# THE WEIERSTRASS $E$-FUNCTION FOR PROBLEMS OF THE CALCULUS OF VARIATIONS IN SPACE[*]

BY

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For the problem of minimizing the definite integral

$$\int F(x, y, z, p, q, r)\, dt,$$

$$(1) \qquad p = \frac{dx}{dt}, \qquad q = \frac{dy}{dt}, \qquad r = \frac{dz}{dt},$$

where the function $F$ has the homogeneity property

$$(2) \qquad F(x, y, z, kp, kq, kr) = kF(x, y, z, p, q, r) \qquad (k > 0),$$

the Weierstrass $E$-function

$$E(x, y, z;\ p, q, r;\ p', q', r')$$

has the form

$$\begin{aligned} E = F(x, y, z, p', q', r') &- p' F_p(x, y, z, p, q, r) \\ &- q' F_q(x, y, z, p, q, r) \\ &- r' F_r(x, y, z, p, q, r). \end{aligned}$$

Behaghel[†] has deduced a very useful expression for the $E$-function in terms of the quadratic form

$$(3) \qquad \begin{aligned} &Q(x, y, z;\ p, q, r;\ \xi, \eta, \zeta) \\ &\qquad = F_{pp}\, \xi^2 + F_{qq}\, \eta^2 + F_{rr}\, \zeta^2 + 2F_{qr}\, \eta\zeta + 2F_{rp}\, \zeta\xi + 2F_{pq}\, \xi\eta. \end{aligned}$$

In his proof he makes use of functions, in his notation $A$, $B$, $C$, which are analogous to one which is used for a similar problem in the plane. For the space problem, unfortunately, these functions may become infinite even for very simple integrals of the type (1), and in such cases his proof of the formula would not be valid.[‡]

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* Presented to the Society, December 26, 1913.

† Mathematische Annalen, vol. 73 (1913), p. 596.

‡ For the function

$$F = \sqrt{p^2 + q^2 + (q + r)^2}$$

the expression for $A$ is

$$A = -F_{qr}/qr = \left(1 - \frac{p^2}{qr}\right) / [p^2 - q^2 + (q + r)^2]^{3/2},$$

and this is infinite for any direction in the $y$-plane or $z$-plane not perpendicular to the $x$-axis.

In the following sections it is proposed to give a proof of a formula somewhat more elegant than that of Behaghel and which can be reduced to his by a simple transformation. The objections mentioned above are avoided, and the special case when the directions $(p, q, r)$ and $(p', q', r')$ are in the same line but opposite, treated separately in Behaghel's paper, is here quite unexceptional.* The method used applies at once to problems in space of any number of dimensions. In § 2 some useful consequences of the formula are deduced, and in § 3 some peculiarities of the extension of the results found to spaces of higher dimensions are explained.

## 1. The Formula for the $E$-function

It will be supposed that for a fixed set of values $(x, y, z, p, q, r)$ the function $F$ has continuous first and second derivatives for arguments $(p', q', r')$ defining directions in a cone shaped region $P_\epsilon$, shown in Fig. 1

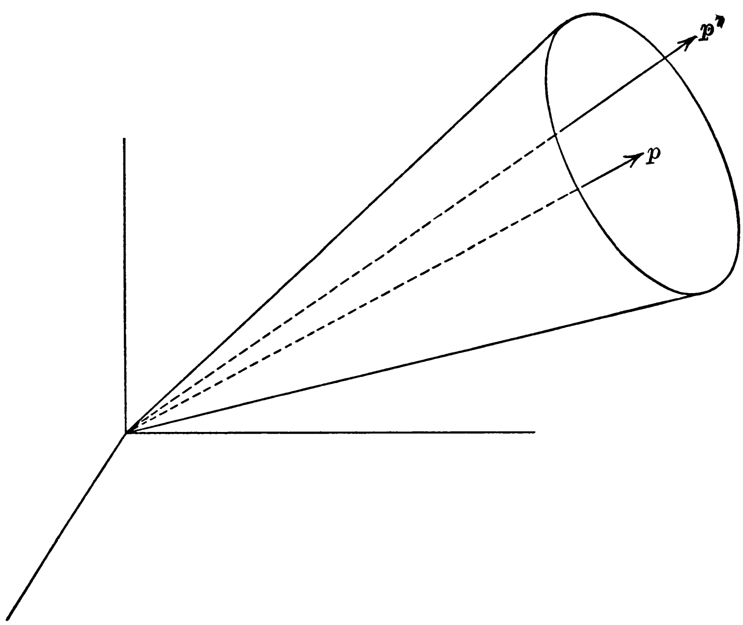


Fig. 1.

about an initial direction $p, q, r$. This and other directions will be referred to hereafter by their first letters only, as illustrated in the figure. Analytically the region is defined by an inequality of the form

$$(4) \qquad |\omega| < \epsilon$$

* A formula given by Mason and Bliss, these Transactions, vol. 9 (1908), p. 459, is less effective than the one proved in the text for a similar reason. It does not necessarily hold when the two directions $x', y', z'$ and $x_u, y_u, z_u$ are opposite.

where $\omega$ is the angle between $p$ and $p'$ defined by the formula

$$\cos \omega = \frac{pp' + qq' + rr'}{\sqrt{p^2 + q^2 + r^2}\sqrt{p'^2 + q'^2 + r'^2}} \qquad (0 \leqq \omega \leqq \pi).$$

In as much as the properties of the $E$-function to be considered in this section do not depend upon the arguments $x, y, z$, the latter may be omitted in most of the equations. It is understood that for the direction $p$ the quantities $p, q, r$ are not all zero, and a similar remark applies for other directions.

The function $F$ satisfies the relation

$$pF_p + qF_q + rF_r = F, \tag{5}$$

found by differentiating (2) for $k$ and putting $k = 1$, and also the relations

$$\begin{aligned} pF_{pp} + qF_{pq} + rF_{pr} &= 0, \\ pF_{qp} + qF_{qq} + rF_{qr} &= 0, \\ pF_{rp} + qF_{rq} + rF_{rr} &= 0, \end{aligned} \tag{6}$$

which follow by differentiating (5) for $p, q, r$, respectively. Furthermore the first derivatives of $F$ are positively homogeneous with order zero. For by differentiating (2) with respect to $p$, for example, it is found that

$$F_p(x, y, z, kp, kq, kr) = F_p(x, y, z, p, q, r) \qquad (k > 0).$$

From these results it follows readily that the $E$-function itself has the homogeneity property

$$E(x, y, z;\, kp, kq, kr;\, k'p', k'q', k'r') = k' E(x, y, z;\, p, q, r;\, p', q', r') \qquad (k > 0,\ k' > 0),$$

and the direction ratios which occur among its arguments may be taken without loss of generality to be direction cosines.

It will be helpful to use the notation

$$\begin{aligned} \Phi(p, \xi, \xi') = {} & \xi'(F_{pp}\xi + F_{pq}\eta + F_{pr}\zeta) \\ & + \eta'(F_{qp}\xi + F_{qq}\eta + F_{qr}\zeta) \\ & + \zeta'(F_{rp}\xi + F_{rq}\eta + F_{rr}\zeta). \end{aligned}$$

The form $\Phi$ has the following properties to be used later, the first of which gives an expression for the quadratic form $Q$ in terms of $\Phi$:

$$Q(x, y, z;\, p, q, r;\, \xi, \eta, \zeta) = \Phi(p, \xi, \xi), \tag{7}$$

$$\Phi(p, \xi, \xi') = \Phi(p, \xi', \xi), \tag{8}$$

$$\Phi(p, \xi, u\xi' + v\xi'') = u\Phi(p, \xi, \xi') + v\Phi(p, \xi, \xi''), \tag{9}$$

$$\Phi(p, \xi, p) = 0, \tag{10}$$

where in the third formula $u\xi' + v\xi''$ is a symbol for the values

$$u\xi' + v\xi'', \qquad u\eta' + v\eta'', \qquad u\zeta' + v\zeta''.$$

The first three of these formulas follow readily from the definition of $\Phi$, while the last is a consequence of the homogeneity properties (6).

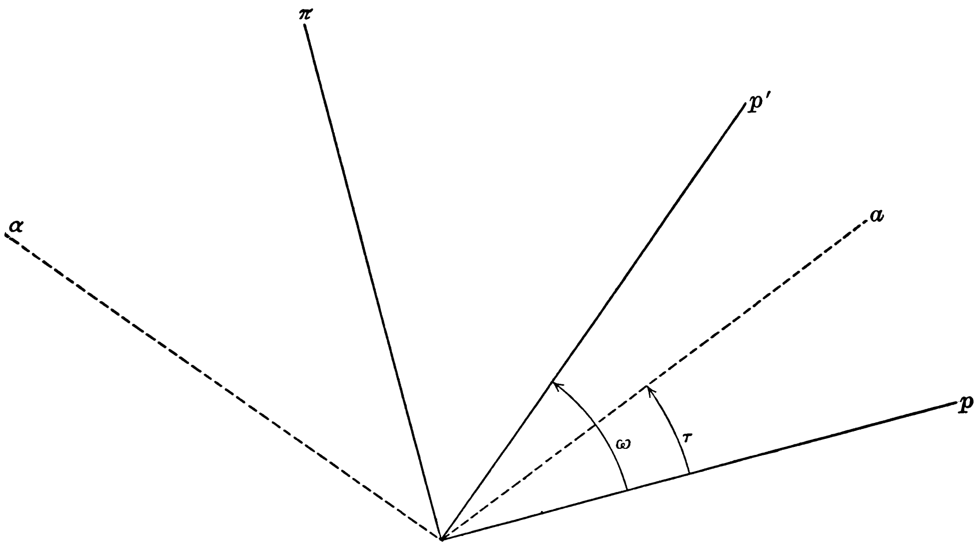


FIG. 2.

Consider now the two directions $p$ and $p'$ indicated in Fig. 2. There will always be a direction $\pi$ which is orthogonal to $p$ and co-planar with $p$ and $p'$. Let $a$ and $\alpha$ be two directions orthogonal to each other in the same plane with $p$, $p'$, and $\pi$. Their direction cosines are expressible in the form

$$\begin{aligned} a &= p\cos\tau + \pi\sin\tau, & \alpha &= -p\sin\tau + \pi\cos\tau, \\ b &= q\cos\tau + \kappa\sin\tau, & \beta &= -q\sin\tau + \kappa\cos\tau, \\ c &= r\cos\tau + \rho\sin\tau, & \gamma &= -r\sin\tau + \rho\cos\tau, \end{aligned} \tag{11}$$

where $\tau$ is the angle between $a$ and $p$. By changing the sense of the direction $\pi$, if necessary, it can be effected that for $\tau = \omega$ the direction $a$ coincides with $p'$.

The $E$-function has then the expressions

$$\begin{aligned} E(p, p') &= \int_0^\omega \frac{d}{d\tau}\{p' F_p(a) + q' F_q(a) + r' F_r(a)\} d\tau \\ &= \int_0 \Phi(a, \alpha, p') d\tau, \end{aligned}$$

which follow readily from (5), (11), and the definition of $\Phi$. From the relations

$$p' = a\cos(\omega-\tau) + \alpha\sin(\omega-\tau),$$
$$q' = b\cos(\omega-\tau) + \beta\sin(\omega-\tau),$$
$$r' = c\cos(\omega-\tau) + \gamma\sin(\omega-\tau),$$

and the formulas (9) and (10) it is seen that

$$E(p, p') = \int_0^\omega \sin(\omega-\tau)\Phi(a, \alpha, \alpha)\,d\tau,$$

or, by applying the mean value theorem for a definite integral,

$$(12) \qquad E(p, p') = (1-\cos\omega)\Phi(a^*, \alpha^*, \alpha^*),$$

where $a^*$ is the direction a corresponding to a suitably chosen value $\tau^*$ between 0 and $\omega$, and $\alpha^*$ is the corresponding direction perpendicular to $a^*$.

*Hence the function*

$$E(x, y, z;\ p, q, r;\ p', q', r')$$

*is expressible in the form*

$$(13) \qquad E = (1-\cos\omega)Q(x, y, z;\ a, b, c;\ \alpha, \beta, \gamma).$$

*In this formula $\omega$ is the angle $(0 \leqq \omega \leqq \pi)$ between the directions $(p, q, r)$ and $(p', q', r')$, while $(a, b, c)$ and $(\alpha, \beta, \gamma)$ are directions orthogonal to each other in the plane of the other two and defined by the formulas* (11) *for a suitably chosen intermediate value of $\tau$ satisfying the inequality* $0 < \tau < \omega$. The relations between these directions is clearly shown in Fig. 2.

It is easy to derive Behaghel's formula from this result. The relations (11) give

$$p = a\cos\tau - \alpha\sin\tau, \qquad q = b\cos\tau - \beta\sin\tau, \qquad r = c\cos\tau - \gamma\sin\tau,$$

and with the help of (9), (10), and (8) it follows that

$$\Phi(a, p, p) = \sin^2\tau\,\Phi(a, \alpha, \alpha).$$

Hence the expression (12) can be put into the form

$$E(p, p') = (1-\cos\omega)\frac{\Phi(a^*, p, p)}{\sin^2\tau^*}$$
$$= (1-\cos\omega)\frac{Q(x, y, z;\ a^*, b^*, c^*;\ p, q, r)}{\sin^2\tau^*},$$

which is the formula required.

## 2. Consequences of the Formula

Consider an arc $C$ defined by the continuous functions

$$(C) \qquad x = x(t), \qquad y = y(t), \qquad z = z(t) \qquad (t_1 \leqq t \leqq t_2)$$

and having a continuously turning tangent whose direction cosines will be represented by $p(t)$, $q(t)$, $r(t)$. The neighborhood $C_\epsilon$ of order zero of this arc is the totality of values $(x, y, z, p, q, r)$ for each of which the inequalities

$$(C_\epsilon) \qquad \begin{gathered} |x - x(t)| < \epsilon, \qquad |y - y(t)| < \epsilon, \qquad |z - z(t)| < \epsilon, \\ (p, q, r) \neq (0, 0, 0) \end{gathered}$$

are true for at least one value of $t$ between $t_1$ and $t_2$. The neighborhood $C'_\epsilon$ of order one is similarly defined by the conditions

$$(C'_\epsilon) \qquad \begin{gathered} |x - x(t)| < \epsilon, \qquad |y - y(t)| < \epsilon, \qquad |z - z(t)| < \epsilon, \\ (p, q, r) \neq (0, 0, 0), \qquad |\theta| < \epsilon, \end{gathered}$$

where $\theta$ is the angle $(0 \leqq \theta \leqq \pi)$ between the directions $p$ and $p(t)$. Besides the region $P_\epsilon$ of § 1 and the two neighborhoods just defined, there is a fourth region involved in the following theorems and defined by the conditions

$(R)$ $(x, y, z)$ *in a continuum of xyz-points*; $(p, q, r) \neq (0, 0, 0)$.

For each of the theorems the function $F$ is supposed to be continuous and to have continuous first and second derivatives with respect to $p$, $q$, $r$ in some region including the one involved in the statement of the theorem.

*If for a fixed set of values* $(x, y, z, p, q, r)$ *the condition*

$$E(x, y, z;\ p, q, r;\ p', q', r') \geqq 0 \tag{14}$$

*is satisfied for all directions* $(p', q', r')$ *in a neighborhood* $P_\epsilon$ *of the direction* $(p, q, r)$, *then the condition*

$$Q(x, y, z;\ p, q, r;\ \xi, \eta, \zeta) \geqq 0 \tag{15}$$

*is also satisfied for any values whatsoever of* $\xi, \eta, \zeta$. *In other words the necessary condition of Weierstrass, for either a strong or a weak minimum, at any point of a given curve, implies the necessity of Legendre's condition also.*

If in the formula (13) the direction $p'$ approaches the direction $p$ while remaining in a fixed plane through the latter, the directions $a$ and $\alpha$ approach, respectively, $p$ and a direction $\pi$ normal to $p$ in the plane. Hence the inequality (15) must be true when the direction $\xi$ coincides with $\pi$. But $p'$ may be made to approach $p$ in any plane through $p$, and consequently $\pi$ may be any normal to $p$. If $\xi$ lies in a plane with $p$ and $\pi$ and makes an angle $\tau$ with $p$, the formulas

$$\xi = p \cos \tau + \pi \sin \tau, \qquad \eta = q \cos \tau + \kappa \sin \tau, \qquad \zeta = r \cos \tau + \rho \sin \tau,$$

together with (9) and (10) show that

$$Q(x, y, z;\ p, q, r;\ \xi, \eta, \zeta) = \sin^2 \tau\, Q(x, y, z;\ p, q, r;\ \pi, \kappa, \rho).$$

Hence the inequality (15) is true for all values of $\xi, \eta, \zeta$.

The form $Q$ in the expression (13) is said to be regular in a region of values $(x, y, z;\ p, q, r)$ if it does not take opposite signs for different values of its arguments and vanishes only when $\xi, \eta, \zeta$ are all zero, or when the directions $(\xi, \eta, \zeta)$ and $(p, q, r)$ are in the same line.

*If the function $Q$ is regular at all values $(x, y, z, p, q, r)$ on an arc $C$, then there exists a neighborhood $C'_\delta$ in which it remains regular and does not take opposite signs.*

For $Q$ can always be transformed by a homogeneous orthogonal linear transformation of $\xi, \eta, \zeta$* into a sum of squares whose coefficients $\lambda_1, \lambda_2, \lambda_3$ are the roots of the equation

$$\begin{vmatrix} F_{pp} + \lambda & F_{pq} & F_{pr} \\ F_{qp} & F_{qq} + \lambda & F_{qr} \\ F_{rp} & F_{rq} & F_{rr} + \lambda \end{vmatrix} = 0. \tag{16}$$

The term independent of $\lambda$ in this equation is zero since it is the determinant of the coefficients of $p, q, r$ in the equations (6). Hence one of the roots, say $\lambda_3$, is always zero; and the other two are different from zero and of the same sign for any set of values $(x, y, z, p, q, r)$ corresponding to a point of the arc $C$, since for such values $Q$ is regular and can vanish for one direction $\xi, \eta, \zeta$ only. Since the roots $\lambda_1$ and $\lambda_2$ are continuous functions of the arguments $x, y, z, p, q, r$, they will remain different from zero and will not change sign in a sufficiently small neighborhood $C'_\delta$.

*Suppose that $Q$ is regular along the arc $C$ and that the inequality*

$$E[x(t), y(t), z(t);\ p(t), q(t), r(t);\ p', q', r'] > 0$$

*is satisfied at every point of $C$ whenever the direction $(p', q', r')$ is distinct from that of $C$. Then there exists a neighborhood $C'_\epsilon$ such that in the region of values $(x, y, z;\ p, q, r;\ p', q', r')$ defined by the relations*

$$(x, y, z, p, q, r) \text{ in } C'_\epsilon, \qquad (p', q', r') \neq (0, 0, 0)$$

*the function $E$ does not take opposite signs and vanishes only when the directions $(p, q, r)$ and $(p', q', r')$ coincide.*

To prove this consider the function $E_1$ defined by the equations

$$\begin{aligned} E_1(x, y, z;\ p, q, r;\ \pi, \kappa, \rho;\ \omega) &= E/(1 - \cos\omega) \qquad (0 < \omega \leqq \pi), \\ E_1(x, y, z;\ p, q, r;\ \pi, \kappa, \rho;\ 0) &= Q(x, y, z;\ p, q, r;\ \pi, \kappa, \rho), \end{aligned} \tag{17}$$

* See Kowalewski, *Einfuhrung in die Determinantentheorie*, § 116.

the first of which is formed by replacing the values $(p', q', r')$ in $E$ by expressions similar to (11) with $\tau = \omega$. From its definition $E_1$ is continuous for all values of its arguments for which the point $(x, y, z)$ is sufficiently near to $C$, and $0 < \omega \leqq \pi^r$,* while the other arguments are the direction cosines of two perpendicular directions. It is continuous also for $\omega = 0$, since the formula (13) shows that as $\omega$ approaches zero the values of the quotient in the first part of the definition of $E_1$ approach continuously the values defined by the second of equations (17). Along the arc $C$ the quotient $E_1$ is different from zero for directions $\pi$ normal to $p(t)$, and for $0 \leqq \omega \leqq \pi^r$, on account of the hypothesis of the theorem and the expressions (17). Since it is continuous it retains these properties when the values $(x, y, z, p, q, r)$ are in a neighborhood $C'_\epsilon$ of the arc $C$. The function $E$ has therefore the properties described in the theorem, since $E$ is expressible as a non-vanishing factor times $E_1$ whenever $p$ and $p'$ are distinct.

*Consider a problem which is regular in a region $R$ of the kind described in the first paragraph of this section, that is, a problem such that the form $Q$ is everywhere regular in $R$. Then the function $E(x, y, z;\ p, q, r;\ p', q', r')$ does not take opposite signs when $(x, y, z, p, q, r)$ and $(x, y, z, p', q', r')$ are in $R$, and vanishes only when the directions $(p, q, r)$ and $(p', q', r')$ are coincident.*

At every point of the region $R$ the quadratic form $Q$ is regular, and hence the expression (13) for the $E$-function vanishes only when $p$ and $p'$ coincide. Further any two sets of values $(x, y, z;\ p, q, r;\ p', q', r')$ can be joined by a continuous path over which $p$ and $p'$ are nowhere coincident. Hence the sign of the $E$-function is always the same at points where it does not vanish.

The regularity or non-regularity of the quadratic form $Q$ is controlled by the behavior of a function $F_1$ defined by the equations

$$\begin{aligned} p^2 F_1 &= F_{qq} F_{rr} - F_{qr}^2, & qrF_1 &= F_{pq} F_{pr} - F_{pp} F_{qr}, \\ q^2 F_1 &= F_{rr} F_{pp} - F_{rp}, & rpF_1 &= F_{qr} F_{qp} - F_{qq} F_{rp}, \\ r^2 F_1 &= F_{pp} F_{qq} - F_{pq}^2, & pqF_1 &= F_{rp} F_{rq} - F_{rr} F_{pq}. \end{aligned}$$

The existence and continuity of this function are consequences of the relations (6). The coefficient of the first power of $\lambda$ in the equation (16) is precisely $F_1$, and consequently $\lambda_1$ and $\lambda_2$ will be different from zero and have the same sign if and only if $F_1$ is positive.

*A necessary and sufficient condition that the quadratic form $Q$ be regular at a set of values $(x, y, z, p, q, r)$ is*

$$F_1(x, y, z, p, q, r) > 0.$$

* The symbol $\pi^r$ stands for $\pi$ radians.

## 3. Extension of the Preceding Results to Higher Spaces

The formula (13) which was the goal of the discussion of § 1 has an analogue for an integral of the form

$$\int F(y, p)\, dt,$$

where $y$ and $p$ are symbols for sets of elements of the form

$$y = (y_1, y_2, \cdots, y_n),$$

$$p = (p_1, p_2, \cdots, p_n) = \left(\frac{dy_1}{dt}, \frac{dy_2}{dt}, \cdots, \frac{dy_n}{dt}\right).$$

The function $F$ is supposed to have a homogeneity property with respect to the arguments $p$ similar to (2), and relations analogous to (5) and (6) are readily derived.

The $E$-function and the quadratic form $Q$ have the forms

$$E(y, p, p') = F(y, p') - \sum_{i=1}^{n} p_i F_i,$$

$$Q(y, p, \xi) = \sum_{i,j=1}^{n} F_{ij}(y, p)\, \xi_i \xi_j$$

where $F_i$ and $F_{ij}$ are symbols for the derivatives $\partial F/\partial p_i$ and $\partial^2 F/\partial p_i p_j$, respectively. The $E$-function has the homogeneity property

$$E(y, \kappa p, \kappa' p') = \kappa' E(y, p, p') \qquad (\kappa > 0,\ \kappa' > 0),$$

and as before there will be no loss of generality if all the directions $p$ are supposed to be normed, that is, to satisfy the relation

$$p_1^2 + p_2^2 + \cdots + p_n^2 = 1.$$

The condition for the orthogonality of two directions, $p$ and $\pi$, in $n$-dimensional space is

$$p_1 \pi_1 + p_2 \pi_2 + \cdots + p_n \pi_n = 0.$$

The auxiliary form $\Phi$ has the definition

$$\Phi(p, \xi, \xi') = \sum_{i,j=1}^{n} F_{ij}(y, p)\, \xi_i \xi'_j$$

and satisfies four relations analogous to (7)–(10) of § 1.

There will always be at least $n - 2$ independent directions $q', q'', \cdots, q^{(n-2)}$ orthogonal to both $p$ and $p'$, which with $p$ itself form a system of $n - 1$ independent directions; and the last system of $n - 1$ directions determines a unique direction $\pi$ orthogonal to each of them. The directions $\pi$ and $p$ are independent, and orthogonal to each of the directions $q', q'', \cdots, q^{(n-2)}$. Since every other direction having this property must be expressible linearly in terms of $p$ and $\pi$, it follows that $p'$ must have the form

$$p'_i = p_i \cos\omega + \pi_i \sin\omega \quad (i = 1, 2, \cdots, n; 0 \leqq \omega \leqq \pi^\tau).$$

In order to satisfy the last restriction in the parenthesis it may be necessary to change the sign of the direction $\pi$.

If the directions $a$ and $\alpha$ are defined by the equations

$$a_i = p_i \cos\tau + \pi_i \sin\tau, \qquad \alpha_i = -p_i \sin\tau + \pi_i \cos\tau \quad (i = 1, 2, \cdots, n),$$

the argument proceeds exactly as in § 1, and the formula found for the $E$-function is

$$E(y, p, p') = (1 - \cos\omega) Q(y, a, \alpha),$$

where $a$ and $\alpha$ are defined by the equations just given for a suitably chosen intermediate value of $\tau$ between 0 and $\omega$.

All of the results of § 2 except the last can be proved for $n$ dimensions in a manner analogous to that which is effective for three dimensions. The analogue of $F_1$ is a function defined by relations of the form

$$p_i p_k F_1 = A_{ik} \qquad (i, k = 1, 2, \cdots, n),$$

where $A_{ik}$ is the cofactor of $F_{ik}$ in the matrix of these derivatives. In the equation corresponding to (16) for $n$ dimensions the term independent of $\lambda$ vanishes as before, and $F_1$ is the coefficient of the first power of $\lambda$. Hence if $Q$ is regular at a set of values $(y, p)$ it will remain so in any neighborhood of those values in which $F_1$ is different from zero. But more than the requirement that $F_1$ shall have a certain sign is needed in order to insure the regularity of $Q$.

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