THE RESOLUTION INTO PARTIAL FRACTIONS OF THE RECIPROCAL OF AN ENTIRE FUNCTION OF GENUS ZERO*

BY

J. F. RITT

It will be desirable, in the paper which follows this one,† to have information as to the possibility of separating into partial fractions, the reciprocal of the entire function

$$\zeta(z) = \left(1 - \frac{z}{a_1}\right)^{p_1}\left(1 - \frac{z}{a_2}\right)^{p_2}\cdots\left(1 - \frac{z}{a_n}\right)^{p_n}\cdots,$$

where the exponents p_n are positive integers, and the zeros a_n any complex numbers except zero such that

$$\sum_{n=1}^{\infty}\frac{p_n}{|a_n|}$$

is convergent.

Proceeding as in the case of a polynomial, we might form the sum of the principal parts in the Laurent developments of $1/\zeta(z)$ at the poles a_n, and write

$$\frac{1}{\zeta(z)} = \sum_{n=1}^{\infty}\left[\frac{f_{n,1}}{1 - \dfrac{z}{a_n}} + \frac{f_{n,2}}{\left(1 - \dfrac{z}{a_n}\right)^2} + \cdots + \frac{f_{n,p_n}}{\left(1 - \dfrac{z}{a_n}\right)^{p_n}}\right],$$

where the expressions for the numbers f are readily found.

An example to be given later will show, however, that the series so found may be divergent. Thus, a discussion of the validity of the development is certainly in order.

Cauchy‡ discussed the resolution of meromorphic functions into simple elements, making suitable hypotheses for the behavior of the function on a sequence of closed curves.

Borel§ has considered the problem in the light of the more recent develop-

* Presented to the Society, April 29, 1916.

† The reader can go as far as the eighth article of the next paper without reading the present paper. We advise him, in fact, to do this.

‡ Cauchy, *Oeuvres Complètes*, 2d series, vol. 7, p. 324. For other references, see Lindelöf, *Calcul des Résidus*, Chapter II.

§ Borel, Annales de l'école normale, ser. 3, vol. 18 (1901); Acta Mathematica, vol. 24 (1901); also, *Fonctions Méromorphes*, Chapter IV.

ments in the theory of functions, and has extended Cauchy's results in several directions.

The case where the meromorphic function is the reciprocal of an entire function of genus zero does not appear to have received special notice, nor does it seem that a specialization of more general discussions would lead to the result of this paper.

We shall confine ourselves to the case where $\zeta(z)$ has only a finite number of multiple zeros. The theorem of this paper may then be stated:

If there exists an integer r such that, for $n \geqq r$,

$$\left|\frac{a_{n+1}}{a_n}\right| > 1 + \frac{k}{n},$$

where $k > 2$, *the formal development of* $1/\zeta(z)$ *converges absolutely and uniformly to* $1/\zeta(z)$ *in every bounded domain in which* $1/\zeta(z)$ *is regular.**

If $\zeta(z)$ has no multiple zeros, the development to be considered is readily seen to be

$$\sum_{n=1}^{\infty} \frac{-1}{a_n \zeta'(a_n)} \frac{1}{1 - \dfrac{z}{a_n}},$$

where $\zeta'(z)$ is the first derivative of $\zeta(z)$.† Since $|a_n|$ goes to infinity with n, it is clear that this series will be absolutely and uniformly convergent in every bounded domain in which $1/\zeta(z)$ is regular, if

$$\sum_{n=1}^{\infty} \frac{1}{a_n \zeta'(a_n)}$$

is absolutely convergent. As to the case where there exist a finite number of multiple zeros, a similar statement is possible if we reject a finite number of terms of the development.

Let us examine now the development of the reciprocal of

$$\xi(z) = \frac{\sin \pi z^{\frac{1}{2}}}{\pi z^{\frac{1}{2}}} = \left(1 - \frac{z}{1^2}\right)\left(1 - \frac{z}{2^2}\right)\cdots\left(1 - \frac{z}{n^2}\right)\cdots.$$

Which value of $z^{\frac{1}{2}}$ we take is immaterial, provided that the same value is taken in both numerator and denominator. By direct calculation, the development for $1/\xi(z)$ is found to be

* If the condition of this theorem is satisfied, it will continue to be satisfied when equal zeros are written with distinct subscripts.

† Find the reciprocal of the first term in the development of $1/\zeta(z)$ for $z = a_n$.

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{1-\dfrac{z}{n^2}},$$

which is divergent for every value of z.

Upon this example, we shall base our discussion of the general case. Writing equal roots now with distinct subscripts, we have

$$-a_n\zeta'(a_n)=\left(1-\frac{a_n}{a_1}\right)\cdots\left(1-\frac{a_n}{a_r}\right)\cdots\left(1-\frac{a_n}{a_{n-1}}\right)\left(1-\frac{a_n}{a_{n+1}}\right)\cdots.$$

Let us suppose that we can choose an integer r such that, for $n \geqq r$,

$$\left|\frac{a_{n+1}}{a_n}\right|>1+\frac{k}{n},$$

where $k > 2$. Let h be any number less than k and greater than 2. Then, for n sufficiently great,

$$1+\frac{k}{n}>e^{h/n}.$$

Suppose r to have been chosen initially so that this last inequality holds for $n \geqq r$. Then, for $n \geqq r$,

$$\left|\frac{a_{n+1}}{a_n}\right|>e^{h/n},\quad \left|\frac{a_{n+2}}{a_{n+1}}\right|>e^{h/(n+1)},\quad \cdots,\quad \left|\frac{a_{n+q}}{a_{n+q-1}}\right|>e^{h/(n+q-1)}.$$

Multiplying all these inequalities together,

$$\left|\frac{a_{n+q}}{a_n}\right|>e^{h\left(\frac{1}{n}+\frac{1}{n+1}+\cdots+\frac{1}{n+q-1}\right)}$$
$$>e^{h\int_n^{n+q}\frac{dx}{x}}=e^{h[\log(n+q)-\log n]},$$

so that, finally,

$$\left|\frac{a_{n+q}}{a_n}\right|>\left(\frac{n+q}{n}\right)^h.$$

We have

$$-n^2\xi'(n^2)=\frac{(-1)^{n+1}}{2}$$
$$=\left(1-\frac{n^2}{1^2}\right)\cdots\left(1-\frac{n^2}{r^2}\right)\cdots\left(1-\frac{n^2}{\overline{n-1}^2}\right)\left(1-\frac{n^2}{\overline{n+1}^2}\right)\cdots.$$

Now, for $q > n \geqq r$,

$$\left|1-\frac{a_n}{a_q}\right|>1-\left(\frac{n}{q}\right)^h>1-\frac{n^2}{q^2},$$

so that

$$(1) \qquad \left| \prod_{q=n+1}^{\infty} \left(1 - \frac{a_n}{a_q}\right) \right| > \prod_{q=n+1}^{\infty} \left(1 - \frac{n^2}{q^2}\right).$$

Also, for $r \leqq q < n$,

$$\left| 1 - \frac{a_n}{a_q} \right| \geqq \left| \frac{a_n}{a_q} \right| - 1 > \left(\frac{n}{q}\right)^h - 1 > \left(\frac{n}{q}\right)^{h-2} \left(\frac{n^2}{q^2} - 1\right).$$

Thus,

$$\left| \prod_{q=r}^{q=n-1} \left(1 - \frac{a_n}{a_q}\right) \right| > \prod_{q=r}^{q=n-1} \left(\frac{n}{q}\right)^{h-2} \left(\frac{n^2}{q^2} - 1\right)$$

$$> \left[\frac{(r-1)!\, n^{n-r}}{(n-1)!} \right]^{h-2} \left| \prod_{q=r}^{q=n-1} \left(1 - \frac{n^2}{q^2}\right) \right|.$$

By Stirling's theorem,

$$(n-1)! = \sqrt{2\pi n}\, n^{n-1} e^{-n+\theta/12n},$$

where $0 < \theta < 1$, so that, for n sufficiently large,

$$(n-1)! < n^n e^{-n}.$$

Then, dropping the factor $(r-1)!$,

$$(2) \qquad \left| \prod_{q=r}^{q=n-1} \left(1 - \frac{a_n}{a_q}\right) \right| > (n^{-r} e^n)^{h-2} \left| \prod_{q=r}^{q=n-1} \left(1 - \frac{n^2}{q^2}\right) \right|.$$

Now, observing that for n sufficiently large, the first $r - 1$ factors of

$$-a_n \zeta'(a_n)$$

are each greater in absolute value than unity, we have, from (1) and (2),

$$|a_n \zeta'(a_n)| > \left| \frac{(n^{-r} e^n)^{h-2}}{\left(1 - \frac{n^2}{1^2}\right)\left(1 - \frac{n^2}{2^2}\right) \cdots \left(1 - \frac{n^2}{\overline{r-1}^2}\right)} n^2 \xi'(n^2) \right|$$

$$> \frac{(n^{-r} e^n)^{h-2}}{2n^{2r-2}},$$

or finally,

$$|a_n \zeta'(a_n)| > \frac{n^{2-rh} e^{n(h-2)}}{2}.$$

From this last inequality, the absolute convergence of

$$\sum \frac{1}{a_n \zeta'(a_n)}$$

follows without difficulty, so that the development of $1/\zeta(z)$ is absolutely and uniformly convergent in every bounded domain in which $1/\zeta(z)$ is regular.

It is not difficult now to show that the series of fractions converges to $1/\zeta(z)$.* Let $\chi(z)$ be the entire function whose zeros are the moduli of the zeros of $\zeta(z)$. Then, ignoring the finite number of multiple zeros of $\chi(z)$,

$$\sum \frac{1}{|a_n \chi'(|a_n|)|}$$

is convergent. Let

$$\zeta_m(z) = \left(1 - \frac{z}{a_1}\right)\left(1 - \frac{z}{a_2}\right)\cdots\left(1 - \frac{z}{a_m}\right).$$

It is easily seen that for $n \leqq m$,

$$|a_n \zeta_m'(a_n)| \geqq |a_n \chi'(|a_n|)|.$$

Hence, the sum of the last q terms in the partial fraction development of $1/\zeta_{m+q}(z)$ goes to zero for every value of z as m and q go to infinity independently of each other. We can take m so as to make small at pleasure, for any particular value of z, the sum of the terms in the development of $1/\zeta(z)$ which involve zeros with subscripts greater than m. As q increases indefinitely, m remaining fixed, $1/\zeta_{m+q}(z)$ approaches $1/\zeta(z)$ and the coefficients in the development of $1/\zeta_{m+q}(z)$ approach those in the development of $1/\zeta(z)$. It follows readily that the development of $1/\zeta(z)$ converges to $1/\zeta(z)$.

Through considerations similar to those which precede, it can be shown that, if the zeros of $\zeta(z)$ are all real and positive, if an infinite number of them are of order one, and if, after arranging the zeros in order of magnitude (increasing), there exists an integer r such that, for $n \geqq r$,

$$\frac{a_{n+1}}{a_n} < 1 + \frac{k}{n},$$

where $0 < k < 2$, the partial fraction development of $1/\zeta(z)$ is divergent.

We have thus determined a value of the rate of increase of the moduli of the zeros of $\zeta(z)$ which is critical with respect to the convergence of the

* This fact is not important for the following paper.

development of $1/\zeta(z)$. It would be easy, however, to give examples where the zeros do not increase in absolute value as rapidly as we have supposed, and where the partial fraction development is convergent. For instance, if the zeros of $\zeta(z)$ are real and of alternating signs, weaker conditions will suffice to insure the convergence.*

* For example, the reciprocal of

$$\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^3}\right)$$

can be developed formally.

Columbia University