ON THE EQUIVALENCE OF ÉCART AND VOISINAGE*

BY

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1. Fréchet has suggested that being given a class (V) for which a " voisinage " is defined it should be possible to give a definition of " écart " in this class such that the convergent sequences and their limits remain the same whether limit is defined in terms of voisinage or in terms of the corresponding écart.† He also observes that a result of Hahn:‡ there exists on every class (V) at least one continuous non-constant function, might lead to a proof of equivalence between écart and voisinage. It is the purpose of this paper to supply a proof of the correctness of these suppositions.

2. For each pair A, B of elements of a class (V) there exists a number $[A, B]$, the voisinage of A and B, which is non-negative, symmetric in A and B, and satisfies the following conditions:

(*a*) $[A, B] = 0$, if and only if $A = B$;

(*b*) There exists a positive function $f(e)$ such that

$$\underset{e \to 0}{L} f(e) = 0,$$

and if

$$[A, B] < e, \qquad [B, C] < e,$$

then

$$[A, C] < f(e).$$

The écart (A, B) of two elements A, B of a class (E) differs from the voisinage $[A, B]$ only in that condition (*b*) is replaced by:

(*c*) If A, B, C are any three elements of E, then§

$$(A, C) \leqq (A, B) + (B, C).$$

3. Given a class H of elements of (V), denote by $\overline{H}$ *the class of all elements*

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† *Les ensembles abstraits et le calcul fonctionnel*, Rendiconti del Circolo Matematico di Palermo, vol. 30 (1910), pp. 22–23.

‡ Monatshefte für Mathematik und Physik, vol. 19 (1908), pp. 247–257.

§ Cf. Fréchet, *Sur quelques points du calcul fonctionnel*, Rendiconti del Circolo Matematico di Palermo, vol. 22 (2d semester, 1906), pp. 1–74.

*which are limits of sequences of elements of H.** Then $\bar{H}$ contains H and H'. In a class (V), $\bar{H} = H + H'$. A finite set

$$H_1, H_2, \cdots, H_n$$

of subclasses of (V) will be called a Hahn sequence if for $|i-j|>1$, $\bar{H}_i$ and $\bar{H}_j$ have no element in common.

From a given Hahn sequence a Hahn sequence may be derived by subdivision of the classes H_i as follows. Denote by $[A_i, \bar{H}_j]$ the greatest lower bound of $[A_i, A_j]$ for fixed A_i in H_i and every A_j in $\bar{H}_j$. If

$$[A_i, \bar{H}_{i-1}] \leqq [A_i, \bar{H}_{i+1}],$$

A_i is assigned to H_{i0}, otherwise to H_{i1}. H_1 and H_n remain undivided. The sequence

$$H_1, H_{10}, H_{11}, \cdots, H_{n-1,1}, H_n$$

is a Hahn sequence. We will first show that $\bar{H}_{i0}$, $\bar{H}_{i+1}$ have no element in common. Suppose the contrary. Then for every integer n there exist elements A_n in H_{i0}, B_n in H_{i+1} such that for some A_0 independent of n,

$$[A_n, A_0] < \frac{1}{n}, \qquad [B_n, A_0] < \frac{1}{n}.$$

For every small positive e there is n_e such that for all $n \geqq n_e$, $f(1/n) < e$. Therefore for $n \geqq n_e$,

$$[A_n, B_n] < e.$$

Since A_n lies in H_{i0}, we have

$$[A_n, \bar{H}_{i-1}] \leqq [A_n, \bar{H}_{i+1}].$$

Hence for every e, n there is C_n in $\bar{H}_{i-1}$ such that

$$[A_n, C_n] < [A_n, B_n] + e.$$

If n exceeds n_e and $1/2e$, then we have

$$[A_n, C_n] < 2e, \qquad [A_n, A_0] < 2e,$$

from which

$$[C_n, A_0] < f(2e).$$

Therefore $\bar{\bar{H}}_{i-1}$ and $\bar{\bar{H}}_{i+1}$ have a common element. But property (b) of the voisinage implies that $\bar{\bar{H}} = \bar{H}$. A contradiction has been obtained.

Since H_{i+1} contains $H_{i+1,0}$ and $H_{i+1,1}$ the separation is established for these cases. The case of H_i and $H_{i+1,1}$ is readily treated by the analogous argument. All other separations follow immediately from those assumed for the original sequence $H_1 \cdots H_n$.

* The proof given in this article covers cases more general in form than the case of classes (V). For example, in this proof no use is made of uniqueness of the relation limit of a sequence.

If $|i-j|>1$, H_i and H_j are non-adjacent classes of a sequence $H_1, \cdots H_n$. Suppose that for every non-adjacent pair of classes H_i, H_j

$$[A_i, A_j] \geqq a$$

for every pair A_i, A_j of elements from the respective classes H_i, H_j. Then there exists $a_1 < a$ such that $f(a_1) < a$, similarly effective with respect to the sequence derived from $H_1 \cdots H_n$. If not, there will exist, for every m, elements A_m, B_m in non-adjacent classes of H_1, H_{10}, $\cdots$, H_n, such that

$$[A_m, B_m] < \frac{1}{m}.$$

If $(1/m) < a$, A_m, B_m lie in adjacent classes H_{i_m}, H_{i_m+1}. Suppose A_m lies in H_{i_m0}. Then C_m exists in H_{i_m-1} such that

$$[A_m, C_m] \leqq [A_m, B_m] < \frac{1}{m}.$$

If $f(1/m) < a$, then

$$[B_m, C_m] < a,$$

contrary to hypothesis. A similar result follows the hypothesis that A_m lies in H_{i1}. The existence of a_1 is established. In fact a_1 may be supposed to denote the greatest value satisfying the conditions $a_1 \leqq a/2, f(a_1) \leqq a/2$, and is completely independent of the particular Hahn sequence concerned. This fact is of great importance in the sequel.

4. If (V) is a singular class, voisinage and écart are identical. If (V) contains at least two elements it is evident that there is a number $s > 0$ such that to every A there corresponds a B for which

$$[A, B] > s.$$

Let s be fixed. Choose $r < s$ so that $f(r) < s$. For fixed A_0, K_r denotes the class of all elements A in the relation

$$[A, A_0] \leqq r,$$

K_s the class of all elements A in the relation

$$[A, A_0] \geqq s,$$

K the remaining elements of (V).

The sequence K_r, K, K_s is a Hahn sequence. For $\overline{K}_r$, $\overline{K}_s$ have no common element. Otherwise there is an element A, and there are elements A_n in K_r, B_n in K_s, such that

$$[A_n, A] < \frac{1}{n}, \qquad [B_n, A] < \frac{1}{n},$$

and therefore

$$[A_n, B_n] < f\left(\frac{1}{n}\right).$$

But $[A_0, A_n] \leqq r$; and, if $f(1/n) \leqq r$, we have

$$[A_0, B_n] < f(r) < s,$$

contrary to the hypothesis B_n in K_s.

From K_r, K, K_s we obtain by successive subdivision of K relative to K_r and K_s a development of (V) in Hahn sequences. The mth stage of the process is the Hahn sequence

$$K_r, K_{0,0,0\cdots 0}, K_{0,0\cdots 0,1}, \cdots K_{i_1, i_2, \cdots i_m} \cdots K_s,$$

where i assumes the values $0, 1$ only.

There exists a sequence $[a_m]$ of positive numbers such that $a_m < a_{m-1}$, $f(a_m) < a_{m-1}$, $a_1 < a$, $f(a_1) < a$, where $a < r$, $f(a) < r$. Then if A, B are in non-adjacent classes of stage m,

$$[A, B] > a_m.$$

This is easily established by induction.

For each element A of K there is a unique sequence $i_1, i_2, \cdots, i_m, \cdots$, of indices, the first m of which determine the class $K_{i_1 \cdots i_m}$ which contains A.

5. In terms of the development of (V) thus obtained we define a Hahn function $F(A)$. If A lies in K_r, $F(A) = 0$; if A lies in K_s, $F(A) = 1$; otherwise,

$$F(A) = \frac{i_1}{2} + \frac{i_2}{2^2} + \cdots + \frac{i_m}{2^m} + \cdots,$$

where $i_1, \cdots, i_m$, are the indices associated with A in the development of K.

$F(A)$ is easily seen to be continuous. In fact if $[A, B] < a_m$, A and B lie in adjacent classes of stage m, and therefore

$$|F(A) - F(B)| \leqq \frac{1}{2^{m-1}}.$$

The function $F(A)$ thus defined is dependent upon A_0, r, and s. Given a sequence $[s_n]$ of numbers decreasing to zero $(s_1 = s)$, we may suppose a corresponding sequence $[r_n]$ and for each n a function $F_n(A)$ relative to A_0, s_n, r_n. Henceforth we denote by $F(A; A_0)$ the function

$$F(A; A_0) = \sum_1^\infty \frac{1}{2^n} F_n(A),$$

which vanishes only for $A = A_0$, is positive if $0 < [A, A_0] < s_1$; and if $[A, A_0] \geqq s_1 = s$, then $F(A, A_0) = 1$.

It is evident that the sequences $[s_n]$, $[r_n]$ are independent of A_0. In the following discussion they will be supposed to have been determined once for all. In fact we may suppose $s_n = s/n$.

6. In terms of the functions $F(A; A_0)$ we define the écart (A, B) of two elements as follows: (A, B) *is the least upper bound of the differences*

$$|F(A; A_0) - F(B; A_0)|$$

for all possible A_0.

The function (A, B) *is an écart.* For (A, B) is non-negative and equal to (B, A); (A, A) is evidently zero; and if A and B are distinct, $F(B; A) > 0$, while $F(A; A) = 0$, so that $(A, B) > 0$. Furthermore

$$(A, B) \leqq (A, C) + (B, C),$$

since

$$|F(A; A_0) - F(B; A_0)| \leqq |F(A; A_0) - F(C; A_0)| \xi + |F(B; A_0) - F(C; A_0)|,$$

and the upper bound on the left is not greater than the sum of the upper bounds on the right.

The equivalence of $[A, B]$ and (A, B) in respect to limit must be established.

Given $L_k(A_k, A) = 0$, $L_k[A_k, A] \neq 0$, a contradiction arises. A positive number e_0 and a subsequence $[A'_k]$ of $[A_n]$ exist such that for every k

$$[A'_k, A] > e_0 > 0.$$

Choose n so that $s_n < e_0$. Then for every k the relations hold in order:

$$F_n(A'_k; A) = 1; \qquad F(A'_k; A) > \frac{1}{2^n}; \qquad (A'_k, A) > \frac{1}{2^n}.$$

The desired contradiction has been obtained.

The proof that $L_k[A_k, A] = 0$, $L_k(A_k, A) \neq 0$ is impossible, is more difficult. We may suppose, without loss of generality, that $(A_k, A) > e_0$ for every k. From the definition of (A_k, A) there is for every k, B_k such that

$$(1) \qquad |F(A_k; B_k) - F(A; B_k)| > e_0.$$

Since $L_k F(A_k; A) = 0$ there is a k_0 such that $k > k_0$ implies B_k is distinct from A. There exists a_0 such that for $k > k_0$, $[B_k, A] > a_0$. Otherwise a subsequence $\{B'_k\}$ of $\{B_k\}$ would exist with limit A. In this case, for every n, $k_n > k_0$ exists such that for $k \geqq k_n$, $[B'_k, A] \leqq r_{n+1}$. Select n_0 so that

$$\sum_{n_0}^{\infty} \frac{1}{2^n} < \frac{1}{2} e_0,$$

and $k_1 > k_{n_0}$ such that $[A_{k_1}, A] < r_{n_0+1}$. Then

$$[B'_{k_1}, A_{k_1}] < f(r_{n_0+1}) < r_{n_0},$$

from which it follows that

$$F_n(A_{k_1}; B'_{k_1}) = 0, \qquad F_n(A; B_{k_0}) = 0$$

for every $n \leqq n_0$, while

$$|F(A_{k_1}; B'_{k_1}) - F(A; B'_{k_1})| < \sum_{n_0}^{\infty} \frac{1}{2^n} < e_0,$$

a result contrary to the assumed inequality (1) above.

The number a_0 being given, choose n_0 so that $s_{n_0} < a_0$ and $f(s_{n_0}) < a_0$. There is an integer $k'_0 \geqq k_0$ such that if both k_1, k_2 exceed k'_0 then

$$[A_{k_1}, B_{k_2}] > s_{n_0}.$$

This is an immediate consequence of the existence of a_0. Then for every $k > k'_0$, $n > n_0$,

$$F_n(A_k; B_k) = F_n(A; B_k) = 1.$$

From this and inequality (1),

$$\left| \sum_{1}^{n_0} \frac{1}{2^n} \{F_n(A_k; B_k) - F_n(A; B_k)\} \right| > e_0.$$

Hence for every k there is $n_k \leqq n_0$ such that

$$|F_{n_k}(A_k; B_k) - F_{n_k}(A; B_k)| > \frac{2^{n_k}}{n_0} e_0.$$

The index k is unlimited, while $n_k \leqq n_0$. A single number $\bar{n}$ must correspond to an infinity of values k_j $(j = 1, 2, 3, \cdots)$ of k. Choose m so that

$$\frac{1}{2^{m-1}} < e_0 \frac{2^{\bar{n}}}{n_0}.$$

There is a number a_m such that if $[A_1, A_2] < a_m$, then A_1, A_2 lie in adjacent classes of stage m of the Hahn development of (V) relative to A_0, $s_{\bar{n}}$, $r_{\bar{n}}$, whatever element of (V) A_0 may be. If $k_j > k'_0$ is sufficiently large,

$$[A_{k_j}, A] < a_m.$$

Therefore

$$|F_{\bar{n}}(A_{k_j}; B_{k_j}) - F_{\bar{n}}(A; B_{k_j})| < \frac{1}{2^{m-1}} < e_0 \frac{2^{\bar{n}}}{n_0}.$$

This contradicts the preceding inequalities, which hold for every $k > k'_0$. The proof of equivalence of écart and voisinage is complete.

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