ON THE THEORY OF CURVED SURFACES, AND CANONICAL SYSTEMS IN PROJECTIVE DIFFERENTIAL GEOMETRY*

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In a series of five memoirs,† Wilczynski has developed a projective theory of curved surfaces, employing in his study a method which is peculiarly his own. This consists in the consideration of a certain system of partial differential equations and the theory of its invariants and covariants. The analysis, however, is often extremely complicated, so that any simplification of the problem is welcome. In the following pages will be indicated a procedure which, though carried out in detail for the particular problem of the theory of surfaces only, seems to be of very general applicability.

In the theory of curved surfaces, a very natural plan presents itself, viz., to assume that the surface is referred to its asymptotic curves. This is done by Wilczynski in his first three memoirs, and on this assumption he calculates the complete system of invariants and covariants. But the determination of the asymptotic lines on a surface requires the integration of a quadratic partial differential equation of the first order, and it is in general impossible to carry out this integration explicitly. There is, then, an apparent restriction in generality in supposing known the asymptotic lines. To remove this objection, Wilczynski, in his fourth memoir, started to set up a system of invariants for the most general differential equations of the unrestricted problem. Notwithstanding the complications involved in the calculations, the method and the results are very elegant. The question of the completeness of the system of invariants is, however, left open.

The invariants of the first three memoirs are functions of the coefficients of a canonical system, i. e., of a system of differential equations whose independent variables are parameters of the asymptotic lines of the surface. Their invariantive property holds with respect to all those transformations of the independent variables which preserve the asymptotic lines as para-

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† *Projective differential geometry of curved surfaces.* These Transactions, vols. 8–10 (1907–1909). We shall have occasion to refer to the *First Memoir*, vol. 8 (1907), pp. 233–260, and the *Fourth Memoir*, vol. 10 (1909), pp. 176–200.

meter curves. The invariants of the fourth memoir, on the other hand, are functions of the coefficients of an arbitrary system, and their invariantive property holds with reference to arbitrary transformations of the independent variables. It is clear, however, from the geometric significance of the invariants, as well as on general invariant- and group-theoretic grounds, that the two complete systems of invariants must be equivalent. In fact, the invariants of the first three memoirs are merely the canonical forms which those of the fourth memoir assume, when the general system of partial differential equations there considered is supposed to be reduced to its canonical form.

In the present paper, we propose to show how a complete system of concomitants for the general theory of surfaces may be constructed. The method employed enables us to avail ourselves of all the simplifications which would be possible if the asymptotic curves were known; in fact, we shall show that there is no loss of generality whatever in supposing, as Wilczynski does in his first three memoirs, that the surface is referred to its asymptotic curves.

The application of the principles developed in this paper to any problem in projective differential geometry will be best understood after their application to the theory of curved surfaces has been developed in detail. We therefore proceed at once to the consideration of the differential equations for this particular problem.

The most general system of two linear homogeneous partial differential equations of the second order in one dependent and two independent variables is of the form

$$(1)\qquad \begin{aligned} Ay_{uu} + By_{uv} + Cy_{vv} + Dy_u + Ey_v + Fy &= 0, \\ A'y_{uu} + B'y_{uv} + C'y_{vv} + D'y_u + E'y_v + F'y &= 0. \end{aligned}$$

In all cases which are of interest in the theory of curved surfaces, system (1) may be reduced to one of the form

$$(2)\qquad \begin{aligned} y_{uu} &= a\, y_{uv} + b\, y_u + c\, y_v + d\, y, \\ y_{vv} &= a'\, y_{uv} + b'\, y_u + c'\, y_v + d'\, y, \end{aligned}$$

where $aa' - 1$ is not identically equal to zero, the reduction requiring only operations which are always practically possible.* Our object is to set up a complete system of invariants and covariants for system (2), in the sense in which these terms are used by Wilczynski.

Let us, with Wilczynski, transform the independent variables in (2) by the transformation

$$(3)\qquad \bar{u} = \phi(u, v), \qquad \bar{v} = \psi(u, v).$$

* E. J. Wilczynski, *First Memoir*, p. 234.

We find

(4) $$y_u = \phi_u \bar{y}_u + \psi_u \bar{y}_v, \qquad y_v = \phi_v \bar{y}_u + \psi_v \bar{y}_v,$$

(5) $$\begin{aligned} y_{uu} &= \phi_u^2 \bar{y}_{uu} + 2\phi_u \psi_u \bar{y}_{uv} + \psi_u^2 \bar{y}_{vv} + \phi_{uu} \bar{y}_u + \psi_{uu} \bar{y}_v, \\ y_{uv} &= \phi_u \phi_v \bar{y}_{uu} + (\phi_u \psi_v + \psi_u \phi_v) \bar{y}_{uv} + \psi_u \psi_v \bar{y}_{vv} + \phi_{uv} \bar{y}_u + \psi_{uv} \bar{y}_v, \\ y_{vv} &= \phi_v^2 \bar{y}_{uu} + 2\phi_v \psi_v \bar{y}_{uv} + \psi_v^2 \bar{y}_{vv} + \phi_{vv} \bar{y}_u + \psi_{vv} \bar{y}_v, \end{aligned}$$

in which $\bar{y}_u$, $\bar{y}_v$, etc., denote $\partial y/\partial \bar{u}$, $\partial y/\partial \bar{v}$, etc.

Substituting in (2), we obtain the equations

(6) $$\begin{aligned} (\phi_u^2 - a\phi_u \phi_v) \bar{y}_{uu} &+ (\psi_u^2 - a\psi_u \psi_v) \bar{y}_{vv} \\ &= [a(\phi_u \psi_v + \psi_u \phi_v) - 2\phi_u \psi_u] \bar{y}_{uv} \\ &+ (b\phi_u + c\phi_v - \phi_{uu} + a\phi_{uv}) \bar{y}_u \\ &+ (b\psi_u + c\psi_v - \psi_{uu} + a\psi_{uv}) \bar{y}_v + d\bar{y}, \\ (\phi_v^2 - a'\phi_u \phi_v) \bar{y}_{uu} &+ (\psi_v^2 - a'\psi_u \psi_v) \bar{y}_{vv} \\ &= [a'(\phi_u \psi_v + \psi_u \phi_v) - 2\phi_v \psi_v] \bar{y}_{uv} \\ &+ (b'\phi_u + c'\phi_v - \phi_{vv} + a'\phi_{uv}) \bar{y}_u \\ &+ (b'\psi_u + c'\psi_v - \psi_{vv} + a'\psi_{uv}) \bar{y}_v + d'\bar{y}. \end{aligned}$$

If these be solved for $\bar{y}_{uu}$ and $\bar{y}_{vv}$, one obtains a system of the same form as (2); denoting its coefficients by $\bar{a}$, $\bar{b}$, etc., we find that

(7) $$\begin{aligned} \bar{a} &= \frac{a\psi_v^2 - 2\psi_u \psi_v + a'\psi_u^2}{\phi_u \psi_v + \phi_v \psi_u - a\phi_v \psi_v - a'\phi_u \psi_u}, \\ \bar{a}' &= \frac{a\phi_v^2 - 2\phi_u \phi_v + a'\phi_u^2}{\phi_u \psi_v + \phi_v \psi_u - a\phi_v \psi_v - a'\phi_u \psi_u}. \end{aligned}$$

We may therefore make $\bar{a}$ and $\bar{a}'$ vanish by taking for ϕ and ψ solutions of the two factors of the differential equation

$$a\chi_v^2 - 2\chi_u \chi_v + a'\chi_u^2 = 0,$$

i. e., by choosing ϕ and ψ so as to satisfy the differential equations*

(8) $$(1 + \sqrt{1 - aa'})\phi_u - a\phi_v = 0, \qquad (1 - \sqrt{1 - aa'})\psi_u - a\psi_v = 0,$$

which are distinct, since $aa' - 1 \neq 0$.

Suppose this to have been done. The new system will take the form

(9) $$\begin{aligned} \bar{y}_{uu} &= \bar{b}\ \bar{y}_u + \bar{c}\ \bar{y}_v + \bar{d}\ \bar{y}, \\ \bar{y}_{vv} &= \bar{b}'\bar{y}_u + \bar{c}'\bar{y}_v + \bar{d}'\bar{y}. \end{aligned}$$

* *First Memoir*, p. 243, where it is also proved that the denominator in equations (7) will not vanish identically.

It is for this system of differential equations that Wilczynski, in his *First Memoir,* calculates a complete system of invariants and covariants. But the reduction of system (2) to the form (9)—which is equivalent to the determination of the asymptotic curves of the surface defined by any fundamental system of solutions of (2)—requires the integration of equations (8). In general an explicit integration of (8) is of course impossible, so that apparently a theory based on system (9) is less general in form than one based on system (2). Regarded geometrically, however, the projective theory of a surface is identical with that of its asymptotic curves; for any property of the one, there must be a corresponding property of the other. It is to be expected therefore, that the invariants and covariants as calculated for system (9) must suffice to express all projective properties of the integrating surfaces of (2). The question to be answered, then, is whether the invariants and covariants of equations (9) are expressible *explicitly and in a simple manner* in terms of the coefficients of equations (2), so that the integration of equations (8) may be rendered unnecessary. We shall see that this is indeed possible, so that the identity, geometrically, of the theory of a surface and that of its asymptotic lines has its complete analytic analogue.

If we put

$$\mu = \sqrt{1 - aa'}, \qquad \mu \neq 0, \tag{10}$$

equations (8) become

$$(1 + \mu)\phi_u - a\phi_v = 0, \qquad (1 - \mu)\psi_u - a\psi_v = 0,$$

or, multiplying the second of these by $1 + \mu$ and dividing out the factor a* which appears because $1 - \mu^2 = aa'$, we obtain the more symmetrical equations

$$(1 + \mu)\phi_u - a\phi_v = 0, \qquad (1 + \mu)\psi_v - a'\psi_u = 0.$$

By means of these, we may express ϕ_v in terms of ϕ_u, and ψ_u in terms of ψ_v:

$$\phi_v = \frac{1 + \mu}{a}\phi_u, \qquad \psi_u = \frac{1 + \mu}{a'}\psi_v. \tag{11}$$

By differentiation, we find also

$$\begin{aligned} \phi_{uv} &= \left(\frac{1 + \mu}{a}\right)_u \phi_u + \frac{1 + \mu}{a}\phi_{uu}, \qquad & \psi_{uv} &= \left(\frac{1 + \mu}{a'}\right)_v \psi_v + \frac{1 + \mu}{a'}\psi_{vv}, \\ \phi_{vv} &= \left(\frac{1 + \mu}{a}\right)_v \phi_u + \frac{1 + \mu}{a}\phi_{uv}, \qquad & \psi_{uu} &= \left(\frac{1 + \mu}{a'}\right)_u \psi_v + \frac{1 + \mu}{a'}\psi_{uv}, \end{aligned} \tag{12}$$

so that the second derivatives of ϕ are expressible in terms of ϕ_u and ϕ_{uu}, and those of ψ in terms of ψ_v and ψ_{vv}. Substituting in (6), we obtain the equations

* The special cases $a = 0$ or $a' = 0$ are easily discussed and may therefore be excluded from the present consideration.

$$
\begin{aligned}
(13)\quad & -\mu\phi_u^2\,\bar{y}_{uu} + \frac{\mu(1+\mu)^2}{a'^2}\psi_v^2\,\bar{y}_{vv} \\
&= \left[\left\{b + \frac{1+\mu}{a}c + a\left(\frac{1+\mu}{a}\right)_u\right\}\phi_u + \mu\phi_{uu}\right]\bar{y}_u \\
&\quad + \left[\left\{\frac{1+\mu}{a'}b + c - \left(\frac{1+\mu}{a'}\right)_u\right\}\psi_v - \frac{\mu(1+\mu)}{a'}\psi_{uv}\right]\bar{y}_v + d\bar{y}, \\
& \frac{\mu(1+\mu)^2}{a^2}\phi_u^2\,\bar{y}_{uu} - \mu\psi_v^2\,\bar{y}_{vv} \\
&= \left[\left\{b' + \frac{1+\mu}{a}c' - \left(\frac{1+\mu}{a}\right)_v\right\}\phi_u - \frac{\mu(1+\mu)}{a}\phi_{uv}\right]\bar{y}_u \\
&\quad + \left[\left\{\frac{1+\mu}{a'}b' + c' + a'\left(\frac{1+\mu}{a'}\right)_v\right\}\psi_v + \mu\psi_{vv}\right]\bar{y}_v + d'\,\bar{y}.
\end{aligned}
$$

Solving these equations for $\bar{y}_{uu}$ and $\bar{y}_{vv}$, we obtain a system of form (9), which we may write*

$$
(14)\quad
\begin{aligned}
\bar{y}_{uu} + 2\alpha\;\bar{y}_u + 2\beta\;\bar{y}_v + \gamma\;\bar{y} &= 0, \\
\bar{y}_{vv} + 2\alpha'\,\bar{y}_u + 2\beta'\,\bar{y}_v + \gamma'\,\bar{y} &= 0.
\end{aligned}
$$

The coefficients in these equations are of the form

$$
(15)\quad
\begin{aligned}
\alpha &= \frac{1}{\phi_u}(\{\alpha\} + \tfrac{1}{2}\xi), & \beta &= \frac{\psi_v}{\phi_u^2}\{\beta\}, & \gamma &= \frac{1}{\phi_u^2}\{\gamma\}, \\
\alpha' &= \frac{\phi_u}{\psi_v^2}\{\alpha'\}, & \beta' &= \frac{1}{\psi_v}(\{\beta'\} + \tfrac{1}{2}\eta), & \gamma' &= \frac{1}{\psi_v^2}\{\gamma'\},
\end{aligned}
$$

where

$$
(16)\quad \xi = \frac{\phi_{uu}}{\phi_u}, \qquad \eta = \frac{\psi_{vv}}{\psi_v},
$$

and $\{\alpha\}$, $\{\beta\}$, etc., are the following expressions in the coefficients of (2), entirely free from ϕ and ψ:

$$
\begin{aligned}
\{\alpha\} = -\frac{(1-\mu)^2}{8\mu^2}\Bigg\{ & b + \frac{1+\mu}{a}c + \frac{a(1-3\mu)}{(1-\mu)^2}\left(\frac{1+\mu}{a}\right)_u \\
& + \frac{(1+\mu)^2}{a'^2}\left[b' + \frac{1+\mu}{a}c' - \left(\frac{1+\mu}{a}\right)_v\right]\Bigg\}, \\
\{\beta\} = -\frac{(1-\mu)^2}{8\mu^2}\Bigg\{ & \frac{1+\mu}{a'}b + c - \left(\frac{1+\mu}{a'}\right)_u + \frac{1+\mu}{a'}\left(\frac{1+\mu}{a'}\right)_v \\
& + \frac{(1+\mu)^2}{a'^2}\left(\frac{1+\mu}{a'}b' + c'\right)\Bigg\},
\end{aligned}
$$

* These are the same as equations (27) of Wilczynski's *First Memoir*, except that we have used Greek letters for corresponding Roman letters to avoid confusion.

$$
(17)\qquad
\begin{aligned}
\{\alpha'\} &= -\frac{(1-\mu)^2}{8\mu^2}\left\{\frac{(1+\mu)^2}{a^2}\left(b+\frac{1+\mu}{a}c\right)+b'+\frac{1+\mu}{a}c'\right.\\
&\qquad\left.+\frac{1+\mu}{a}\left(\frac{1+\mu}{a}\right)_u-\left(\frac{1+\mu}{a}\right)_v\right\},\\
\{\beta'\} &= -\frac{(1-\mu)^2}{8\mu^2}\left\{\frac{(1+\mu)^2}{a^2}\left[\frac{1+\mu}{a'}b+c-\left(\frac{1+\mu}{a'}\right)_u\right]\right.\\
&\qquad\left.+\frac{1+\mu}{a'}b'+c'+\frac{a'(1-3\mu)}{(1-\mu)^2}\left(\frac{1+\mu}{a'}\right)_v\right\},\\
\{\gamma\} &= -\frac{(1-\mu)^2}{4\mu^2}\left[d+\frac{(1+\mu)^2}{a'^2}d'\right],\\
\{\gamma'\} &= -\frac{(1-\mu)^2}{4\mu^2}\left[\frac{(1+\mu)^2}{a^2}d+d'\right].
\end{aligned}
$$

The coefficients in equations (14) depend of course on ϕ and ψ, as shown by (15); to determine these coefficients, equations (8) must actually be integrated. But if, in the expressions for the fundamental invariants of (14)— to be written presently— we substitute for α, β, etc., their expressions as given by (15), we shall find that ϕ and ψ come out as factors of the form $\phi_u^k \psi_v^l$. For absolute invariants, $k = l = 0$. The quantities α' and β are relative invariants of system (14); the vanishing of either expresses the condition that a corresponding family of asymptotic curves of the surface consists of straight lines. The vanishing of $\{\alpha'\}$ or $\{\beta\}$, as given by (17), expresses the same condition in terms of the coefficients of system (2). The product, $\{\alpha'\}\{\beta\}$, is essentially the invariant θ of Wilczynski's *Fourth Memoir,* whose vanishing is the condition that the surface be ruled.

Besides β and α', there are two more invariants, which form with these a fundamental system. If we put

$$
(18)\qquad f = \gamma - \alpha_{\bar u} - \alpha^2 - 2\beta\beta', \qquad g = \gamma' - \beta'_{\bar v} - \beta'^2 - 2\alpha\alpha',
$$

then the quantities

$$
(19)\qquad
\begin{aligned}
h &= \beta^2\left[f+\beta_{\bar v}-\frac{1}{4}\frac{\partial}{\partial \bar u}\left(\frac{\beta_{\bar u}}{\beta}\right)+\frac{1}{16}\frac{\beta_{\bar u}^2}{\beta^2}\right],\\
k &= \alpha'^2\left[g+\alpha'_{\bar u}-\frac{1}{4}\frac{\partial}{\partial \bar v}\left(\frac{\alpha'_{\bar v}}{\alpha'}\right)+\frac{1}{16}\frac{\alpha'^2_{\bar v}}{\alpha'^2}\right]
\end{aligned}
$$

are the invariants sought. In calculating their expressions in terms of the coefficients, we must note that in (19) occur differentiations with respect to the transformed variables $\bar u$ and $\bar v$. The relation between such differentiations and those with respect to the original variables is given by the formulæ (4), in which y is to be interpreted as any function of u, v. Solving for $\bar{y}_u$ and $\bar{y}_v$, we have

$$(20)\qquad \bar{y}_u = -\frac{1}{\phi_u}\left(\frac{1-\mu}{2\mu}y_u - \frac{a}{2\mu}y_v\right), \qquad \bar{y}_v = \frac{1}{\psi_v}\left(\frac{a'}{2\mu}y_u - \frac{1-\mu}{2\mu}y_v\right).$$

Consequently,

$$\alpha_{\bar{u}} = -\frac{1}{\phi_u}\left[\frac{1-\mu}{2\mu}\,\frac{\{\alpha\}_u + \frac{1}{2}\xi_u - \{\alpha\}\xi - \frac{1}{2}\xi^2}{\phi_u} - \frac{a}{2\mu}\,\frac{\{\alpha\}_v + \frac{1}{2}\xi_v - \frac{\phi_{uv}}{\phi_u}(\{\alpha\} + \frac{1}{2}\xi)}{\phi_u}\right].$$

But ξ_v in this expression may be found in terms of ξ_u and ξ. In fact, we have

$$(21)\qquad \begin{aligned} \xi_v &= \frac{\partial}{\partial v}\left(\frac{\phi_{uu}}{\phi_u}\right) = \frac{\phi_{uuv} - \xi\phi_{uv}}{\phi_u} \\ &= \left(\frac{1+\mu}{a}\right)_{uu} + \xi\left(\frac{1+\mu}{a}\right)_u + \frac{1+\mu}{a}\xi_u, \end{aligned}$$

by the first of equations (12). Therefore

$$\alpha_{\bar{u}} = -\frac{1}{\phi_u^2}\left[\frac{1-\mu}{2\mu}\{\alpha\}_u - \frac{a}{2\mu}\left[\{\alpha\}_v - \{\alpha\}\left(\frac{1+\mu}{a}\right)_u + \frac{1}{2}\left(\frac{1+\mu}{a}\right)_{uu}\right] - \frac{1}{2}\xi_u + \{\alpha\}\xi + \frac{1}{2}\xi^2\right],$$

so that

$$(22)\qquad f = \frac{1}{\phi_u^2}(\{f\} - \tfrac{1}{2}\xi_u + \tfrac{1}{4}\xi^2 - \{\beta\}\eta),$$

where

$$(23)\qquad \begin{aligned} \{f\} = \{\gamma\} - 2\{\beta\}\{\beta'\} - \{\alpha\}^2 + \frac{1-\mu}{2\mu}\{\alpha\}_u \\ - \frac{a}{2\mu}\left[\{\alpha\}_v - \{\alpha\}\left(\frac{1+\mu}{a}\right)_u + \frac{1}{2}\left(\frac{1+\mu}{a}\right)_{uu}\right]. \end{aligned}$$

It may also be found without difficulty that

$$(24)\qquad \beta_{\bar{u}} = -\frac{\psi_v}{\phi_u^3}(\{\beta_{\bar{u}}\} + 2\{\beta\}\xi), \qquad \beta_{\bar{v}} = \frac{1}{\phi_u^2}(\{\beta_{\bar{v}}\} + \{\beta\}\eta),$$

where the quantities

$$(25)\qquad \begin{aligned} \{\beta_{\bar{u}}\} &= \left[\frac{a}{\mu}\left(\frac{1+\mu}{a}\right)_u + \frac{1-\mu}{2\mu}\left(\frac{1+\mu}{a'}\right)_v\right]\{\beta\} + \frac{1-\mu}{2\mu}\{\beta\}_u - \frac{a}{2\mu}\{\beta\}_v, \\ \{\beta_{\bar{v}}\} &= \left[\frac{1-\mu}{\mu}\left(\frac{1+\mu}{a}\right)_u + \frac{a'}{2\mu}\left(\frac{1+\mu}{a'}\right)_v\right]\{\beta\} + \frac{a'}{2\mu}\{\beta\}_u - \frac{1-\mu}{2\mu}\{\beta\}_v \end{aligned}$$

are the parts of $\beta_{\bar{u}}$ and $\beta_{\bar{v}}$ independent of ϕ and ψ. Continuing to use braces $\{\,\}$ to denote that part of the enclosed expression which does not contain ϕ or ψ, we have

$$(26)\qquad \frac{\partial}{\partial \bar{u}}\frac{\beta_{\bar{u}}}{\beta} = \frac{1}{\phi_u^2}\left(\left\{\frac{\partial}{\partial \bar{u}}\frac{\beta_u}{\beta}\right\} - 2\xi_u + \frac{\{\beta_{\bar{u}}\}}{\{\beta\}}\xi + 2\xi^2\right),$$

where

$$(27)\qquad \left\{\frac{\partial}{\partial \bar{u}}\frac{\beta_{\bar{u}}}{\beta}\right\} = \frac{1-\mu}{2\mu}\frac{\partial}{\partial u}\frac{\{\beta_{\bar{u}}\}}{\{\beta\}} - \frac{a}{2\mu}\left[\frac{\partial}{\partial v}\frac{\{\beta_{\bar{u}}\}}{\{\beta\}} - \left(\frac{1+\mu}{a}\right)_u + 2\left(\frac{1+\mu}{a}\right)_{uu}\right].$$

Using (15), (22), (24), and (26), the first of the invariants (19) becomes

$$(28)\qquad k = \frac{\psi_v^2}{\phi_u^6}\{h\},$$

where

$$(29)\qquad \{h\} = \{\beta\}^2\left[\{f\} + \{\beta_{\bar{\imath}}\} - \frac{1}{4}\left\{\frac{\partial}{\partial \bar{u}}\frac{\beta_{\bar{u}}}{\beta}\right\} + \frac{1}{16}\frac{\{\beta_{\bar{u}}\}^2}{\{\beta\}^2}\right].$$

Similarly, we find for the invariant k

$$(30)\qquad k = \frac{\phi_u^2}{\psi_v^2}\{k\},$$

where the formula for $\{h\}$ may be used to get the expression for $\{k\}$, merely by replacing any letter in the symbol

$$\begin{pmatrix} \alpha & \beta & \gamma & a & u & \phi & \xi & f \\ \beta' & \alpha' & \gamma' & a' & v & \psi & \eta & g \end{pmatrix}$$

by the letter just above or just below it.

From the four invariants β, α', h, k, that is,

$$(31)\qquad \beta = \frac{\psi_v}{\phi_u^2}\{\beta\}, \qquad \alpha' = \frac{\phi_v}{\psi_v^2}\{\alpha'\}, \qquad h = \frac{\psi_v^2}{\phi_u^6}\{h\}, \qquad k = \frac{\phi_u^2}{\psi_u^6}\{k\},$$

a complete system of invariants may be calculated. Let us put, with Wilczynski,*

$$(32)\qquad A = \alpha'\beta^2, \qquad B = \alpha'^2\beta, \qquad H = \alpha' h, \qquad K = \beta k;$$

then

$$(33)\qquad A = \frac{1}{\phi_u^3}\{A\}, \qquad B = \frac{1}{\psi_v^3}\{B\}, \qquad H = \frac{1}{\phi_u^5}\{H\}, \qquad K = \frac{1}{\psi_v^5}\{K\},$$

where $\{A\}$, $\{B\}$, $\{H\}$, and $\{K\}$ again denote functions independent of ϕ and ψ. Wilczynski has proved† that all invariants of the system (14) may be calculated from A, B, H, and K by combination and repetition of certain differ-

* *First Memoir*, p. 250.

† *First Memoir*, pp. 250–255.

entiation processes. First, we note that if system (14) be transformed by

$$\bar{u} = U(\bar{u}), \qquad \bar{v} = V(\bar{v}),$$

a new system (which we may call $(\overline{14})$), of the same form is obtained, the invariants $\bar{A}, \bar{B}, \bar{H}, \bar{K}$ of which are related to the invariants A, B, H, K of (14) by the equations*

$$(34) \qquad \bar{A} = \frac{1}{U_{\bar{u}}^3} A, \qquad \bar{B} = \frac{1}{V_{\bar{v}}^3} B, \qquad \bar{H} = \frac{1}{U_{\bar{u}}^5} H, \qquad \bar{K} = \frac{1}{V_{\bar{v}}^5} K.$$

We shall call any invariants L and M of system (14) which are connected with the corresponding quantities for the transformed system $(\overline{14})$ by equations of the form

$$(35) \qquad \bar{L} = \frac{1}{U_{\bar{u}}^l} L, \qquad \bar{M} = \frac{1}{V_{\bar{v}}^m} M,$$

U- and V-invariants respectively.

Then the operator

$$(36) \qquad \mathbf{U} = \alpha' \frac{\partial}{\partial \bar{u}},$$

applied to a V-invariant, gives another V-invariant, and the operator

$$(37) \qquad \mathbf{V} = \beta \frac{\partial}{\partial \bar{v}},$$

applied to a U-invariant, gives another U-invariant. In fact, we find

$$(38) \qquad \bar{\mathbf{V}}(\bar{L}) = \frac{1}{U_{\bar{u}}^{l+2}} \mathbf{V}(L), \qquad \bar{\mathbf{U}}(\bar{M}) = \frac{1}{V_{\bar{v}}^{m+2}} \mathbf{U}(M).$$

Again, from two U-invariants P and Q which satisfy the relations

$$(39) \qquad \bar{P} = \frac{1}{U_{\bar{u}}^p} P, \qquad \bar{Q} = \frac{1}{V_{\bar{v}}^q} Q,$$

may be formed their wronskian with respect to $\bar{u}$, viz.,

$$(40) \qquad (QP_{\bar{u}}) = qQP_{\bar{u}} - pPQ_{\bar{u}},$$

an invariant satisfying the equation

$$(41) \qquad (\overline{QP}_{\bar{u}}) = \frac{1}{U_{\bar{u}}^{p+q+1}} (QP_u).$$

Similarly, from two V-invariants may be formed their wronskian with respect to $\bar{v}$.

* *First Memoir,* pp. 249, 250. Our equations (34) are Wilczynski's equations (52), with α, β, u, v replaced by $U, V, \bar{u}, v$.

Wilczynski's theorem is, that *all invariants of system* (14) *may be calculated from the fundamental ones, A, B, H, K, by combination and repetition of the wronskian-, U-, and V-processes.* It remains for us to show that *the invariants thus found are expressible explicitly in terms of the coefficients and variables of the original system of differential equations* (2).

The proof is a very simple induction. Assume that invariants L and M of system (14) which satisfy (35) have the following expressions in terms of the coefficients and variables of system (2):

$$L = \frac{1}{\phi_u^l}\{L\}, \qquad M = \frac{1}{\psi_v^m}\{M\}. \tag{42}$$

This assumption is verified by the invariants A, B, H, K, as is evident on comparing equations (33) and (34). From equations (20) we have

$$\begin{aligned} \mathbf{U} &= \frac{1}{\psi_v^2}\{\alpha'\}\left(\frac{a}{2\mu}\frac{\partial}{\partial v} - \frac{1-\mu}{2\mu}\frac{\partial}{\partial u}\right), \\ \mathbf{V} &= \frac{1}{\phi_u^2}\{\beta\}\left(\frac{a'}{2\mu}\frac{\partial}{\partial u} - \frac{1-\mu}{2\mu}\frac{\partial}{\partial v}\right), \end{aligned} \tag{43}$$

and it is easy to verify, if use be made of equations (12), that

$$\mathbf{U}(M) = \frac{1}{\psi_v^{m+2}}\{\mathbf{U}(M)\}, \qquad \mathbf{V}(L) = \frac{1}{\phi_u^{l+2}}\{\mathbf{V}(L)\}, \tag{44}$$

where

$$\begin{aligned} \{\mathbf{U}(M)\} &= \{\alpha'\}\left[\frac{a}{2\mu}\{M\}_v - \frac{1-\mu}{2\mu}\{M\}_u + \frac{1-\mu}{2\mu}\left(\frac{1+\mu}{a'}\right)_v\{M\}\right], \\ \{\mathbf{V}(L)\} &= \{\beta\}\left[\frac{a'}{2\mu}\{L\}_u - \frac{1-\mu}{2\mu}\{L\}_v + \frac{1-\mu}{2\mu}\left(\frac{1+\mu}{a}\right)_u\{L\}\right]. \end{aligned} \tag{45}$$

Similarly for the wronskian, it may be shown that if

$$P = \frac{1}{\phi_u^p}\{P\}, \qquad Q = \frac{1}{\phi_u^q}\{Q\}, \tag{46}$$

then

$$(QP_{\bar{u}}) = \frac{1}{\phi_u^{p+q+1}}\{(QP_{\bar{u}})\}, \tag{47}$$

where

$$\begin{aligned} \{(QP_{\bar{u}})\} = q\{Q\}&\left(\frac{a}{2\mu}\{P\}_v - \frac{1-\mu}{2\mu}\{P\}_u\right) \\ &- p\{P\}\left(\frac{a}{2\mu}\{Q\}_v - \frac{1-\mu}{2\,\mu}\{Q\}_u\right). \end{aligned} \tag{48}$$

Now, equations (44) and (47) resemble equations (38) and (41) just as equations (42) and (46) resemble by assumption equations (35) and (39). But this assumed resemblance exists for the fundamental invariants A, B, H, K; consequently it will be preserved for all invariants formed from these by the wronskian-, U-, and V-processes. By this method, however, all invariants of system (14) may be formed,* so that the complete system of invariants so constructed is expressible explicitly in terms of the coefficients of system (2), except of course for extraneous factors of the form $\phi_u^s \psi_v^t$ in the case of relative invariants.

In the same way, the four fundamental covariants of Wilczynski's theory may be expressed in terms of the coefficients and variables of system (2). We take as these covariants†

$$y, \qquad Z = 6Az + A_{\bar{u}}\, y, \qquad R = 6B\rho + B_{\bar{v}}\, y,$$

$$S = 36AB\sigma + 6AB_{\bar{v}}\, z + 6BA_{\bar{u}}\, \rho + A_{\bar{u}}\, B_v\, y,$$

where

$$z = y_{\bar{u}} + \alpha y, \qquad \rho = y + \beta' y,$$

$$\sigma = y_{\bar{u}\bar{v}} + \beta' y_{\bar{u}} + \alpha y_{\bar{v}} + \tfrac{1}{2}(\alpha_{\bar{v}} + \beta'_{\bar{u}} + 2\alpha\beta')\, y.$$

It may be verified without difficulty that

$$Z = \frac{1}{\phi_u^4}\{Z\}, \qquad R = \frac{1}{\psi_v^4}\{R\}, \qquad S = \frac{1}{\phi_u^4 \psi_v^4}\{S\},$$

where the quantities $\{Z\}$, $\{R\}$, $\{S\}$ are easily calculated expressions in the coefficients and variables of system (2), and do not contain either ϕ or ψ.

It is therefore possible to express all of the concomitants, as calculated by Wilczynski for system (14), in terms of the coefficients and variables of system (2), even though the reduction of (2) to (14) requires the integration of the partial differential equations (8). That this must be so becomes sufficiently obvious if one inquires into the geometric significance of the invariants and covariants. As calculated for system (14), these invariants and covariants have interpretations connected with the asymptotic lines. But the properties of the asymptotic lines so expressed are such that these properties may be recognized, not only if the asymptotic lines themselves are known, but even if merely the two asymptotic directions at each point of the surface are known. These asymptotic directions, however, may be determined without any integration, so that no integration ought to be necessary in expressing the properties of the surface, as defined by (2), in terms of the asymptotic lines.

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* Wilczynski, *First Memoir*, p. 255.

† The quantities z, ρ, σ are the semi-covariants given by equations (40) of the *First Memoir*. Of the quantities (69) of that memoir, the first three are covariants; Professor Wilczynski has remarked to the writer that the fourth of (69) is not a covariant, and suggests the substitution of the quantity S instead. It should be remembered that the Greek letters α, β, γ replace the Roman a, b, c of the *First Memoir*.

Our analytic work proves that this is in fact the case. As an example, take the condition that the surface be ruled. In terms of the asymptotic lines, this is the condition that one of the families of asymptotic lines be composed of straight lines. Now, given the ∞^2 asymptotic directions corresponding to a family of asymptotic curves, it is evident that no integration is required to determine whether these ∞^2 directions give ∞^1 straight lines. Whether a similar statement may be made for *any* property of the surface which may be expressed in terms of the concomitants is not of course *a priori* obvious; our analytic results, however, actually prove that this is so. *All properties of the surface expressible in terms of the invariants and covariants of equations* (2) *are such that corresponding properties of the asymptotic lines of the surface may be enunciated in terms of the same invariants and covariants, even though it be impossible to determine the asymptotic lines themselves.*

A discussion analogous to that of the present paper finds a place in the study of many configurations by means of Wilczynski's general method. According to this method, a projective theory is equivalent to the theory of the invariants and covariants of a completely integrable system of differential equations, say S. It is, however, frequently convenient to use instead of S a canonical form, say S', on account of analytic simplifications introduced thereby. But to throw S into the canonical form S' may require an impossible integration. Nevertheless, the invariants and covariants calculated for the system S' may be calculable explicitly in terms of the coefficients and variables of the original system S. In that case—already exemplified by the calculations of the present paper—it is possible to make use of the simplifications, due to the canonical form, in the actual calculation of the concomitants for the general system S.

The above remarks, however, find their chief importance in the obvious deduction that, without any loss of generality whatever, the canonical system S' may be used instead of the original system S. The restriction of the problem consists merely in the omission of calculations which it is always possible to carry out in practice.

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