_4}{\partial u} & 0 \\ \frac{\partial y_0}{\partial v} & \cdots & \frac{\partial y_4}{\partial v} & 0 \\ \frac{\partial z_0}{\partial u} & \cdots & \frac{\partial z_4}{\partial u} & 0 \\ \frac{\partial z_0}{\partial v} & \cdots & \frac{\partial z_4}{\partial v} & 0 \end{vmatrix}^2 = \begin{vmatrix} E & F & e & f' \\ F & G & f & g \\ e & f & E' & F' \\ f' & g & F' & G' \end{vmatrix} = 0.$$

We saw in theorem D that two circles in general position will be cospherical and orthogonal with two other circles, the latter being in bi-involution, i. e., all spheres through the one are orthogonal to all through the other. We start with two adjacent circles of our congruence; the circles orthogonal to them will be determined by two mutually orthogonal spheres through the first, each orthogonal to one of two mutually orthogonal spheres through the second. Let the spheres through the first circle be $\cos\varphi\,(y)+\sin\varphi\,(z)$, $-\sin\varphi\,(y)+\cos\varphi\,(z)$, while those through the second are $\lambda\,(y+dy)+u\,(z+dz)$, $-\mu\,(y+dy)+\lambda\,(z+dz)$; writing the conditions for criss-cross orthogonality we have two linear homogeneous equations in $\lambda:\mu$. Equating the discriminant to zero, and remembering that

$$(y\,dy)=-\tfrac{1}{2}(dy\,dy),\quad (z\,dz)=-\tfrac{1}{2}(dz\,dz),\quad (y\,dz)+(z\,dy)=-(dy\,dz),$$

we have

$$\begin{vmatrix} [1-\frac{1}{2}(dy\,dy)]\sin\varphi-(z\,dy)\cos\varphi & (y\,dz)\sin\varphi-[1-\frac{1}{2}(dz\,dz)]\cos\varphi \\ -[1-\frac{1}{2}(dz\,dz)]\sin\varphi-(y\,dz)\cos\varphi & (z\,dy)\sin\varphi+[1-\frac{1}{2}(dy\,dy)]\cos\varphi \end{vmatrix} = 0.$$

Expanding, and casting aside infinitesimals of higher order, we have

$$\begin{aligned}(16)\qquad &[e\,du^2+(f+f')\,du\,dv+g\,dv^2](\cos^2\varphi-\sin^2\varphi)+[(E-E')\,du^2\\ &\quad+2(F-F')\,du\,dv+(G-G')\,dv^2]\sin\varphi\cos\varphi=0.\end{aligned}$$

In order to discuss this equation, we write the two

$$e\,du^2+(f+f')\,du\,dv+g\,dv^2=0,$$

$$(E-E')\,du^2+2(F-F')\,du\,dv+(G-G')\,dv^2=0.$$

They are equivalent to

$$(dy\,dz)=(dy\,dy)-(dz\,dz)=0.$$

Suppose then that the foci of our circle are (α), (γ). We may write

$$(17)\qquad \alpha_i=\frac{-\sqrt{-1}\,y_i+z_i}{2},\qquad \gamma_i=-\frac{\sqrt{-1}\,y_i+z_i}{2};$$

$$y_i=\sqrt{-1}\,(\alpha_i+\gamma_i),\qquad z_1=\alpha_i-\gamma_i.$$

Our pair of differential equations may thus be replaced by

$$(d\alpha\,d\alpha)=(d\gamma\,d\gamma)=0.$$

We have the following possibilities with regard to these two equations:

A. They have, in general, no common root. The isotropic curves do not correspond on the surfaces by the foci. We shall call these *non-conformal congruences*.

B. The equations have one common root, in general. Then on the two surfaces of foci (which must not be confused with the focal surfaces to which our circles are tangent) one system of isotropic curves will be in correspondence. These congruences shall be called *semi-conformal*.

C. The two equations are identical. The two surfaces of foci when they exist are conformally related, and the congruence is said to be *conformal*.

Let us begin with the non-conformal congruence. Here if $du:dv$ be given we usually get a unique value of $\tan 2\varphi$, that is two mutually orthogonal spheres, and so the two circles in bi-involution required. On the other hand if φ be given we have a quadratic in $du:dv$ so that on each sphere through a circle in general position of a non-conformal congruence will lie two circles orthogonal to this circle, cospherical and orthogonal with two adjacent circles of the congruence. These circles will fall together if the equation in $du : dv$ have equal roots, i. e., if

$$\begin{aligned}&[eg - \tfrac{1}{2}(f+f')^2](\tan^2\varphi - 1)^2\\ (18)\qquad &+ [e(G-G') - (F-F')(f+f') + g(E-E')](\tan^2\varphi - 1)\tan\varphi\\ &+ [(E-E')(G-G') - (F-F')^2]\tan^2\varphi = 0.\end{aligned}$$

This equation is unaltered if we replace φ by $\pi/2 + \varphi$.

THEOREM 12. *Through each circle in general position of a non-conformal congruence will pass two pairs of mutually orthogonal spheres on each of which there is but one circle orthogonal to the given circle, cospherical and orthogonal to an adjacent circle of the congruence.*

These spheres shall be called the *limiting spheres*. They correspond to maximum and minimum values for $\tan\varphi$. We see, in fact, that if we equate to zero the partial derivatives of (16) with regard to du and dv, we get

$$\left[edu + \frac{f+f'}{2}dv\right](\tan^2\varphi - 1) - [(E-E')du + (F-F')dv]\tan\varphi = 0,$$

$$\left[\frac{f+f'}{2}du + gdv\right](\tan^2\varphi - 1) + [(F-F')du + (G-G')dv]\tan\varphi = 0.$$

Eliminating $du : dv$ we fall back on (18). In the case of a real congruence, the spheres containing real circles orthogonal and cospherical to the adjacent circle will lie in determinate angular openings between these limiting spheres. We next make the further assumption that our congruence is focal. The following equations are characteristic of a focal sphere

$$\cos\varphi\, y_i + \sin\varphi\, z_i \equiv \cos(\varphi + d\varphi)(y_i + dy_i) + \sin(\varphi + d\varphi)(z_i + dz_i),$$

$$dy_i\cos\varphi + dz_i\sin\varphi - (y_i\sin\varphi - z_i\cos\varphi)\,d\varphi = 0.$$

Multiplying through by y_i and summing, we have

$$d\varphi = (ydz) = -(zdy).$$

Substituting this value of $d\varphi$ in the last equation, we get

$$\left[\left(\frac{\partial y_i}{du}\cos\varphi + \frac{\partial z_i}{\partial u}\sin\varphi\right) - (y_i\sin\varphi - z_i\cos\varphi)\left(y\frac{\partial z}{\partial u}\right)\right]du$$

$$+\left[\left(\frac{\partial y_i}{\partial v}\cos\varphi + \frac{\partial z_i}{\partial v}\sin\varphi\right) - (y_i\sin\varphi - z_i\cos\varphi)\left(y\frac{\partial z}{\partial v}\right)dv = 0.\right.$$

Multiplying through by $\partial z_i/\partial u$ and summing, then doing the same for $\partial z_i/\partial v$,

$$[edu + fdv]\cos\varphi + [E'du + F'dv]\sin\varphi = 0,$$
$$[f'du + gdv]\cos\varphi + [F'du + G'dv]\sin\varphi = 0.$$

Similarly

$$[edu + f'dv]\sin\varphi + [Edu + Fdv]\cos\varphi = 0,$$
$$[fdu + gdv]\sin\varphi + [Fdu + Gdv]\cos\varphi = 0.$$

Eliminating φ,

$$(19)\quad \begin{aligned} &(E'f' - F'e)\,du^2 + [E'g - F'(f - f') - G'e]\,du\,dv \\ &\qquad + (F'g - G'f)\,dv^2 = 0, \\ &(Ef - Fe)\,du^2 + [Eg - F(f' - f) - Ge]\,du\,dv + (Fg - Gf')\,dv^2 = 0. \end{aligned}$$

Eliminating $du : dv$,

$$(20)\quad \begin{aligned} &(E'G' - F'^2)\tan^2\varphi + [E'g - F'(f + f') + G'e]\tan\varphi + (eg - ff') = 0, \\ &(eg - ff')\tan^2\varphi + [Eg - F(f + f') + Ge]\tan\varphi + (EG - F^2) = 0, \end{aligned}$$

$$(21)\quad \begin{aligned} &[(eg - ff') - (E'G' - F'^2)]\tan^2\varphi + [(E - E')g - (F - F') \\ &\times (f + f') + (G - G')e]\tan\varphi + [(EG - F^2) - (eg - ff')] = 0. \end{aligned}$$

It is necessary, in order that a congruence should be focal, that the two equations (19) and the two equations (20) should be equivalent. It should be noted further that the middle coefficient is the same in (18) and (21). This coefficient will vanish when the corresponding pairs of spheres make equal angles with (y). Now if two spheres be coaxal with a third, and make equal angles therewith, they are mutually inverse therein and conversely, and we have

THEOREM 13. *In a focal congruence, the pairs of limiting spheres are mutually inverse in the same two spheres as are the two focal spheres.*

The necessary and sufficient conditions that a congruence should be focal are best reached in another form. Retaining (α) and (γ) for our foci there must, by (4) and (5), be one sphere through all the points (α), (γ), $(\alpha) + (d\alpha)$, $(\gamma) + (d\gamma)$. This gives the equations

$$\left|\frac{\partial\gamma}{\partial u}\ \alpha\ \gamma\ \frac{\partial\alpha}{\partial u}\ \frac{\partial\alpha}{\partial v}\right| = \left|\frac{\partial\gamma}{\partial v}\ \alpha\ \gamma\ \frac{\partial\alpha}{\partial u}\ \frac{\partial\alpha}{\partial v}\right| = 0,$$

which are equivalent to

$$\left|\frac{\partial z}{\partial u}\ y\ z\ \frac{\partial y}{\partial u}\ \frac{\partial y}{\partial v}\right| = \left|\frac{\partial z}{\partial v}\ y\ z\ \frac{\partial y}{\partial u}\ \frac{\partial y}{\partial v}\right| = 0.$$

Squaring we have

$$
(22) \qquad \begin{vmatrix} E & F & e \\ F & G & f \\ e & f & E' \end{vmatrix} = \begin{vmatrix} E & F & f' \\ F & G & g \\ f' & g & G' \end{vmatrix} = 0 .
$$

The condition that two adjacent circles intersect is that their common orthogonal sphere should be null. This will give the equation

$$
(Edu^2 + 2Fdu\,dv + Gdc^2)\,(E'\,du^2 + 2F'\,du\,dv + G'\,dv^2) - (edu^2 + (f + f')\,du\,dv + g'\,dv^2)^2 = 0 .
$$

The fact that this equation is of the fourth degree proves theorem 1. For a focal congruence, it must be the square of a quadratic expression. When we know that our congruence is focal, and take u and v as the focal parameters, we have the simple equations

$$
(23) \qquad \begin{gathered} \frac{\partial z_i}{\partial u} = ay_i + ac\,z_i + c\,\frac{\partial y_i}{\partial u}, \qquad \frac{\partial z_i}{\partial v} = a'\,y_i + a'\,c'\,z_i + c'\,\frac{\partial y_i}{\partial v}, \\ E' = ce, \qquad F' = cf' = c'f, \qquad G' = c'g, \\ E = \frac{1}{c}e, \qquad F = \frac{1}{c}f = \frac{1}{c'}f', \qquad G = \frac{1}{c'}g . \end{gathered}
$$

The focal spheres will be

$$
c\,(y) - (z), \qquad c'\,(y) - (z).
$$

The condition for a normal congruence is thus, by theorem 6,

$$
cc' + 1 = 0 .
$$

This may be written in the invariant form

$$
(24) \qquad (EG - F^2) = (E'\,G' - F'^2) = -\,(eg - ff').
$$

These equations and (20) will give necessary and sufficient conditions for a normal congruence.

The normal congruence of circles seems superficially the most natural analog of the normal line congruence. From our present point of view, however, the natural analog is that circle congruence which corresponds to a normal congruence of lines in four-dimensional space. A little reflection shows that such a congruence has the following interpretation in circle geometry. Through each circle (perhaps only in a limited region) will pass one sphere whose points of contact with the two nappes of its envelop are mutually inverse in that circle. (The extension of inversion to the geometry on a sphere is immediate. Two points of a sphere are inverse in a circle thereof, if all circles of the sphere through them are orthogonal to that circle.) Suppose that such a sphere have the coördinates

$$y'_i = \cos\varphi\, y_i + \sin\varphi\, z_i .$$

Then every sphere through the points of contact of (y') with its envelop cuts (y') in a circle orthogonal to the given circle, i. e., every such sphere is orthogonal to the sphere through the given circle orthogonal to (y'). In algebraic form, every sphere

$$\left[(y') + \left(\frac{\partial y'}{\partial u}\right)\right] du + \left(\frac{\partial y'}{\partial v}\right) dv$$

is orthogonal to the sphere

$$-\sin\varphi\,(y) + \cos\varphi\,(z).$$

This yields the equations

$$(25)\qquad \begin{aligned} &-\left(y\frac{\partial z}{\partial u}\right)\sin^2\varphi + \left(z\frac{\partial y}{\partial u}\right)\cos^2\varphi + \frac{\partial\varphi}{\partial u} \\ &\qquad = -\left(y\frac{\partial z}{\partial v}\right)\sin^2\varphi + \left(z\frac{\partial y}{\partial v}\right)\cos^2\varphi + \frac{\partial\varphi}{\partial v} = 0, \\ &\frac{\partial\varphi}{\partial u} = \left(y\frac{\partial z}{\partial u}\right), \qquad \frac{\partial\varphi}{\partial v} = \left(y\frac{\partial z}{\partial v}\right). \end{aligned}$$

The condition of compatibility will be

$$(26)\qquad f = f'.$$

The differential equations (25) are equally well solved by φ and $\varphi + k$ where k is any constant.

THEOREM 14. *If through each circle of a congruence it be possible to pass one sphere whose points of contact with the two nappes of the envelop are mutually inverse in that circle, then an infinite number of such spheres may be passed through each circle. The spheres generate a one-parameter family of congruences, the corresponding spheres of any two congruences will intersect at a constant angle.*

We shall say in this case that our congruence of circles is *pseudo-normal.* Suppose that we have a congruence which is of this sort, and also is focal. We find from (21)

$$(c - c')f = 0.$$

If

$$c = c',$$

$$dz_i = py_i + qz_i + rdy_i ,$$

any two adjacent circles of the congruence will be cospherical, i. e., all the circles lie on a sphere, or pass through two points, two cases which we may rule out. Hence

$$f = f' = 0.$$

Solving (20) in view of (21) we find both roots are solutions of (18), hence

THEOREM 15. *The necessary and sufficient condition that a focal congruence should be pseudo-normal is that the focal spheres should coincide with a pair of limiting spheres.*

Let us see what relations subsist between the foci of the circles of a pseudo-normal congruence. We see from (17) that if we have

$$\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v}\right) = \left(\frac{\partial y}{\partial v}\frac{\partial z}{\partial u}\right),$$

we also have

$$\left(\frac{\partial \alpha}{\partial u}\frac{\partial \gamma}{\partial v}\right) = \left(\frac{\partial \alpha}{\partial v}\frac{\partial \gamma}{\partial u}\right).$$

If then our congruence be focal as well as pseudo-normal, and (β) be the sphere whose points of contact with its envelop are (α) and (γ), by theorem 4 we see that the circle orthogonal to (β) at the points (α) and (γ) (and so in bi-involution with the given circle) will, by equation (12), generate a normal congruence. We next notice that if u and v be the focal parameters for the pseudo-normal congruence, our equations (23) lead to the equations used in proving theorem 8, so that they are focal parameters for the normal congruence also and give the lines of curvature of the (α) and (γ) surfaces.

THEOREM 16. *The necessary and sufficient condition that a focal congruence of circles should be pseudo-normal is that the lines of curvature should correspond in the surfaces traced by the foci of the circles.*

We may find still another necessary and sufficient condition for a pseudo-normal congruence. Revert to our equations (23); the focal spheres were

$$c\,(y) - (z), \qquad c'\,(y) - (z).$$

The spheres orthogonal to our circle through the pairs of focal points are

$$\left(\frac{\partial y}{\partial u}z\right)z_i - \frac{\partial y_i}{\partial u}, \qquad \left(\frac{\partial y}{\partial v}z\right)z_i - \frac{\partial y_i}{\partial v}.$$

These will be mutually orthogonal if

$$F = f = f' = 0.$$

When, however, two spheres are orthogonal to a circle and to one another, their pairs of intersections with the circle separate one another harmonically, and vice versa.

THEOREM 17. *The necessary and sufficient condition that a focal congruence of circles should be pseudo-normal is that the pairs of focal points should separate one another harmonically.*

We get at once from theorem 16

THEOREM 18. *If a normal congruence be given, the congruence of circles whose foci are the pairs of intersections of the circles of the normal congruence with any two orthogonal trajectories is focal and pseudo-normal.*

THEOREM 19. *If a congruence be focal and pseudo-normal, the foci are the pairs of intersections of the circles of a normal congruence with two orthogonal trajectories.*

When a normal and pseudo-normal congruence are related to one another in this way, we shall call them *associated.* The normal congruence has the parametric form

$$x_i = t^2 \alpha_i + t\beta_i + \gamma_i, \qquad \rho\beta_i = \left| y_j \; z_k \; \frac{\partial y_l}{\partial u} \; \frac{\partial y_m}{\partial v} \right|.$$

Here u and v are supposed to be the common focal parameters of both congruences. Let (y') be one of those spheres whose points of contact with the envelop are mutually inverse in the (y) (z) circle. We write

$$y'_i = \cos\varphi\, y_i + \sin\varphi\, z_i,$$

$$\frac{\partial y'_i}{\partial u} = \cos\varphi \frac{\partial y_i}{\partial u} + \sin\varphi \frac{\partial z_i}{\partial u} + \left(y \frac{\partial z}{\partial u} \right)(-\, y_i \sin\varphi + z_i \cos\varphi),$$

$$\frac{\partial y'_i}{\partial v} = \cos\varphi \frac{\partial y_i}{\partial v} + \sin\varphi \frac{\partial z_i}{\partial v} + \left(y \frac{\partial z}{\partial v} \right)(-\, y_i \sin\varphi + z_i \cos\varphi),$$

as we see with the aid of (25). We also see from (17) and the above value of (β) that all points of the circle of the normal congruence lie on the spheres $(\partial y'/\partial u)$, $(\partial y'/\partial v)$, so that, in particular, the points of contact of (β) with its envelop lie on this circle. The circle is orthogonal to (y'), since $(\partial y'/\partial u)$ and $(\partial y'/\partial v)$ are orthogonal thereto.

THEOREM 20. *If a focal and pseudo-normal congruence be given, the spheres whose pairs of points of contact with their envelops are mutually inverse in the circles of the congruence will generate the orthogonal trajectories of the circles of the associated normal congruence.*

Since our two congruences have the same focal parameters

THEOREM 21. *If a normal and pseudo-normal congruence be associated, the annular surfaces will correspond in the two.*

THEOREM 22. *If a normal congruence be given, not consisting in the circles through two points, the pairs of intersections with any two orthogonal trajectories may be taken as foci of the circles of an associated pseudo-normal congruence.*

The other orthogonal trajectories will then be paired in such a way that the intersections of each circle with a pair of trajectories are mutually inverse in the associated circle.

The pseudo-normal circle congruence enjoys a sort of indestructibility, akin to that of the normal line congruence. Let our congruence be determined by the spheres (y) and (z). We may determine (z') in such a way that

$$(z'z') = 1, \qquad (z'y) = 0, \qquad (z'z) = \cos\theta, \qquad \left(\frac{\partial z'}{\partial u}\frac{\partial y}{\partial v}\right) = \left(\frac{\partial z'}{\partial v}\frac{\partial y}{\partial u}\right).$$

Here θ is supposed to have a fixed value, the spheres (y) and (z') will determine a second pseudo-normal congruence of such sort that each of its circles is cospherical with one circle of the original pseudo-normal congruence, and makes therewith a fixed angle. The sphere through the circle common to (z) and (z'), making an angle φ with the former is (z''), where

$$z_i'' = \frac{\sin(\theta - \varphi)}{\sin\theta} z_i + \frac{\sin\varphi}{\sin\theta} z_i'.$$

This cuts (y) in a circle coaxal with the two circles already determined thereon, and making an angle φ with the first of them. If, then, φ be constant

$$\left(\frac{\partial z''}{\partial u}\frac{\partial y}{\partial v}\right) = \left(\frac{\partial z''}{\partial v}\frac{\partial y}{\partial u}\right).$$

THEOREM 23. *If two pseudo-normal congruences be so related that corresponding circles are cospherical and make a fixed angle, then the congruence of circles coaxal with them, and making fixed angles with both, is also pseudo-normal.*

Another theorem of the same sort is obtained as follows. Suppose that we have an analytical complex (three-parameter family) of spheres. A sphere of the complex and its next neighbors are all orthogonal to one sphere which we may speak of as *correlative* to the first. The complex generated by these spheres shall be said to be *correlative* to the first complex. Analytically, if our complex be given parametrically in the form

$$x_i = x_i(u, v, w),$$

the correlative sphere is (y), where

$$\rho y_i = \left| x_j \; \frac{\partial x_k}{\partial u} \; \frac{\partial x_l}{\partial v} \; \frac{\partial x_m}{\partial w} \right|,$$

Since

$$(xy) = \left(y\frac{\partial x}{\partial u}\right) = \left(y\frac{\partial x}{\partial v}\right) = \left(y\frac{\partial x}{\partial w}\right) = \left(x\frac{\partial y}{\partial u}\right) = \left(x\frac{\partial y}{\partial v}\right) = \left(x\frac{\partial y}{\partial w}\right) = 0,$$

we see that (x) and (y) bear a reciprocal relation, or each is the correlative of the other. This is true in general; there are special cases where (y) may depend on less than three parameters, but these we explicitly exclude by saying that our complex is *non-developable.* Suppose, then, that we have a pseudo-normal congruence. Through each circle will pass at least one sphere of any chosen non-developable complex. We take this for our sphere (y), the correlative sphere is (t), where

$$\rho t_i = \left| y_j \frac{\partial y_k}{\partial u} \frac{\partial y_l}{\partial v} s_m \right|,$$

where s_m is a function of u, v, w. We further put

$$z'_i = \frac{\sin(\varphi - \theta)}{\sin\theta} t_i + \frac{\sin\varphi}{\sin\theta} z_i,$$

so that θ is the angle of (z) and (t), while φ is the angle of (z') and (t), and assume finally that the ratio of $\sin\varphi$ to $\sin\theta$ has the constant value k. We have

$$(yt) = \left(y\frac{\partial t}{\partial u}\right) = \left(y\frac{\partial t}{\partial v}\right) = 0,$$

$$\left(\frac{\partial z'}{\partial u}\frac{\partial y}{\partial v}\right) - \left(\frac{\partial z'}{\partial v}\frac{\partial y}{\partial u}\right) = k\left[\left(\frac{\partial z}{\partial u}\frac{\partial y}{\partial v}\right) - \left(\frac{\partial z}{\partial v}\frac{\partial y}{\partial u}\right)\right],$$

and this last equation gives the four-dimensional equivalent of the Malus-Dupin theorem.

THEOREM 24. *Through each circle c of a pseudo-normal congruence a sphere is passed belonging to a given non-developable complex in such a way as to generate an analytic congruence of that complex. This sphere meets the correlative sphere in c', and a circle c'' is found coaxal with c and c' so that the sines of the angles of c and c'' with c' bear to one another a constant ratio. The circles c'' will also generate a pseudo-normal congruence.*

We now pass from the non-conformal congruence to the other types noted above. The semi-conformal type has the property that each circle in general position is tangent or paratactic to one adjacent circle. The members of such a congruence are, however, essentially imaginary, and we pass on to the more interesting conformal congruences. Here we have

$$(E - E'):(F - F'):(G - G') = e:\frac{f+f'}{2}:g. \tag{27}$$

Equations (16) become

$$[\,edu^2 + (f + f')\,du\,dv + gdv^2]\,[\cos 2\varphi + k\sin 2\varphi] = 0.$$

The roots of the second factor give two mutually orthogonal spheres, which we may take for our spheres (y) and (z). A circle cospherical and orthogonal to a circle in general position and one of its next neighbors must therefore, in general, lie either on (y) or (z). The sphere (y) is thus orthogonal to (z) and to $(z)+(dz)$ so that

$$(28)\quad \begin{aligned} (ydz) = (zdy) = -\tfrac{1}{2}(dy\,dz) = 0, \\ e = f + f' = g = 0. \end{aligned}$$

Reverting to (17) we see

$$\left(\frac{\partial\alpha}{\partial u}\frac{\partial\alpha}{\partial u}\right) = \left(\frac{\partial\gamma}{\partial u}\frac{\partial\gamma}{\partial u}\right), \quad \left(\frac{\partial\alpha}{\partial u}\frac{\partial\alpha}{\partial v}\right) = \left(\frac{\partial\gamma}{\partial u}\frac{\partial\gamma}{\partial v}\right), \quad \left(\frac{\partial\alpha}{\partial v}\frac{\partial\alpha}{\partial v}\right) = \left(\frac{\partial\gamma}{\partial v}\frac{\partial\gamma}{\partial v}\right).$$

This shows that if (α) and (γ) really depend on two parameters, their surfaces are conformally related. There is the additional possibility that all of these terms should vanish identically; (α) and (γ) would then trace two minimal curves. Conversely, let (α) and (γ) trace two conformally related surfaces. We have

$$(d\alpha\, d\alpha) = \sigma\,(d\gamma\, d\gamma).$$

Replacing α_i by $\sqrt[4]{\sigma}\,\alpha_i$ and γ_i by $1/\sqrt[4]{\sigma}\,\gamma_i$

$$(d\alpha\, d\alpha) = (d\gamma\, d\gamma), \qquad (dy\, dz) = 0,$$

we have surely a conformal congruence. We next write

$$y_i' = y_i \cos\varphi + z_i \sin\varphi, \qquad y_i'' = y_i\cos\varphi - z_i\sin\varphi,$$

where φ is constant

$$(dy'\, dy') = (dy''\, dy'').$$

Conversely, if this equation holds, and if φ be a fixed angle

$$(dy\, dz) = 0.$$

We shall say that two congruences of spheres are *conformally related* if the infinitesimal angle of two adjacent spheres of the one is equal to the corresponding infinitesimal angle of the other. We thus get

THEOREM 26. *Through each circle of a conformal congruence we may, in an infinite number of ways, pass two spheres, making a constant angle, and generating conformally related congruences as the circle describes the given congruence.*

THEOREM 25. *If two sphere congruences be conformally related, and corresponding spheres in the two meet at a fixed angle, then the circles of intersection of such spheres will generate a conformal congruence.*

We now make the additional assumption that our circle congruence is focal. The isotropic curves will correspond on the two surfaces generated by the foci. If we take these as our parameter curves u and v we have two conceivable cases

$$(A) \qquad \frac{\partial \gamma_i}{\partial u} = p\alpha_i + q\gamma_i + r\frac{\partial \alpha_i}{\partial v}, \qquad \frac{\partial \gamma_i}{\partial v} = p'\alpha_i + q'\gamma_i + r'\frac{\partial \alpha_i}{\partial u},$$

$$(B) \qquad \frac{\partial \gamma_i}{\partial u} = p\alpha_i + q\gamma_i + r\frac{\partial \alpha_i}{\partial u}, \qquad \frac{\partial \gamma_i}{\partial v} = p'\alpha_i + q'\gamma_i + r'\frac{\partial \alpha_i}{\partial v}.$$

Leaving aside for the moment the question of whether both of these cases actually exist, let us take them up in turn. In case A we have

$$\left(\frac{\partial \alpha}{\partial u}\frac{\partial \gamma}{\partial v}\right) = \left(\frac{\partial \alpha}{\partial v}\frac{\partial \gamma}{\partial u}\right).$$

Our congruence is pseudo-normal, the focal parameters u' and v' will give also the focal directions for the associated normal congruence (by 21), so that they give mutually orthogonal directions of advance for (α) and (γ). We have the partial differential equations, analogous to those previously found for a normal congruence,

$$\frac{\partial \gamma_i}{\partial u'} = bc\alpha_i + b\gamma_i + c\frac{\partial \alpha_i}{\partial u'}, \qquad \frac{\partial \gamma_i}{\partial v'} = b'c'\alpha_i + b'\gamma_i + c'\frac{\partial \alpha_i}{\partial v'}.$$

The relation among the coefficients of α_i, γ_i comes from the equations

$$\left(\alpha\frac{\partial \gamma}{\partial u'}\right) + \left(\gamma\frac{\partial \alpha}{\partial u'}\right) = \left(\alpha\frac{\partial \gamma}{\partial v'}\right) + \left(\gamma\frac{\partial \alpha}{\partial v'}\right) = 0.$$

The condition that our surfaces (α) and (γ) should be conformally related is

$$\left(\frac{\partial \alpha}{\partial u'}\frac{\partial \alpha}{\partial u'}\right)du'^2 + \left(\frac{\partial \alpha}{\partial v'}\frac{\partial \alpha}{\partial v'}\right)dv'^2 = \left(\frac{\partial \gamma}{\partial u'}\frac{\partial \gamma}{\partial u'}\right)du'^2 + \left(\frac{\partial \gamma}{\partial v'}\frac{\partial \gamma}{\partial v'}\right)dv'^2.$$

This gives $c^2 = c'^2$. Now if $c = c'$ we have at once

$$d\gamma_i = P\alpha_i + Q\gamma_i + R\,d\alpha_i,$$

each two adjacent circles are cospherical, and we have the circles through two points or on one sphere, which cases we may exclude. Hence we must have

$$c + c' = 0.$$

This shows that the focal spheres, whose coördinates are $c(\alpha) - (\gamma)$, $c'(\alpha) - (\gamma)$ are mutually orthogonal, and the congruence is, by theorem 6, a normal one. Our processes here are entirely reversible, and we have the theorems

THEOREM 27. *If a congruence be both conformal and pseudo-normal, it is normal.*

THEOREM 28. *If a congruence be both normal and conformal it is pseudo-normal, or else consists in the circles touching a given circle at a given point.*

With regard to the existence of such congruences, we construct an example as follows. Let a cylinder be given whose elements are all parallel to a given plane, and of such cross section that a sphere whose center moves perpendicularly to the direction of the elements, while it remains tangent to the cylinder and the plane, will trace paths of equal length on the two. The points of contact of the spheres with plane and cylinder will be the required foci. As a matter of fact, however, these congruences are well enough known.* In type B we see that the isotropic parameters are the focal ones. We thus see by theorem B that each circle is tangent to two adjacent circles, or the circles of the congruence are the osculating circles of two different one-parameter families of curves. This much is true, if the congruence exist; unfortunately the present writer has signally failed in all his attempts to answer this interesting question.

§ 4. Complexes of Circles.

Our leading idea in approaching our circle geometry by the Kummer method was to interpret in three dimensions those theorems of the line geometry of four dimensions which are easily reached by an extension of the well-known methods of line geometry. The idea lies close at hand that in changing our line geometry from three to four dimensions, we might have done well to allow ourselves an extra parameter, in other words, to busy ourselves with three-parameter circle systems, or complexes. To such systems we now give our attention. We write

$$y_i = y_i\,(u_1\,u_2\,u_3), \qquad z_i = z_i\,(u_1\,u_2\,u_3), \qquad (yy) = (zz) = 1, \qquad (yz) = 0. \tag{29}$$

$$\begin{aligned} (dy\,dy) - (zdy)^2 &\equiv \Sigma a_{ij}\,du_i\,du_j, \qquad a_{ij} = a_{ji}, \\ (dz\,dz) - (ydz)^2 &\equiv \Sigma b_{ij}\,du_i\,du_j, \qquad b_{ij} = b_{ji}, \\ (dydz) &= \Sigma c_{ij}\,du_i\,du_j. \end{aligned} \tag{30}$$

$$\begin{aligned} \left(\frac{\partial y}{\partial u_i}\frac{\partial y}{\partial u_i}\right) - \left(z\frac{\partial y}{\partial u_i}\right)^2 &= a_{ii}, \\ \left(\frac{\partial y}{\partial u_i}\frac{\partial y}{\partial u_j}\right) - \left(z\frac{\partial y}{\partial u_i}\right)\left(z\frac{\partial y}{\partial u_j}\right) &= a_{ij} + a_{ji} = 2a_{ij} = 2a_{ji} \quad (i \neq j). \end{aligned} \tag{31}$$

* The problem of finding a conformal focal congruence is sometimes called the problem of Ribaucour, for it amounts to finding a congruence of spheres which establish a conformal relation between the two nappes of their envelop. For an interesting discussion of the congruences of type A see Darboux, *Sur les surfaces isothermiques*, Annales de l'Ecole Normale, series 3, vol. 16 (1899), pp. 498 ff. Darboux there proves that this is the only type of conformal focal congruence, but his proof is erroneous, as he has acknowledged in a letter to the present writer. He doubted, however, whether any congruences of type B really existed. The theorem that congruences of type A are normal was casually mentioned by Cosserat, *Sur le problème di Ribaucour*, Bulletin de l'Académie des Sciences de Toulouse, vol. 3 (1900), pp. 267 ff.

$$(32)\qquad \left(\frac{\partial z}{\partial u_i}\frac{\partial z}{\partial u_i}\right) - \left(y\frac{\partial z}{\partial u_i}\right)^2 = b_{ii},$$

$$\left(\frac{\partial z}{\partial u_i}\frac{\partial z}{\partial u_j}\right) - \left(y\frac{\partial z}{\partial u_i}\right)\left(y\frac{\partial z}{\partial u_j}\right) = b_{ij} + b_{ji} = 2b_{ij} = 2b_{ji} \quad (i \neq j).$$

$$(33)\qquad \left(\frac{\partial y}{\partial u_i}\frac{\partial z}{\partial u_j}\right) = c_{ij}.$$

The coefficients are connected by various relations

$$(34)\qquad \sum_{ijk}\begin{vmatrix}\frac{\partial y_i}{\partial u_1}\frac{\partial y_j}{\partial u_1}\frac{\partial y_k}{\partial u_1}\\ \frac{\partial y_i}{\partial u_2}\frac{\partial y_j}{\partial u_2}\frac{\partial y_k}{\partial u_2}\\ \frac{\partial y_i}{\partial u_3}\frac{\partial y_j}{\partial u_3}\frac{\partial y_k}{\partial u_3}\end{vmatrix}\cdot\begin{vmatrix}\frac{\partial z_i}{\partial u_1}\frac{\partial z_j}{\partial u_1}\frac{\partial z_k}{\partial u_1}\\ \frac{\partial z_i}{\partial u_2}\frac{\partial z_j}{\partial u_2}\frac{\partial z_k}{\partial u_2}\\ \frac{\partial z_i}{\partial u_3}\frac{\partial z_j}{\partial u_3}\frac{\partial z_k}{\partial u_3}\end{vmatrix} = \begin{vmatrix}c_{11} & c_{12} & c_{13}\\ c_{21} & c_{22} & c_{23}\\ c_{31} & c_{32} & c_{33}\end{vmatrix},$$

$$(35)\qquad \left|z\,y\,\frac{\partial y}{\partial u_1}\,\frac{\partial y}{\partial u_2}\,\frac{\partial y}{\partial u_3}\right|^2 = |a_{ij}|, \qquad \left|y\,z\,\frac{\partial z}{\partial u_1}\,\frac{\partial z}{\partial u_2}\,\frac{\partial z}{\partial u_3}\right|^2 = |\,b_{ij}\,|,$$

$$z_i = \frac{\left|y_j\,\frac{\partial z_k}{\partial u_1}\,\frac{\partial z_l}{\partial u_2}\,\frac{\partial z_m}{\partial u_3}\right|}{\sqrt{|b_{ij}|}}, \qquad y_i = \frac{\left|z_j\,\frac{\partial y_k}{\partial u_1}\,\frac{\partial y_l}{\partial u_2}\,\frac{\partial y_m}{\partial u_3}\right|}{\sqrt{|\,a_{ij}\,|}}.$$

If, then, we write

$$B_{kl} \equiv \frac{\partial\,|\,b_{ij}\,|}{\partial b_{kl}}, \qquad A_{kl} \equiv \frac{\partial\,|\,a_{ij}\,|}{\partial a_{kl}},$$

we shall have

$$(36)\qquad a_{ij} = -\frac{\sum_{kl} c_{ik}\,c_{jl}\,B_{kl}}{|\,b_{ij}\,|}, \qquad b_{ij} = -\frac{\sum_{kl} c_{ik}\,c_{jl}\,A_{kl}}{|\,a_{ij}\,|}.$$

$$(37)\qquad |\,a_{ij}\,|\cdot|\,b_{ij}\,| = |\,c_{ij}\,|^2.$$

We next look for the circles cospherical and orthogonal to a circle in general position, and to one of its next neighbors. Following the method which previously led up to (16) we now have

$$(38)\qquad \sum_{ij} c_{ij}\,du_i\,du_j\,(\cos^2\varphi - \sin^2\varphi) + \sum_{ij}(a_{ij} - b_{ij})\,du_i\,du_j \sin\varphi\cos\varphi = 0.$$

As in the case of the congruence, we must here consider the equations

$$(39)\qquad (d\alpha\,d\alpha) = 0, \qquad (d\gamma\,d\gamma) = 0,$$

where (α) and (γ) are the foci of our circle. When these equations are identical, the complex of circles effect a conformal transformation of space, and the complex is said to be *conformal;* if they have four or fewer common solutions, the complex is called *non-conformal*. As the equations are irreducible, there is no type of complex corresponding to the semi-conformal congruence. We begin with the non-conformal complex. The circles cospherical and orthogonal to two given circles are indeterminate when, and only when, the given circles are cospherical or are paratactic. The four common solutions of the equations (39) will give four circles adjacent to a given circle and either tangent or paratactic thereto. To determine which of these cases arises, let us see how many circles adjacent to a circle in general position of a complex are cospherical therewith. For cospherical adjacent circles we must have

$$dy_i \cos\varphi + dz_i \sin\varphi \sin\varphi - (y_i \sin\varphi - z_i \cos\varphi)\, d\varphi = 0,$$

$$(ydz) = -(zdy) = d\varphi,$$

$$(40)\qquad \sum_{n=1}^{n=3}\left[\frac{\partial y_i}{\partial u_n}\cos\varphi + \frac{\partial z_i}{\partial u_n}\sin\varphi - (y_i\sin\varphi - z_i\cos\varphi)\left(y\frac{\partial z}{\partial u_n}\right)\right]du_n = 0.$$

Multiplying through by $\partial z_i/\partial u_1$, $\partial z_i/\partial u_2$, $\partial z_i/\partial u_3$ and summing, we find

$$(41)\qquad \sum_{j=1}^{j=3} c_{ij}\, du_j + \left[\sum_{j=1}^{j=3} b_{ij}\, du_j\right]\tan\varphi = 0 \qquad (i = 1, 2, 3),$$

$$(42)\qquad \begin{vmatrix} (c_{11} + b_{11}\tan\varphi) & \cdots \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ \cdots & (c_{33} + b_{33}\tan\varphi) \end{vmatrix} = 0.$$

These equations show that a circle in general position will not, usually, be cospherical with more than three adjacent circles. Conversely, if we have the equations (41), (42) and replace the right-hand side of (40) by c_i while $(ydz) = d\varphi$ we find

$$(yt) = (zt) = \left(\frac{\partial y}{\partial u_1}t\right) = \left(\frac{\partial y}{\partial u_2}t\right) = \left(\frac{\partial y}{\partial u_3}t\right) = 0,$$

hence

$$t_i \equiv 0 \qquad (i = 0 \cdots 4).$$

Lastly, we see that the determination of these three circles depends upon the solution of a (usually) irreducible cubic equation, and so must give different results from the solution of the irreducible quartic arising from (39).

THEOREM 29. *A circle in general position in a non-conformal complex lies usually on three annular surfaces thereof.**

THEOREM 30. *A circle in general position in a non-conformal complex is usually paratactic with four adjacent circles of the complex.*

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* This theorem is due to COSSERAT, *Sur le cercle considéré comme élément générateur de l'espace*, Annales de la Faculté des Sciences de Toulouse, vol. 3 (1889) p. E 33.

The qualification *pseudo-normal* may be applied to complexes as well as to congruences of circles. A complex of circles shall be said to be *pseudo-normal* if it be composed of the circles of intersection of corresponding spheres of two correlative complexes. To find the condition that a given complex should be of this type, we put

$$y'_i = y_i \cos \varphi + z_i \sin \varphi, \qquad z'_i = - y_i \sin \varphi + z_i \cos \varphi.$$

The complex will be pseudo-normal if

$$(y' \, dz') = (z' \, dy') = 0.$$

Proceeding exactly as before, we find

$$(ydz) = d\varphi, \tag{43}$$

$$c_{ij} = c_{ji}. \tag{44}$$

We note that if the differential equation for φ have one solution, it will have an infinite number differing by an additive constant.

THEOREM 31. *The circles of a pseudo-normal complex are the intersections of corresponding spheres in a singly infinite system of pairs of correlative complexes. The spheres of any two of these complexes through the same circle make with one another a constant angle.*

We get the necessary and sufficient geometrical conditions for a pseudo-normal complex as follows. Let the three sets of solutions of equations (41), (42) be

$$d^{(1)} u_1 \, d^{(1)} u_2 \, d^{(1)} u_3 \, \varphi_1, \qquad d^{(2)} u_1 \, d^{(2)} u_2 \, d^{(2)} u_3 \, \varphi_2, \qquad d^{(3)} u_1 \, d^{(3)} u_2 \, d^{(3)} u_3 \, \varphi_3.$$

Multiply the three equations (41) through by the corresponding factors $d^{(m)} u_1 \, d^{(m)} u_2 \, d^{(m)} u_3$ and add; we get equations of the form

$$\sum_{ij} c_{ij} \, d^{(m)} u_i \, d^{(n)} u_j + \tan \varphi_m \sum_{ij} b_{ij} \, d^{(m)} u_i \, d^{(n)} u_j = 0 \qquad (m \neq n),$$

$$\sum_{ij} c_{ij} \, d^{(n)} u_i \, d^{(m)} u_j + \tan \varphi_n \sum_{ij} b_{ij} \, d^{(n)} u_i \, d^{(m)} u_j = 0 \qquad (m \neq n).$$

If now our complex be pseudo-normal, so that (44) are satisfied, we get, by subtraction,

$$\sum_{ij} b_{ij} \, d^{(m)} u_i \, d^{(n)} u_j = 0 \qquad (m \neq n). \tag{45}$$

Conversely, when these equations hold, the complex is pseudo-normal. On the other hand if we mean by the *focal points* of a circle the pairs of points where it meets an adjacent cospherical circle, that is, its pairs of points of contact with the envelop on an annular surface of the complex, we find the spheres through these pairs of focal points, orthogonal to the given circle exactly as we did for the congruence. We wish to find a linear combination of (y) and (z), which is also a linear combination of $(y)+(dy)$ and $(z)+(dz)$, the differentials being determined by a set of solutions to our equations (41). The three spheres sought will thus be

$$\sum_{k=1}^{k=3}\left[\left(\frac{\partial z}{\partial u_k}y\right)z_i-\frac{\partial z_i}{\partial u_k}\right]du_k^{(1)},\qquad \sum_{k=1}^{k=3}\left[\left(\frac{\partial z}{\partial u_k}y\right)z_i-\frac{\partial z_i}{\partial u_k}\right]du_k^{(2)},$$

$$\sum_{k=1}^{k=3}\left[\left(\frac{\partial z}{\partial u_k}y\right)z_i-\frac{\partial z_i}{\partial u_k}\right]du_k^{(3)}.$$

The equations above give us the necessary and sufficient conditions that these should be mutually orthogonal in pairs.

THEOREM 32. *A necessary and sufficient condition that a complex should be pseudo-normal is that the pairs of focal points on a circle in general position should separate one another harmonically.*

On an arbitrary sphere there will lie a finite number of circles belonging to a given complex, unless all are orthogonal to one sphere. Let two pseudo-normal complexes be determined by the three-parameter sphere systems (y) (z) and (y) (z') where (z) and (z') make with one another a fixed angle. Then if

$$z_i'' = \frac{\sin(\varphi-\theta)}{\sin\theta}z_i + \frac{\sin\varphi}{\sin\theta}z_i',$$

where φ and θ have constant values, we see that (y) and (z'') determine a pseudo-normal complex.

THEOREM 33. *If two pseudo-normal complexes be so related that corresponding circles are cospherical and make a fixed angle with one another, then the complex of circles coaxial respectively with these corresponding pairs, and making fixed angles with them, is also pseudo-normal.*

Suppose next, that (s) is the sphere correlative to the sphere (y); we may write

$$\rho s_i = \left| y_j \frac{\partial y_k}{\partial u_1}\frac{\partial y_l}{\partial u_2}\frac{\partial y_m}{\partial u_3}\right|.$$

If, then, we put

$$z_i' = \frac{\sin(\varphi-\theta)}{\sin\theta}s_i + \frac{\sin\varphi}{\sin\theta}z_i,$$

and require the ratio of $\sin\varphi$ to $\sin\theta$ to be constant, we see that (y) and (z') determine a pseudo-normal complex.

THEOREM 34. *Through each circle of a pseudo-normal complex a sphere is passed belonging to a determinate non-developable complex and the original circle* *is replaced by such a circle coaxial with it and the circle cut by the correlative sphere, that the sines of the angles formed therewith by the original and the replacing circle have a fixed ratio; then will the replacing circles also generate a pseudo-normal complex.*

The consideration of the circles adjacent to and cospherical with a circle in general position possesses special features when the complex is conformal, we therefore turn our attention to complexes of that type. Suppose that a pair of foci, which are corresponding points in a conformal transformation of space, are (α) and (γ), and that these are concyclic with the adjacent foci $(\alpha) + (d\alpha)$, $(\gamma) + (d\gamma)$, the circles being then cospherical. If we follow our transformation with an inversion which interchanges (α) and (γ), we have a conformal transformation leaving invariant (α) and the circle through (α), (γ), $(\alpha + d\alpha)$. We are thus led to the consideration of the invariant lineal elements in a conformal transformation of space with a fixed point. We have four possibilities:

1. One proper and two isotropic fixed lineal elements.
2. One proper fixed lineal element, and all elements orthogonal thereto also fixed.
3. All lineal elements of an isotropic plane fixed.
4. All lineal elements fixed.

These four cases for the transformation where (α) is fixed give the possible cases for our given conformal transformation, and a fixed isotropic lineal element through (α) corresponds to the case where (α), (γ), $(\alpha) + (d\alpha)$, $(\gamma) + (d\gamma)$ lie on a null circle, i. e., where adjacent circles are tangent to one another. We thus have

THEOREM 35. *If a circle in general position in a conformal complex be cospherical with but three adjacent circles, it will touch two of them.*

THEOREM 36. *If a circle in general position in a conformal complex be cospherical with three adjacent circles but not tangent to any one of them, it is cospherical with a one-parameter family of adjacent circles, including two of the given ones at least.*

THEOREM 37. *If a circle in general position in a conformal complex be tangent to but one adjacent circle, it will be cospherical with a one-parameter family of adjacent circles including this one, and with no others.*

THEOREM 38. *If a circle in general position in a conformal complex be tangent to three adjacent circles it is cospherical with every adjacent circle, and the complex consists in the totality of circles on a sphere.*

Of course these last four theorems could be restated is more precise form; when we say that adjacent circles are cospherical we mean that our circles can be assembled into the generators of annular surfaces, when we say that they are tangent we mean that we can assemble them into the osculating circles of curves.

Two sphere complexes are said to be *conformally related* when the angle of two adjacent spheres of one is equal to the corresponding infinitesimal angle in the other. We prove exactly as in the case of the conformal congruence:

THEOREM 39. *If a conformal complex be given we may, in an infinite number of ways, pass two spheres through each circle which shall generate conformally related complexes, as the circle generates the given complex.*

THEOREM 40. *If two sphere complexes be conformally related, and corresponding spheres make a fixed angle, their circles of intersection generate a conformal complex.*

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