PROOF OF A THEOREM OF HASKINS*

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The theorem of this note, with its corollary, is due to Professor C. N. Haskins, and it is at his suggestion that the following concise proof, which was communicated to him by the present writer, is published. It is perhaps not superfluous to remark that the proof would hardly have been evolved in this form if the author had not just had the pleasure of listening to Professor de la Vallée Poussin's course on Lebesgue Integrals at Harvard University.

THEOREM. *Let $f(x)$ and $\phi(x)$ be two functions which are bounded and integrable in the sense of Lebesgue for*† $a \leqq x \leqq b$, *and let*

$$h \leqq f(x) \leqq H, \qquad h \leqq \phi(x) \leqq H,$$

for all values of x in the interval. Let these functions be such that

$$\int_a^b [f(x)]^n \, dx = \int_a^b [\phi(x)]^n \, dx$$

for all positive integral values of n.

Let $E_{\alpha\beta}(f)$ be the set of points of (a, b) at which $\alpha \leqq f \leqq \beta$, and let $E_{\alpha\beta}(\phi)$ be the corresponding set of points for ϕ. Then

$$mE_{\alpha\beta}(f) = mE_{\alpha\beta}(\phi),$$

for all pairs of numbers (α, β) in the interval from h to H, if mE denotes the measure of E.

Let $\omega_{\alpha\beta}(y)$ be a function which is equal to 1 for $\alpha \leqq y \leqq \beta$, and equal to 0 elsewhere in the interval $h \leqq y \leqq H$. Let $P_m(y)$, $m = 1, 2, \cdots$, be a sequence of polynomials such that

$$\lim_{m=\infty} P_m(y) = \omega_{\alpha\beta}(y)$$

throughout the interval $h \leqq y \leqq H$. The convergence of the sequence will of course not be uniform, but it will be possible to choose the polynomials so that they are uniformly bounded,

$$|P_m(y)| < G$$

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† It will be clear that any measurable set might be substituted for the interval (a, b).

for all values of y in the interval and all values of m, where G is independent of m and y. For example, let $\epsilon_1, \epsilon_2, \cdots$, be a decreasing sequence of positive quantities approaching zero; let $\omega_{\alpha\beta}(m, y)$ be a continuous function which is equal to $\omega_{\alpha\beta}(y)$ except in the intervals $(\alpha - \epsilon_m, \alpha)$ and $(\beta, \beta + \epsilon_m)$, where it is linear; and let $P_m(y)$ be a polynomial which approximates to $\omega_{\alpha\beta}(m, y)$ uniformly with an error not exceeding ϵ_m. Then

$$\lim_{m=\infty} P_m[f(x)] = \omega_{\alpha\beta}[f(x)], \qquad \lim_{m=\infty} P_m[\phi(x)] = \omega_{\alpha\beta}[\phi(x)],$$

for $a \leqq x \leqq b$, and the functions $P_m[f(x)]$, $P_m[\phi(x)]$, are uniformly bounded. Consequently

$$\lim_{m=\infty} \int_a^b P_m[f(x)]\,dx = \int_a^b \omega_{\alpha\beta}[f(x)]\,dx,$$

$$\lim_{m=\infty} \int_a^b P_m[\phi(x)]\,dx = \int_a^b \omega_{\alpha\beta}[\phi(x)]\,dx.$$

But

$$\int_a^b P_m[f(x)]\,dx = \int_a^b P_m[\phi(x)]\,dx$$

for all values of m, in consequence of the hypothesis. Therefore

$$\int_a^b \omega_{\alpha\beta}[f(x)]\,dx = \int_a^b \omega_{\alpha\beta}[\phi(x)]\,dx.$$

As $\omega_{\alpha\beta}[f(x)]$ is equal to 1 for x in $E_{\alpha\beta}(f)$ and equal to zero elsewhere, and similarly for ϕ, the values of the last two integrals are $mE_{\alpha\beta}(f)$ and $mE_{\alpha\beta}(\phi)$ respectively, and the theorem is proved.

The theorem would remain true, and the proof be essentially unchanged, if the set $E_{\alpha\beta}$ were defined with reference to the interval (α, β) exclusive *of one or both of its extremities, or were replaced by the set E_α where the function takes on a single specified value α.*

It would only be necessary to use a different sequence of functions $\omega_{\alpha\beta}(m, y)$ and of polynomials $P_m(y)$. In the case of the interval $\alpha < y \leqq \beta$, for example, a function $\omega_{\alpha\beta}(m, y)$ would be used which makes the transition from 0 to 1 in the interval from α to $\alpha + \epsilon_m$.

COROLLARY. *If the functions f and ϕ of the theorem are non-decreasing in the interval from a to b, they must be equal to each other except at an enumerable set of points.*

Let $E'_{\alpha\beta}$, with argument f or ϕ, denote the set of values of x for which the value of the function in question lies between α and β, exclusive of the former and inclusive of the latter. Because of the monotone character of the functions, the set $E'_{\alpha\beta}$ is always an interval, with or without its end points, unless it reduces to a single point or contains no point at all. For $\beta = H$, the lengths of $E'_{\alpha H}(f)$ and $E'_{\alpha H}(\phi)$ are equal, as a consequence of the theorem in one of its modified forms, and as both terminate at b, they consist of the same points, except that the left-hand end point may be included in one and not in the other. It follows, more generally, that the sets $E'_{\alpha\beta}(f)$ and $E'_{\alpha\beta}(\phi)$ are identical for any values of α and β, with a possible exception as to their extremities. An *interior* point of one is always an interior point of the other. At such points, as both f and ϕ are included between α and β,

$$|f(x) - \phi(x)| < \beta - \alpha.$$

Now let ϵ be any fixed positive quantity, and let E'_k stand for the set $E'_{(k-1)\epsilon,\, k\epsilon}$. All points of (a, b) are contained in a finite number of sets $E'_k(f)$, which have of course only a finite number of end points. Every point of (a, b) with the exception of these end points is interior to some set $E'_k(f)$, and so interior to the corresponding set $E'_k(\phi)$ as well. Hence

$$|f(x) - \phi(x)| < \epsilon$$

except at a finite number of points. By giving to ϵ a succession of values approaching zero, it is seen that the set of points where $f \neq \phi$ is enumerable.

It is evident that the theorem can be greatly generalized by using trigonometric or other functions of f and ϕ in place of powers of these quantities in the hypothesis.

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