# ON THE EXPRESSIBILITY OF A UNIFORM FUNCTION OF SEVERAL COMPLEX VARIABLES AS THE QUOTIENT OF TWO FUNCTIONS OF ENTIRE CHARACTER*

BY

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## 1. Introduction

It is a classical fact in the theory of functions of one complex variable that any meromorphic function may be expressed as the quotient of two entire functions without common zeros. When $f(x)$ is a uniform function with essential singularities at finite distance, this theorem may be extended, as was shown by Weierstrass† for a finite number of essential singularities, and by Mittag-Leffler in the general case: $f(x)$ is expressible as the quotient of two functions of entire character (that is, uniform and without poles, but generally both having the same essential singularities as $f(x)$) without common zeros.

Before taking up the corresponding question for several variables, it is convenient to recall the following definitions:

The complex variables $x_1, x_2, \cdots, x_n$ are interior to the region $(S_1, S_2, \cdots, S_n)$ when $x_1$ is interior to the region $S_1$ in the $x_1$-plane, $\cdots$, $x_n$ interior to the region $S_n$ in the $x_n$-plane; the regions $S_1, \cdots, S_n$ may be simply or multiply connected.

A *uniform* function $f(x_1, x_2, \cdots, x_n)$ of the complex variables $x_1, x_2, \cdots, x_n$ is *meromorphic* in $(S_1, S_2, \cdots, S_n)$ when, in the vicinity of every point $a_1, a_2, \cdots, a_n$ interior to $(S_1, S_2, \cdots, S_n)$, we have

$$f(x_1, x_2, \cdots, x_n) = \frac{P_1(x_1 - a_1,\, x_2 - a_2,\, \cdots,\, x_n - a_n)}{P_0(x_1 - a_1,\, x_2 - a_2,\, \cdots,\, x_n - a_n)},$$

where $P_0$ and $P_1$ are power series in $x_1 - a_1, x_2 - a_2, \cdots, x_n - a_n$. A *uniform* function $G(x_1, x_2, \cdots, x_n)$ is *of entire character* in $(S_1, S_2, \cdots, S_n)$

* Presented to the Society, October 25, 1913.

† K. Weierstrass, *Zur Theorie der eindeutigen analytischen Functionen, Mathematische Werke*, vol. 2 (Berlin, 1895), pp. 77–124. G. Mittag-Leffler, *Sur la représentation analytique des fonctions monogènes uniformes d'une variable indépendante*, Acta Mathematica, vol. 4 (1884), pp. 1–79.

when holomorphic at every point $a_1, a_2, \cdots, a_n$ interior to $(S_1, S_2, \cdots, S_n)$. Two functions *of entire character* $G_0(x_1, x_2, \cdots, x_n)$ and $G_1(x_1, x_2, \cdots, x_n)$ have *a common divisor* when there exists a point $a_1, a_2, \cdots, a_n$ interior to $(S_1, S_2, \cdots, S_n)$ such that in its vicinity

$$G_0(x_1, x_2, \cdots, x_n) = P(x_1 - a_1, x_2 - a_2, \cdots, x_n - a_n) \cdot P_0(x_1 - a_1, x_2 - a_2, \cdots, x_n - a_n),$$

$$G_1(x_1, x_2, \cdots, x_n) = P(x_1 - a_1, x_2 - a_2, \cdots, x_n - a_n) \cdot P_1(x_1 - a_1, x_2 - a_2, \cdots, x_n - a_n),$$

with $P(0, 0, \cdots, 0) = 0$. Two functions *of entire character* are relatively prime when they have no common divisor.

Poincaré has shown,* by the theory of harmonic functions of four real variables, that when $n = 2$ and $S_1$ and $S_2$ contain all points at finite distance in the $x_1$- and $x_2$-planes respectively, every meromorphic function is expressible as the quotient of two entire functions without common divisor. In a later paper,† he has modified this method and extended it to $n$ variables.

The Cauchy integral was used by Cousin‡ to prove Poincaré's result and extend it to more general regions. His most general results are the following, of which **A** may be regarded as the extension to several variables of Mittag-Leffler's theorem, while **B** generalizes Weierstrass's theorem on the existence of uniform functions with given zeros:

**A.** When for every point $a_1, a_2, \cdots, a_n$ interior to $(S_1, S_2, \cdots, S_n)$ there are given

(1) a region $\Gamma_{a_1, a_2, \cdots, a_n}$ consisting of $n$ circles $|x_\nu - a_\nu| < r_\nu$ $(\nu = 1, 2, \cdots, n)$, each of these circles being interior to the corresponding region $S_\nu$;

(2) a function $f_{a_1, a_2, \cdots, a_n}(x_1, x_2, \cdots, x_n)$ uniform in $\Gamma_{a_1, a_2, \cdots, a_n}$ and such that when two regions $\Gamma_{a_1, a_2, \cdots, a_n}$ and $\Gamma_{a_1', a_2', \cdots, a_n'}$ have a region in common, the difference

$$f_{a_1, a_2, \cdots, a_n}(x_1, x_2, \cdots, x_n) - f_{a_1', a_2', \cdots, a_n'}(x_1, x_2, \cdots, x_n)$$

is holomorphic in the common region;

Then there exists a function $F(x_1, x_2, \cdots, x_n)$ uniform in $(S_1, S_2, \cdots, S_n)$ and such that for every interior point $a_1, a_2, \cdots, a_n$ the difference

$$F(x_1, x_2, \cdots, x_n) - f_{a_1, a_2, \cdots, a_n}(x_1, x_2, \cdots, x_n)$$

is holomorphic in $\Gamma_{a_1, a_2, \cdots, a_n}$.

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* H. Poincaré, *Sur les fonctions de deux variables*, Acta Mathematica, vol. 2 (1883), pp. 97–113.

† H. Poincaré, *Sur les propriétés du potentiel et sur les fonctions Abéliennes*, Acta Mathematica, vol. 22 (1899), pp. 89–178.

‡ P. Cousin, *Sur les fonctions de n variables complexes*, Acta Mathematica, vol. 19 (1895), pp. 1–62.

**B.** When for every point $a_1, a_2, \cdots, a_n$ interior to $(S_1, S_2, \cdots, S_n)$ there are given

(1) a region $\Gamma_{a_1, a_2, \cdots, a_n}$ as in **A**;

(2) a function $u_{a_1, a_2, \cdots, a_n}(x_1, x_2, \cdots, x_n)$ of entire character in $\Gamma_{a_1, a_2, \cdots, a_n}$ and such that when two regions $\Gamma_{a_1, a_2, \cdots, a_n}$ and $\Gamma_{a_1', a_2', \cdots, a'}$ have a region in common, the quotient

$$u_{a_1, a_2, \cdots, a_n}(x_1, x_2, \cdots, x_n)/u_{a_1', a_2', \cdots, a_n'}(x_1, x_2, \cdots, x_n)$$

is holomorphic and different from zero in the common region;

Then there exists a function $G(x_1, x_2, \cdots, x_n)$ of entire character in $(S_1, S_2, \cdots, S_n)$ such that for every interior point $a_1, a_2, \cdots, a_n$ the quotient $G(x_1, x_2, \cdots, x_n)/u_{a_1, a_2, \cdots, a_n}(x_1, x_2, \cdots, x_n)$ is holomorphic and different from zero in $\Gamma_{a_1, a_2, \cdots, a_n}$.

**C.** When a function $f(x_1, x_2, \cdots, x_n)$ is meromorphic in $(S_1, S_2, \cdots, S_n)$, it may be expressed as the quotient of two relatively prime functions of entire character* in $(S_1, S_2, \cdots, S_n)$:

$$f(x_1, x_2, \cdots, x_n) = \frac{G_1(x_1, x_2, \cdots, x_n)}{G_0(x_1, x_2, \cdots, x_n)}.$$

Cousin establishes Theorem **A** in its various stages in an entirely rigorous manner, but his proofs of Theorem **B** (and hence of Theorem **C**, which is a quite elementary consequence of **B**—see Cousin, l. c., §§ 15, 19, and 25) contain a gap (at stages $\alpha$ and $\beta$) which considerably restricts the regions $(S_1, S_2, \cdots, S_n)$ in which they are applicable.

In § 2, the nature of this gap is explained, and Cousin's proofs of **B** are shown to be valid when all, or all but one, of the $n$ regions $S_1, S_2, \cdots, S_n$ are simply connected. On the other hand, it is established by an example that Cousin's construction of $G(x_1, x_2, \cdots, x_n)$ does not always yield a uniform function when two of the regions $S_1, S_2, \cdots, S_n$ are multiply connected.

The question now arises as to the validity of Theorems **B** and **C** in the cases where Cousin's proofs do not apply. In § 3 it is shown by an example that Theorem **C** is false (and consequently Theorem **B**, since **C** would follow from **B**) when two of the regions $S_1, S_2, \cdots, S_n$ are multiply connected, that is, in the very cases where Cousin's proofs fail.

* In his proofs, Cousin proceeds by four stages: first the theorems are derived for any region $(s_1, s_2, \cdots, s_n)$ interior to $(S_1, S_2, \cdots, S_n)$, and this separately for $n = 2$ (stage $\alpha$) and $n$ general (stage $\beta$). Second, a limiting process is used to extend the region of validity of the theorems from $(s_1, s_2, \cdots, s_n)$ to $(S_1, S_2, \cdots, S_n)$, and this separately when all $S_\nu$ are circles (stage $\gamma$) and when $S_\nu$ are quite general (stage $\delta$). For convenient reference, the numbers of Cousin's theorems corresponding to Theorems **A**, **B**, and **C** of the text at the various stages are given below:

| | $\alpha$ | $\beta$ | $\gamma$ | $\delta$ |
|---|---|---|---|---|
| **A** | I | IV | VII, p. 33 | XI |
| **B** | III | VI | IX | XII |
| **C** | — | VII, p. 32 | X | XIV |

Thus the results of the present paper may be summarized in the statement that

*Theorems* **B** *and* **C** *are valid when, and only when,* $n - 1$ *of the* $n$ *regions* $S_1, S_2, \cdots, S_n$ *are simply connected; the remaining region may be simply or multiply connected.*

The author wishes to acknowledge his indebtedness to Professor Osgood, to whom he communicated the example of § 3 in June, 1913, for material assistance in locating the gap in Cousin's proofs.

## 2. The domain of validity of Cousin's proofs of Theorem B

To abridge the notation, we shall write $x$ for the system of $n - 1$ variables $x_1, x_2, \cdots, x_{n-1}$ and $S$ for $(S_1, S_2, \cdots, S_{n-1})$; $x_n$ will be denoted by $y$ and $S_n$ by $S'$. A simply connected part $\Sigma$ of $S$ we define as a system of regions $(\Sigma_1, \Sigma_2, \cdots, \Sigma_{n-1})$ where, for $\nu = 1, 2, \cdots, n - 1$, every interior or boundary point of the simply connected region $\Sigma_\nu$ is interior to or on the boundary of $S_\nu$. The boundaries of $S_1, S_2, \cdots, S_{n-1}, \Sigma_1, \Sigma_2, \cdots, \Sigma_{n-1}$, and $S'$ are assumed to be regular, that is, each is to consist of a finite number of pieces of analytic curves without singular points.

We now assume $S'$ to be subdivided, by a finite number of pieces of regular curves, into a finite number of simply connected regions $R_1, R_2, \cdots, R_p, \cdots$. When $R_n$ and $R_p$ are adjacent regions, we denote by $l_{np}$ their common boundary, or, should this consist of several pieces, any one of these. If any $l_{np}$ is a closed curve, we cut it at three points, thus obtaining three pieces such that no two of them taken together form a closed curve. The direction of $l_{np}$ is that which leaves the interior of the region $R_n$ to the left, so that $l_{np}$ and $l_{pn}$ are the same curve described in opposite directions. Finally, let $T_{np}$ consist of all points in the $y$-plane interior to at least one circle with center on $l_{np}$ and sufficiently small radius $r$, this $r$ being constant not only for different points on $l_{np}$, but also for all the various curves $l_{np}$.

The proof of Theorem **B** now depends on the following lemma:

*Let a function* $u_p(x, y)$ *be given for every region* $R_p$, *uniform and holomorphic in* $(S, R_p)$, *boundaries included, and such that for any two adjacent regions* $R_n$ *and* $R_p$, *the quotient*

$$\frac{u_p(x, y)}{u_n(x, y)} = g_{np}(x, y)$$

*is holomorphic and different from zero in* $(S, T_{np})$. *Then there exists a function* $G(x, y)$ *holomorphic in* $(S, S')$, *uniform in* $(\Sigma, S')$, *where* $\Sigma$ *is any simply connected part of* $S$, *and such that in* $(S, R_p)$ *(boundaries included, except those* $y$ *which are end points of an* $l_{np}$ *and lie on the boundary of* $S'$) *the* *quotient*

$$\frac{G(x, y)}{u_p(x, y)}$$

*is holomorphic and different from zero.*

When $S$ is simply connected, we may evidently let $\Sigma$ coincide with $S$. In his formulation of the lemma (l. c., § 7; proof in § 6) Cousin makes no distinction between $\Sigma$ and $S$, so that, when $S$ is multiply connected (that is, one at least of $S_1, S_2, \cdots, S_{n-1}$ is multiply connected) he tacitly assumes the function $G(x, y)$ to be uniform in $(S, S')$, while the uniformity is proved only in $(\Sigma, S')$.

This constitutes the gap in Cousin's proofs referred to in the introduction. It might also be objected to his proof of the lemma (l. c., § 6) that he operates throughout with the multiform functions $\log u_p(x, y)$ and their differences $\log u_p(x, y) - \log u_n(x, y)$, and that it is not quite clear what branches of these functions are meant at the various points of $(S, S')$; but this objection is met by a modification of Cousin's argument due to Osgood.*

Since $u_p(x, y)$ and $u_n(x, y)$ are uniform in $(S, T_{np})$ by hypothesis, and their quotient $g_{np}(x, y)$ is holomorphic and different from zero in the same region, it follows that writing

$$G_{np}(x, y) = \log g_{np}(x, y),$$

where that branch of $\log g_{np}(x, y)$ is taken which assumes its principal value at some point $x_0, y_0$ interior to $(\Sigma, T_{np})$, the function $G_{np}(x, y)$ is holomorphic in $(S, T_{np})$ and uniform in $(\Sigma, T_{np})$. Next let

$$I_{np}(x, y) = \frac{1}{2\pi i}\int_{l_{np}}^{\cdot} \frac{G_{np}(x, z)\,dz}{z - y},$$

the integral being taken in the positive direction of $l_{np}$. This function is holomorphic for all $y$ at finite or infinite distance, except those on the curve $l_{np}$, and for any $x$ in $S$, and uniform for the same $y$ and any $x$ in $\Sigma$. Moreover, as shown in Cousin §§ 2–3,

$$I_{np}(x, y) = H(x, y) + G_{np}(x, y)\lambda_{np}(y),$$

$$\lambda_{np}(y) = \frac{1}{2\pi i}\log\frac{y - b}{y - a}, \qquad \lambda_{np}(\infty) = 0,$$

where $a$ and $b$ are the end points of $l_{np}$, $\log[(y - b)/(y - a)]$ is that branch of the logarithm which vanishes for $y = \infty$, so that $\lambda_{np}(y)$ is uniform and holomorphic in the whole $y$-plane except on the curve $l_{np}$, and finally $H(x, y)$

* Letter to the author, July 7, 1913. This modified proof is reproduced here with the permission of Professor Osgood.

is holomorphic in $(S, T_{np})$ and uniform in $(\Sigma, T_{np})$. Now write

$$\Phi(x, y) = \sum I_{np}(x, y),$$

where the summation is extended over all the curves $l_{np}$ which are common to the boundaries of two regions $R$ (each curve taken once, and not in the two subscript combinations $l_{np}$ and $l_{pn}$), and define

$$\phi_n(x, y) = \Phi(x, y) \text{ in } (S, R_n).$$

Then $\phi_n(x, y)$ is holomorphic in $(S, R_n)$ and uniform in $(\Sigma, R_n)$, boundaries included except the end points of the various $l_{np}$ belonging to the boundary of $R_n$. Denoting by $\phi_n(x, y)_p$ the analytic continuation of $\phi_n(x, y)$ when $x$ describes any path in $S$ and $y$ a path in $T_{np}$ starting at a point inside $R_n$ and ending at a point inside $R_p$, but not passing through an end point of $l_{np}$, we have (Cousin, l. c., §§ 2–3)

$$\phi_n(x, y)_p = \phi_p(x, y) + G_{np}(x, y). \tag{1}$$

A point $y = b$ interior to $S'$ is called a *vertex* when it is an end point of any $l_{np}$.

Now make

$$\overline{G}_n(x, y) = u_n(x, y)\, e^{\phi_n(x, y)} \text{ in } (S, R_n);$$

then it follows from (1) that $\overline{G}_p(x, y)$ is the analytic continuation of $\overline{G}_n(x, y)$ across $l_{np}$ (the path in the $y$-plane leading from $R_n$ into $R_p$ not crossing $l_{np}$ at a vertex), and consequently the continuation of $\overline{G}_n(x, y)$ along a closed path in the $y$-plane not passing through any vertex brings us back to $\overline{G}_n(x, y)$. We may therefore define a single function $\overline{G}(x, y)$ by the consistent conditions $\overline{G}(x, y) = \overline{G}_n(x, y)$ in $(S, R_n)$, and this $\overline{G}(x, y)$ is visibly uniform in $(\Sigma, S')$. Moreover, the quotient $\overline{G}(x, y)/u_p(x, y)$ is holomorphic and different from zero in $(S, R_p)$, boundaries included, except when $y$ coincides with an end point of an $l_{np}$ while $x$ takes any value inside or on the boundary of $S$.

We shall now modify $\overline{G}(x, y)$ so as to remove the last restriction for those end points of an $l_{np}$ which are vertices. Let $b$ be a vertex, and suppose that, for instance, $R_1, R_2, \cdots, R_m$ are those regions $R$ which are adjacent to this vertex. Let $1 \leqq \nu \leqq m$ and denote by $R'_\nu$ that part of $R_\nu$ which lies within or on the circle $|y - b| = r'$, where $r'$ is less than the radius $r$ of the circles used in defining all $T_{np}$. Then we have in $(S, R'_\nu)$

$$\phi_\nu(x, y) = \Phi(x, y) = A(x, y) + G_{12}(x, y)\lambda_{12}(y) + G_{23}(x, y)\lambda_{23}(y) + \cdots + G_{m-1,m}(x, y)\lambda_{m-1,m}(y) + G_{m1}(x, y)\lambda_{m1}(y),$$

$A(x, y)$ being holomorphic in $(S, |y - b| \leqq r')$ and uniform in $(\Sigma, |y - b| \leqq r')$. Make

$$L_\nu(y - b) = \frac{1}{2\pi i}\log(y - b),$$

where any branch of the logarithm is chosen and rendered uniform by a cut issuing from $y = b$, but having no other point in common with $R'_\nu$ or its boundary. None of the $l_{np}$ abutting at $b$ being closed, we may continue $\lambda_{np}(y)$ analytically from $y = \infty$ to a point inside $R'_\nu$ along a curve intersecting none of these $l_{np}$, and in the relation

$$\lambda_{np}(y) - L_\nu(y-b) = -\frac{1}{2\pi i}\log(y-a),$$

where now $\log(y-a)$ is a definite branch of the logarithm, for $y$ in $R'_\nu$, the right-hand member is holomorphic in the entire region $|y-b| \leqq r'$. Hence we have, *for $y$ interior to $R'_\nu$*,

$$\phi_\nu(x, y) = B_\nu(x, y) + [G_{12}(x, y) + G_{23}(x, y) + \cdots + G_{m-1,m}(x, y) + G_{m1}(x, y)]L_\nu(y-b),$$

where $B_\nu(x, y)$ is holomorphic in $(S, |y-b| \leqq r')$ and uniform in $(\Sigma, |y-b| \leqq r')$. On the other hand, the sum in brackets equals

$$\log\frac{u_2(x, y)}{u_1(x, y)} + \log\frac{u_3(x, y)}{u_2(x, y)} + \cdots + \log\frac{u_m(x, y)}{u_{m-1}(x, y)} + \log\frac{u_1(x, y)}{u_m(x, y)},$$

where each log refers to a definite branch of the function—the branch chosen at the beginning, and this sum therefore equals a definite value of $\log 1$, which we denote by $2\pi i K_b$, the integer $K_b$ being evidently independent of $\nu$. Consequently, for $y$ interior to $R'_\nu$,

$$(y-b)^{-K_b}\bar{G}(x, y) = u_\nu(x, y)e^{\phi_\nu(x, y) - 2\pi i K_b L_\nu(y-b)},$$

or

$$(y-b)^{-K_b}\bar{G}(x, y) = u_\nu(x, y)e^{B_\nu(x, y)};$$

but the expression to the right being holomorphic in $(S, |y-b| \leqq r')$ and uniform in $(\Sigma, |y-b| \leqq r')$, it follows by analytic continuation that the same is true of the left-hand member, and that the quotient of the latter by $u_\nu(x, y)$, which equals $e^{B_\nu(x, y)}$ in $(S, R'_\nu)$, is holomorphic and different from zero in that region.

Finally determine the integer $K_b$ for each vertex $b$ and write

$$G(x, y) = \bar{G}(x, y)\prod_b (y-b)^{-K_b},$$

the product extending over all vertices. It then follows immediately from the preceding argument that $G(x, y)$ has all the properties mentioned in the lemma.

As already stated, Cousin tacitly assumes that from the proven uniformity of $G(x, y)$ in $(\Sigma, S')$ it follows that $G(x, y)$ is also uniform in $(S,')$ $S$ when $S$ is multiply connected.

I shall now show by an example that this conclusion is not legitimate; it is evidently sufficient to assume $n = 2$, so that now $x$ stands for a single variable, and $S$ for a region in the $x$-plane. This example, as well as the one in § 3, is based on the simplest properties of Theta functions of two variables. It is well known that, given the constants $\tau_{11}$, $\tau_{12}$, $\tau_{22}$ such that the real part of $2\pi i(\tau_{11} n_1^2 + 2\tau_{12} n_1 n_2 + \tau_{22} n_2^2)$ is a definite negative quadratic form in $n_1$ and $n_2$, the two expressions*

$$\phi_\nu(v_1, v_2) = \sum_{n_1, n_2=-\infty}^{+\infty} \operatorname{Exp}\left[\left(n_1 - \frac{\nu}{2}\right)^2 \tau_{11} + 2\left(n_1 - \frac{\nu}{2}\right) n_2 \tau_{12} + n_2^2 \tau_{22} - 2\left(n_1 - \frac{\nu}{2}\right) v_1 - 2n_2 v_2\right], \tag{2}$$

where $\nu = 0$ or $1$, define entire functions of $v_1$ and $v_2$ with the properties

$$\begin{aligned} \phi_\nu(v_1 + 1, v_2) &= \phi_\nu(v_1, v_2), \\ \phi_\nu(v_1, v_2 + \tfrac{1}{2}) &= \phi_\nu(v_1, v_2), \\ \phi_\nu(v_1 + \tau_{11}, v_2 + \tau_{12}) &= \operatorname{Exp}(-2v_1 - \tau_{11}) \cdot \phi_\nu(v_1, v_2), \\ \phi_\nu(v_1 + \tau_{12}, v_2 + \tau_{22}) &= \operatorname{Exp}(-2v_2 - \tau_{22}) \cdot \phi_\nu(v_1, v_2) \end{aligned} \qquad (\nu = 0, 1).$$

Assume $\tau_{12} \neq 0$, introduce new variables $w_1$ and $w_2$ by the relations

$$\tau_{12} w_1 = -2\tau_{22} v_1 + 2\tau_{12} v_2, \qquad \tau_{12} w_2 = v_1,$$

and write $\phi_\nu(v_1, v_2) = \psi_\nu(w_1, w_2)$; then $\psi_\nu(w_1, w_2)$ are entire functions of $w_1$ and $w_2$ with the properties

$$\begin{aligned} \psi_\nu(w_1 + 1, w_2) &= \psi_\nu(w_1, w_2), \\ \psi_\nu(w_1, w_2 + 1) &= \operatorname{Exp}(-w_1 - 2\tau_{22} w_2 - \tau_{22}) \cdot \psi_\nu(w_1, w_2), \\ \psi_\nu\left(w_1 - \frac{2\tau_{22}}{\tau_{12}}, w_2 + \frac{1}{\tau_{12}}\right) &= \psi_\nu(w_1, w_2), \qquad (\nu = 0, 1). \\ \psi_\nu\left(w_1 + \frac{2\tau_{12}^2 - 2\tau_{11}\tau_{22}}{\tau_{12}}, w_2 + \frac{\tau_{11}}{\tau_{12}}\right) &= \operatorname{Exp}(-2\tau_{12} w_2 - \tau_{11}) \cdot \psi_\nu(w_1, w_2) \end{aligned} \tag{3}$$

Finally write $\psi(w_1, w_2) = \operatorname{Exp}(\tau_{22} w_2^2) \cdot \psi_0(w_1, w_2)$; then the entire function $\psi(w_1, w_2)$ has the properties

$$\begin{aligned} \psi(w_1 + 1, w_2) &= \psi(w_1, w_2), \\ \psi(w_1, w_2 + 1) &= \operatorname{Exp}(-w_1) \cdot \psi(w_1, w_2). \end{aligned} \tag{4}$$

* To simplify the typography, we shall use the notation $e^{2\pi i x} = \operatorname{Exp}(x)$.

Once more we introduce new variables by the equations

(5) $$x = \text{Exp}\,(w_1), \qquad y = \text{Exp}\,(w_2)$$

and write

(6) $$u(x, y) = \psi(w_1, w_2) = \psi\left(\frac{1}{2\pi i}\log x, \frac{1}{2\pi i}\log y\right);$$

then $u(x, y)$ is holomorphic for all $x, y$ at finite distance, except $x = 0$, $y = y$ and $x = x$, $y = 0$. Starting with some definite branches of $\log x$ and $\log y$, say those that equal zero for $x = 1$ and $y = 1$ respectively, it follows from (4) that $u(x, y)$ is uniform in respect to $x$, while the analytic continuation along a path winding about $y = 0$ once in the positive sense transforms the initial branch $u(x, y)$ into a new branch $\bar{u}(x, y)$ such that

(7) $$\bar{u}(x, y) = \frac{1}{x}u(x, y).$$

Now let us construct the function $G(x, y)$ of the lemma from the following data:

$S$: the circular ring $\frac{1}{2} < |x| < 2$;
$S'$: the circular ring $\frac{1}{2} < |y| < 2$;
$R_1$: the part of $S'$ to the right of the imaginary axis;
$R_2$: the part of $S'$ to the left of the imaginary axis;
$l_{12}$: the straight line segment from $y = 2i$ to $y = \frac{1}{2}i$;
$l'_{12}$: the straight line segment from $y = -\frac{1}{2}i$ to $y = -2i$, so that the common part of the boundaries of $R_1$ and $R_2$ consists of $l_{12}$ and $l'_{12}$;
$u_1(x, y)$: the initial branch of $u(x, y)$ defined above;
$u_2(x, y)$: the analytic continuation of $u_1(x, y)$ across the line $l_{12}$.

Then $u_1(x, y)$ and $u_2(x, y)$ are uniform and holomorphic in $(S, R_1)$ and $(S, R_2)$ respectively, boundaries included. On $l_{12}$,

$$g_{12}(x, y) = \frac{u_2(x, y)}{u_1(x, y)} = 1,$$

while on $l'_{12}$ we have

$$g'_{12}(x, y) = \frac{u_2(x, y)}{u_1(x, y)} = \frac{1}{x}$$

according to (7). We now make

$$G_{12}(x, y) = \log 1 = 0, \qquad G'_{12}(x, y) = -\log x,$$

where that branch of the logarithm is taken which vanishes at $x = 1$; since there are no vertices and therefore no integers $K_b$ to be determined, we may proceed at once to write down $\Phi(x, y)$:

$$\Phi(x, y) = \frac{1}{2\pi i}\int_{-\frac{1}{2}i}^{-2i}\frac{-\log x dz}{z - y} = \frac{1}{2\pi i}\log x \cdot \log\frac{y + \frac{1}{2}i}{y + 2i},$$

where the last logarithm is the branch that vanishes for $y$ infinite. Finally we obtain

$$(8) \qquad G(x, y) = u_p(x, y) \operatorname{Exp}\left(\frac{1}{2\pi i} \cdot \frac{1}{2\pi i} \log x \cdot \log \frac{y + \frac{1}{2}i}{y + 2i}\right)$$

in $(S, R_p)$ for $p = 1, 2$. This $G(x, y)$ now has all the properties indicated in the lemma (as is also readily verified directly in this particular case). Nevertheless, $G(x, y)$ is not uniform in $(S, S')$, for letting $x$ describe a closed path in $S$ starting and ending at $x = 1$, and winding about $x = 0$ once in the positive sense, while $y$ describes a closed path interior to $R_1$, $\log x$ increases by $2\pi i$, while $\log (y + \frac{1}{2}i)/(y + 2i)$ and $u_1(x, y)$ remain unchanged, and we arrive at a branch $\bar{G}(x, y)$ connected with the initial branch $G(x, y)$ by the relation

$$\bar{G}(x, y) = \frac{y + \frac{1}{2}i}{y + 2i} G(x, y).$$

Hence Cousin's lemma, and with it his proofs of Theorem **B**, are valid when, and only when, not more than one of the regions $S_1, S_2, \cdots, S_n$ is multiply connected.

## 3. Example of a function of two variables, meromorphic in a region $(S, S')$, which cannot be expressed as the quotient of two relatively prime functions of entire character

From (3) it is evident that the quotient

$$\frac{\psi_1(w_1, w_2)}{\psi_0(w_1, w_2)} = \frac{\phi_1(v_1, v_2)}{\phi_0(v_1, v_2)}$$

is a meromorphic quadruply periodic function of $w_1$ and $w_2$ with the periods

$$\begin{array}{lllll} 1, & 0, & -\dfrac{2\tau_{22}}{\tau_{12}}, & \dfrac{2\tau_{12}^2 - 2\tau_{11}\tau_{22}}{\tau_{12}} & \text{in } w_1, \\[2ex] 0, & 1, & \dfrac{1}{\tau_{12}}, & \dfrac{\tau_{11}}{\tau_{12}} & \text{in } w_2. \end{array}$$

By (2), $\phi_0(v_1, v_2)$ contains only even, and $\phi_1(v_1, v_2)$ only odd, powers of Exp. $(v_1)$; hence these two functions are linearly independent, and the quotient considered is not a constant. Introducing the variables $x$ and $y$ by (5) and writing

$$f(x, y) = \frac{\psi_1(w_1, w_2)}{\psi_0(w_1, w_2)},$$

$f(x, y)$ is a non-constant, uniform function of $x$ and $y$, meromorphic in the region $(S, S')$, where $S$ consists of all points at finite distance in the $x$-plane, the point $x = 0$ excepted, and $S'$ is defined similarly in the $y$-plane. This function has the properties

$$
(9) \qquad \begin{aligned} f(hx, ky) &= f(x, y), \\ f(lx, my) &= f(x, y), \end{aligned}
$$

where

$$
(10) \qquad \begin{aligned} h &= \operatorname{Exp}\left(-\frac{2\tau_{22}}{\tau_{12}}\right), \qquad k = \operatorname{Exp}\left(\frac{1}{\tau_{12}}\right), \\ l &= \operatorname{Exp}\left(\frac{2\tau_{12}^2 - 2\tau_{11}\tau_{22}}{\tau_{12}}\right), \qquad m = \operatorname{Exp}\left(\frac{\tau_{11}}{\tau_{12}}\right). \end{aligned}
$$

Now let us subject $\tau_{11}$, $\tau_{12}$, $\tau_{22}$ to the further condition that

$$
(11) \qquad l^a m^b \neq h^c k^d
$$

for any integers $a$, $b$, $c$, and $d$ which are not all equal to zero. By (10), this is equivalent to the condition that the equation

$$
(12) \qquad b\tau_{11} + n\tau_{12} + 2c\tau_{22} + 2a(\tau_{12}^2 - \tau_{11}\tau_{22}) - d = 0
$$

shall have no solution in integers $a$, $b$, $c$, $d$, $n$ which are not all equal to zero.* Then $f(x, y)$ cannot be expressed as the quotient of two relatively prime functions of entire character in $(S, S')$. For the purpose of an example, it is sufficient to carry out the proof in a special case, giving numerical values to $\tau_{11}$, $\tau_{12}$, $\tau_{22}$.† Let us make

$$
\tau_{11} = i, \qquad \tau_{12} = \frac{1}{\sqrt[4]{2}}, \qquad \tau_{22} = i\sqrt{2};
$$

then the real part of $2\pi i(\tau_{11} n_1^2 + 2\tau_{12} n_1 n_2 + \tau_{22} n_2^2)$ is $-2\pi(n_1^2 + \sqrt{2}n_2^2)$, a definite negative quadratic form in $n_1$ and $n_2$. Furthermore $\tau_{12} \neq 0$, and (12) gives upon separation of the real and imaginary parts

$$
b + 2c\sqrt{2} = 0, \qquad n + 3a\sqrt[4]{8} - d\sqrt[4]{2} = 0,
$$

whence

$$
b = c = 0, \qquad n^2 + 12ad - (18a^2 + d^2)\sqrt{2} = 0, \qquad a = d = n = 0.
$$

Hence (11) is satisfied, and in particular we have for any integers $\lambda$ and $\mu$, except $\lambda = \mu = 0$,

$$
(13) \qquad h^\lambda k^\mu - 1 \neq 0, \qquad l^\lambda m^\mu - 1 \neq 0.
$$

* In the theory of Theta functions, this condition expresses the fact that the period system $\tau_{11}$, $\tau_{12}$, $\tau_{22}$ is non-singular.

† This has the advantage of simplifying the convergence proof for the series (19).

Now assume that $f(x, y)$ can be expressed in the form*

$$f(x, y) = \frac{G_1(x, y)}{G_0(x, y)}, \tag{14}$$

where $G_0(x, y)$ and $G_1(x, y)$ are of entire character and relatively prime in $(S, S')$; we shall show that this leads to a contradiction. From (9) and (14) it follows that

$$\frac{G_0(hx, ky)}{G_0(x, y)} = \frac{G_1(hx, ky)}{G_1(x, y)}, \qquad \frac{G_0(lx, my)}{G_0(x, y)} = \frac{G_1(lx, my)}{G_1(x, y)},$$

and since $G_0(x, y)$ and $G_1(x, y)$ are relatively prime, we conclude that both these quotients, which are evidently uniform, are holomorphic and different from zero in $(S, S')$.† Let us denote them by $g(x, y)$ and $g'(x, y)$ respectively; then

$$G_\nu(hx, ky) = g(x, y)G_\nu(x, y), \qquad G_\nu(lx, my) = g'(x, y)G_\nu(x, y) \qquad (\nu = 0, 1). \tag{15}$$

Since $g(x, y)$ is of entire character and different from zero in $(S, S')$, we may expand its logarithmic derivatives in Laurent's series‡

$$\frac{\partial \log g(x, y)}{\partial x} = \sum_{\lambda, \mu=-\infty}^{+\infty} a_{\lambda\mu}\, x^\lambda y^\mu, \qquad \frac{\partial \log g(x, y)}{\partial y} = \sum_{\lambda, \mu=-\infty}^{+\infty} b_{\lambda\mu}\, x^\lambda y^\mu,$$

both series being absolutely and uniformly convergent for $\epsilon \leqq |x| \leqq 1/\epsilon$, $\epsilon \leqq |y| \leqq 1/\epsilon$, where $\epsilon$ is as small as we please. From

$$\frac{\partial^2 \log g(x, y)}{\partial y \partial x} = \frac{\partial^2 \log g(x, y)}{\partial x \partial y}$$

it follows that

$$\sum \mu a_{\lambda\mu}\, x^\lambda y^{\mu-1} = \sum \lambda b_{\lambda\mu}\, x^{\lambda-1} y^\mu,$$

so that in particular $\mu a_{-1, \mu} = 0$, $\lambda b_{\lambda, -1} = 0$, whence

$$a_{-1, 0} = a, \qquad a_{-1, \mu} = 0 \qquad (\mu \neq 0),$$

$$b_{0, -1} = b, \qquad b_{\lambda, -1} = 0 \qquad (\lambda \neq 0).$$

---

* The following investigation is closely related to one made by Appell to an entirely different purpose in his paper *Sur les fonctions périodiques de deux variables,* Journal de Mathématiques, ser. 4, vol. 7 (1891), pp. 157–219. See pp. 185–201.

† This is a simple consequence of Weierstrass' preparation theorem; compare Cousin, l. c., § 15, and Appell, l. c., pp. 182–185.

‡ K. Weierstrass, *Einige auf die Theorie der analytischen Funktionen mehrerer Veränderlichen sich beziehende Sätze, Mathematische Werke,* vol. 2 (Berlin, 1895), pp. 135–188. See pp. 183–188.

Treating $g'(x, y)$ in the same way, and integrating, we finally obtain

$$(16)\qquad \begin{aligned} g(x, y) &= x^a y^b \operatorname{Exp}\left(\sum_{\lambda,\mu=-\infty}^{+\infty} A_{\lambda\mu} x^\lambda y^\mu\right), \\ g'(x, y) &= x^c y^d \operatorname{Exp}\left(\sum_{\lambda,\mu=-\infty}^{+\infty} B_{\lambda\mu} x^\lambda y^\mu\right), \end{aligned}$$

the series being absolutely and uniformly convergent as before, and from the uniformity of $g(x, y)$ and $g'(x, y)$ it is evident that $a, b, c, d$ are all integers. We arrive at a relation between $g(x, y)$ and $g'(x, y)$ by observing that according to (15)

$$\frac{G_\nu(hlx, kmy)}{G_\nu(x, y)} = \frac{G_\nu(hlx, kmy)}{G_\nu(lx, my)} \cdot \frac{G_\nu(lx, my)}{G_\nu(x, y)} = g(lx, my)\, g'(x, y),$$

$$\frac{G_\nu(lhx, mky)}{G_\nu(x, y)} = \frac{G_\nu(lhx, mky)}{G_\nu(hx, ky)} \cdot \frac{G_\nu(hx, ky)}{G_\nu(x, y)} = g'(hx, ky)\, g(x, y),$$

whence

$$g(lx, my)\, g'(x, y) = g'(hx, ky)\, g(x, y).$$

Introducing the expressions (16) into this relation, we obtain

$$l^a m^b \operatorname{Exp}\left[\sum (A_{\lambda\mu} l^\lambda m^\mu + B_{\lambda\mu}) x^\lambda y^\mu\right] = h^c k^d \operatorname{Exp}\left[\sum (B_{\lambda\mu} h^\lambda k^\mu + A_{\lambda\mu}) x^\lambda y^\mu\right],$$

which evidently gives

$$(17)\qquad A_{\lambda\mu}(l^\lambda m^\mu - 1) = B_{\lambda\mu}(h^\lambda k^\mu - 1)$$

and $l^a m^b = h^c k^d$. But in the last relation it follows from (11)—and this is the main point of the proof—that the integers $a, b, c$, and $d$ are all equal to zero. Moreover, (13) shows that we may write (17) in the form

$$(18)\qquad \frac{A_{\lambda\mu}}{h^\lambda k^\mu - 1} = \frac{B_{\lambda\mu}}{l^\lambda m^\mu - 1}, \qquad \text{except for } \lambda = \mu = 0.$$

Denote by $\sum'$ a series from which the combination $\lambda = \mu = 0$ is excluded, and write

$$(19)\qquad G(x, y) = \sum_{\lambda,\mu=-\infty}^{+\infty}{}' \frac{A_{\lambda\mu}}{h^\lambda k^\mu - 1} x^\lambda y^\mu = \sum_{\lambda,\mu=-\infty}^{+\infty}{}' \frac{B_{\lambda\mu}}{l^\lambda m^\mu - 1} x^\lambda y^\mu;$$

then (18) shows that the two definitions of $G(x, y)$ are formally consistent. For the convergence proof, separate the terms where $\lambda \neq 0$ from those with $\lambda = 0$; we obtain with the aid of (18)

$$G(x, y) = \sum_{\mu=-\infty}^{+\infty} \sum_{\lambda \neq 0} \frac{A_{\lambda\mu}}{h^\lambda k^\mu - 1} x^\lambda y^\mu + \sum_{\mu \neq 0} \frac{B_{0\mu}}{m^\mu - 1} y^\mu.$$

Introducing the numerical values of $\tau_{11}$, $\tau_{12}$, $\tau_{22}$ in (10), we find

$$h = e^{4\pi\sqrt[4]{2}}, \qquad k = e^{2\pi i\sqrt[4]{2}}, \qquad m = e^{-2\pi\sqrt[4]{2}},$$

and consequently

$$|h^\lambda k^\mu - 1| \geqq \big||h|^\lambda |k|^\mu - 1\big| = |e^{4\pi\sqrt[4]{2}\cdot\lambda} - 1|;$$

the last expression being greater than $e - 1$ or $1 - e^{-1}$ according as $\lambda$ is a positive or negative integer, we have $|h^\lambda k^\mu - 1| > \frac{1}{2}$ for $\lambda \neq 0$, and similarly $|m^\mu - 1| > \frac{1}{2}$ for $\mu \neq 0$. Therefore (19) converges absolutely and uniformly in the same region as (16), that is, for $\epsilon \leqq |x| \leqq 1/\epsilon$, $\epsilon \leqq |y| \leqq 1/\epsilon$. Evidently $G(x, y)$ satisfies the relations

$$(20) \qquad \begin{aligned} G(hx, ky) - G(x, y) &= \textstyle\sum' A_{\lambda\mu} x^\lambda y^\mu, \\ G(lx, my) - G(x, y) &= \textstyle\sum' B_{\lambda\mu} x^\lambda y^\mu. \end{aligned}$$

If we now write

$$G'_\nu(x, y) = \operatorname{Exp}[-G(x, y)] \cdot G_\nu(x, y) \qquad (\nu = 0, 1),$$

$G'_0(x, y)$ and $G'_1(x, y)$ are of entire character (and relatively prime) in $(S, S')$, and by (14)

$$(21) \qquad f(x, y) = \frac{G'_1(x, y)}{G'_0(x, y)}.$$

From (15), (16), and (20) we find, bearing in mind that $a = b = c = d = 0$,

$$(22) \qquad \begin{aligned} G'_\nu(hx, ky) &= \operatorname{Exp}(A_{00}) \cdot G'_\nu(x, y), \\ G'_\nu(lx, my) &= \operatorname{Exp}(B_{00}) \cdot G'_\nu(x, y) \end{aligned} \qquad (\nu = 0, 1).$$

Expanding $G'_0(x, y)$ and $G'_1(x, y)$ in Laurent's series

$$G'_0(x, y) = \sum_{\lambda, \mu = -\infty}^{+\infty} C_{\lambda\mu} x^\lambda y^\mu, \qquad G'_1(x, y) = \sum_{\lambda, \mu = -\infty}^{+\infty} D_{\lambda\mu} x^\lambda y^\mu,$$

the first equation (22) gives

$$C_{\lambda\mu}[h^\lambda k^\mu - \operatorname{Exp}(A_{00})] = D_{\lambda\mu}[h^\lambda k^\mu - \operatorname{Exp}(A_{00})] = 0.$$

Since $G'_0(x, y)$ is not identically zero, one $C_{\lambda\mu}$ at least must be different from zero, say $C_{\rho\sigma}$, so that $h^\rho k^\sigma - \operatorname{Exp}(A_{00}) = 0$. If $h^\lambda k^\mu - \operatorname{Exp}(A_{00}) = 0$, it follows that $h^{\lambda-\rho} k^{\mu-\sigma} - 1 = 0$, whence $\lambda = \rho$, $\mu = \sigma$ by (13). Therefore $h^\lambda k^\mu - \operatorname{Exp}(A_{00}) \neq 0$, and $C_{\lambda\mu} = D_{\lambda\mu} = 0$ except for $\lambda = \rho$, $\mu = \sigma$, and (21) gives

$$f(x, y) = \frac{D_{\rho\sigma} x^\rho y^\sigma}{C_{\rho\sigma} x^\rho y^\sigma} = \text{const.}$$

But we have seen from the definition of $f(x, y)$ that this function is not a constant, and this contradiction shows that Theorem **C** (and consequently

Theorem **B**, since **B** implies **C**) is not valid when two of the regions $S_1$, $S_2$, $\cdots$, $S_n$ are multiply connected.

It is possible however to express our function $f(x, y)$ as the quotient of two functions $G_1(x, y)$ and $G_0(x, y)$ of entire character in $(S, S')$, if we remove the condition that these two functions shall be relatively prime. To prove this, let $\rho = 0$ or $1$ and write

$$\psi_2(w_1, w_2) = \operatorname{Exp}(2\tau_{22} w_2^2) \cdot \psi_\rho(w_1, -w_2);$$

it then follows from (3) that

$$\psi_2(w_1 + 1, w_2) = \psi_2(w_1, w_2),$$
$$\psi_2(w_1, w_2 + 1) = \operatorname{Exp}(w_1 + 2\tau_{22} w_2 + \tau_{22})\psi_2(w_1, w_2),$$

so that

$$\left.\begin{aligned}\psi_2(w_1 + 1, w_2)\psi_\nu(w_1 + 1, w_2) &= \psi_2(w_1, w_2)\psi_\nu(w_1, w_2),\\ \psi_2(w_1, w_2 + 1)\psi_\nu(w_1, w_2 + 1) &= \psi_2(w_1, w_2)\psi_\nu(w_1, w_2)\end{aligned}\right\} \quad (\nu = 0, 1),$$

and consequently, writing

$$G_\nu(x, y) = \psi_2(w_1, w_2)\psi_\nu(w_1, w_2) \qquad (\nu = 0, 1),$$

$G_0(x, y)$ and $G_1(x, y)$ are both *uniform* functions of $x$ and $y$, holomorphic in $(S, S')$. Since $f(x, y) = \psi_1(w_1, w_2)/\psi_0(w_1, w_2)$, we have in

$$f(x, y) = \frac{G_1(x, y)}{G_0(x, y)}$$

a representation of $f(x, y)$ of the required character. Evidently $G_0(x, y)$ and $G_1(x, y)$ have here the common manifold of zeros defined by

$$\psi_2(w_1, w_2) = 0,$$

and from what we have proved before regarding $f(x, y)$, it follows that the common divisor cannot be removed without destroying the uniformity of $G_0(x, y)$ and $G_1(x, y)$.

In a subsequent paper, it will be shown that this representation as the quotient of two functions of entire character with common divisor is possible for any function $f(x, y)$, meromorphic everywhere at finite distance except at the points defined by $G(x, y) = 0$, where $G(x, y)$ is an entire function. The common divisor cannot in general be removed except when $G(x, y)$ is irreducible.