THE CONVERSE OF THE THEOREM CONCERNING THE DIVISION OF A PLANE BY AN OPEN CURVE*

BY

J. R. KLINE

In his paper, *"On the foundations of plane analysis situs,"*† R. L. Moore proved that if l is an open curve‡ and S is the set of all points, then $S - l = S_1 + S_2$, where S_1 and S_2 are connected point sets such that every arc from a point of S_1 to a point of S_2 contains at least one point of l.§ Clearly the sets S_1 and S_2 are non-compact.|| Professor Moore's theorem is proved on the basis of his set of axioms Σ_3. *Thus the theorem is true in certain spaces which are neither metrical, descriptive, nor separable.*

This theorem for open curves is analogous to the theorem of Jordan,¶ that a simple closed curve lying wholly within a plane decomposes the plane into an inside and an outside region. The converse of this theorem for simple closed curves was first formulated by Schoenflies,** who makes use of metrical properties in his proof. A different proof has been given by Lennes,†† who uses straight lines. R. L. Moore has pointed out that, on the basis of Σ_3, an argument similar in large part to that of Lennes can be carried through with the use of arcs and closed curves.

The object of the present paper is to show that the converse of the open curve theorem holds in spaces satisfying Σ_3. The statement of the converse theorem is as follows:

* Presented to the Society, October 28, 1916.

† These Transactions, vol. 17 (1916), pp. 132–64.

‡ An *open curve* is defined by Moore as a closed connected set of points M such that if P is a point of M, then $M - P$ is the sum of two mutually exclusive connected sets of points, neither of which contains a limit point of the other. See R. L. Moore, loc. cit., p. 159.

§ Loc. cit., pp. 160–162.

|| Fréchet calls a set of points M *compact* if every infinite subset of M has at least one limit point. Cf. M. Fréchet, *Sur quelques points du calcul fonctionnel*, Rendiconti del Circolo Matematico di Palermo, vol. 22 (1906), § 9. A set of points which does not possess this property is said to be *non-compact.*

¶ C. Jordan, *Cours d'Analyse*, 2d ed., Paris, 1893, p. 92.

** A. Schoenflies, *Ueber einen grundlegenden Satz der Analysis Situs*, Nachrichten der Göttinger Gesellschaft der Wissenschaften (1902), p. 185.

†† N. J. Lennes, *Curves in non-metrical analysis situs with an application in the calculus of variations*, American Journal of Mathematics, vol. 33 (1911), § 5.

Suppose K is a closed set of points and that $S - K = S_1 + S_2$, where S_1 and S_2 are non-compact point sets such that (1) *every two points of S_i $(i = 1, 2)$ can be joined by an arc lying entirely in S_i;* (2) *every arc joining a point of S_1 to a point of S_2 contains a point of K;* (3) *if O is a point of K and P is a point not belonging to K, then P can be joined to O by an arc having no point except O in common with K. Every point set K that satisfies these conditions is an open curve.*

In order to prove our theorem, we shall first establish the truth of several lemmas.

LEMMA A. *Suppose J is a simple closed curve* such that* (1) *A and B, two distinct points of K, both lie on J and* (2) *$J - A - B = \overline{AF_1B}$† + $\overline{AF_2B}$ where $\overline{AF_1B}$ is a subset of S_1 while $\overline{AF_2B}$ is a subset of S_2. The interior I of J must contain at least one point of K.*

Proof. Suppose Lemma A is false. Then I contains only points of $S_1 + S_2$. Let $\bar{P}$ be any point of I. Join $\bar{P}$ to F_1 by an arc lying except for F_1 entirely in I. Join $\bar{P}$ to F_2 in the same manner. The point set, $\bar{P}F_1 + \bar{P}F_2$, contains as a subset an arc F_1XF_2, which contains no points of K. But this is contrary to our assumption concerning arcs from a point in S_1 to a point in S_2.

LEMMA B. *Under the same hypothesis as in Lemma A, if $[X]$ denotes the set of all points of K, which are in I, then $[X] + A + B$ is a simple continuous arc from A to B.*

Lemma B can be proved by methods similar to those of Lennes.

LEMMA C. *No subset of K is a simple closed curve.*

Proof. Suppose some subset J of K is a simple closed curve. The point set S_1 cannot lie entirely in I, the interior of J. For if every point of S_1 is in I, then every infinite subset‡ of S_1 must have at least one limit point. This is contrary to the supposition that S_1 is non-compact. In like manner, S_2 is not entirely in I.

No point of K belongs to I. For suppose a point F of K belongs to I. Let G be a point of S_1 not in I. Every arc from F to G must contain a

* If A and B are distinct points, a *simple continuous arc* from A to B is defined by Lennes as a bounded, closed, connected set of points containing A and B but containing no connected proper subset that contains both A and B. See N. J. Lennes, loc. cit., p. 308. In the present paper "arc" and "simple continuous arc" will be used synonymously. A *simple closed curve* is a set of points composed of two arcs AXB and AYB that have no common point except A and B.

† If AF_1B is a simple continuous arc, $\overline{AF_1B}$ denotes the point set $AF_1B - A - B$.

‡ See F. Hausdorff, *Grundzüge der Mengenlehre*, Leipzig, 1914. See also E. W. Chittenden, *The Converse of the Heine-Borel Theorem in a Riesz Domain*, Bulletin of the American Mathematical Society, vol. 21 (1915), pp. 179–183, and vol. 20 (1914), p. 461. For a proof that in the presence of certain linear order axioms, the Heine-Borel Theorem is equivalent to the Dedekind-cut Postulate, see O. Veblen, *The Heine-Borel Theorem*, Bulletin of the American Mathematical Society, vol. 10 (1903–4), pp. 436–39.

point of K, different from F, namely a point of J. But this is contrary to hypothesis. Hence I is a subset of $S_1 + S_2$.

Suppose I contains a point P of S_1 and a point Q of S_2. Then P and Q can be joined by an arc lying entirely in I and therefore containing no point of K. But this is contrary to hypothesis.

Suppose I contains a point L of S_1. Let G denote any point of S_1 not in I. Since J consists entirely of points of K, G is not on J. Every arc from L to G must contain a point of K, namely, a point of J. But this is contrary to hypothesis. Hence I contains no points of S_1.

In like manner I contains no point of S_2.

Thus the supposition that J, a subset of K, is a simple closed curve leads to a contradiction.

LEMMA D. *Suppose J_1 and J_2 are simple closed curves such that* (1) *A and B, two distinct points of K, lie on both J_1 and J_2 and* (2) $J_i - A - B = \overline{AF_1^i B} + \overline{AF_2^i B}$ $(i = 1, 2)$ *where $\overline{AF_1^i B}$ is a subset of S_1 while $\overline{AF_2^i B}$ is a subset of S_2. Under these conditions the subset of K within J_1 is the same as the subset of K within J_2.*

Proof. Let K_1 denote the set of all points of K within J_1, while K_2 denotes the set of all points of K within J_2. By Lemma B, $K_1 + A + B$ is a simple continuous arc from A to B, as is also the point set, $K_2 + A + B$.

Suppose Lemma D is false. Four cases may arise:

Case I. K_1 is a proper subset of K_2. Then $K_2 + A + B$ contains a proper connected subset $K_1 + A + B$, which contains both A and B. But this is contrary to the definition of a simple continuous arc from A to B.

Case II. K_2 is a proper subset of K_1. Impossible as in Case I.

Case III. K_2 consists of two non-vacuous point sets, G_1 and G_2, where G_1 denotes those points of K_2 which are points of K_1, while G_2 denotes those points of K_2 which are not points of K_1.

By hypothesis, K_2 contains no point of J_1. Hence all points of K_2 must lie within J_1. For suppose a point P of K_2 is without J_1. Then K_2 can be divided into two mutually exclusive sets, M_1 and M_2, where M_1 is the set of all points of K_2 within J_1 while M_2 is the set of all points of K_2 without J_1. Manifestly neither of these sets can contain a limit point of the other. Hence the supposition that K_2 contains points without J_1 leads to a contradiction.

As all points of K_2 lie within J_1 and are points of K, then, by definition of K_1, either (1) K_2 is the same as K_1, which is contrary to supposition or (2) K_2 is a proper subset of K_1, which is contrary to Case II.

Case IV. K_1 and K_2 have no common point. The point set, $K_1 + K_2 + A + B$, is a simple closed curve composed entirely of points of K. But this is contrary to Lemma C.

DEFINITION 1. *The points A, B and X* $[A \neq B]$ *of K are said to be in* *the order AXB, if, and only if, X is within some simple closed curve J, such that* (1) *A and B are on J and* (2) $J - A - B = \overline{AF_1 B} + \overline{AF_2 B}$, *where $\overline{AF_1 B}$ is a subset of S_1, while $\overline{AF_2 B}$ is a subset of S_2.*

That the set of all points $[X]$ such that AXB is the same for every such closed curve J follows at once from Lemma D.

LEMMA E. *If A and B $[A \neq B]$ are points of K, then there exists a point X of K such that AXB.*

LEMMA F. *If ABC, then CBA.*

LEMMA G. *If ABC, then $A \neq B$ and $B \neq C$.*

LEMMA H. *If ABC, then not BAC.*

Proof. Construct a simple closed curve J such that (1) A and C are on J and (2) $J - A - C = \overline{AF_1 C} + \overline{AF_2 C}$, where $\overline{AF_1 C}$ is a subset of S_1 while $\overline{AF_2 C}$ is a subset of S_2. By Definition 1 and Lemma D, B is within J. Join B to F_1 by an arc lying except for B entirely in S_1. The arc BF_1 has no point in common with the arc $AF_2 C$, which, except for A and C, contains only points of S_2. Let G_1 denote the first point that the arc BF_1 has in common with the arc $AF_1 C$. Join B to F_2 by an arc lying except for B entirely in S_2 and let G_2 denote the first point of the arc BF_2 which is on the arc $AF_2 C$. The point set, $G_1 B + G_2 B$, is a simple continuous arc, lying except for its end points, G_1 and G_2, entirely within J and containing only the point B of K. The point set, $\overline{G_1 AG_2}$, is without the closed curve $BG_1 CG_2 B$.*

Hence, by Lemma D and Definition 1, not BAC.

LEMMA I. *There is but one arc of K from A to C.*

Proof. If there were two such arcs, their sum would contain as a subset at least one simple closed curve. But this is contrary to Lemma C.

LEMMA J. *A necessary and sufficient condition that three distinct points, A, B, and C of K should be in the order ABC, is that B should be on the K-arc from A to C.*

LEMMA K. *If A, B, and C $[A \neq B, B \neq C, C \neq A]$ are points of K, then either ACB, CBA, or BAC.*

Proof. Suppose ACB is false. Let J be a simple closed curve such that (1) A and B are on J, and (2) $J - A - B = \overline{AF_1 B} + \overline{AF_2 B}$, where $\overline{AF_1 B}$ is a subset of S_1 while $\overline{AF_2 B}$ is a subset of S_2. By Definition 1 and Lemma D, the point C is without J. Join C to F_1 by an arc lying except for C entirely in S_1. Join C to F_2 by an arc lying except for C entirely in S_2. Let G_1 denote the first point which the arc CF_1 has in common with the arc $AF_1 B$. Let G_2 denote the first point which the arc CF_2 has in common with the arc $AF_2 B$. The point set, $CG_1 + CG_2$, is a simple continuous arc from G_1 to G_2, lying except for its end points entirely without J and contain-

* See R. L. Moore, loc. cit., Theorem 24, p. 141.

ing only the point C of K. Hence either (1) B is within CG_1AG_2C or (2) A is within CG_1BG_2C.* In Case I, CBA while in Case II, CAB.

LEMMA L. *If A, B, and C ($A \neq B$, $B \neq C$, $C \neq A$) are points of K, in the order ABC, then the K-arc AC = the K-arc AB + the K-arc BC.*

Proof. Consider the figure described in the proof of Lemma H. The interior of $J = \overline{G_1BG_2}$ + the interior of AG_1BG_2A + the interior of BG_1CG_2B.† The point set $\overline{G_1BG_2}$ contains only the point B of K. The points A and B, together with those points of K that lie within AG_1BG_2A, form the K-arc AB, while the points B and C, together with those points of K that lie within BG_1CG_2B, form the K-arc BC. But the points A and C together with the points of K within J form the K-arc AC.

Hence the K-arc AC = the K-arc AB + the K-arc BC.

LEMMA M. *If ABC and BDC, then ADC.*

Proof. By Lemma L, the K-arc AC = K-arc AB + K-arc BC. By Lemma J, D is on the K-arc BC. Hence D is on the K-arc AC. Thus, by Lemma J, ADC.

LEMMA N. *If ABC and ADC and $B \neq D$, then ABD or DBC.*

Proof. By Lemma L, the K-arc AC = K-arc AD + K-arc DC. By Lemma J, B is on the K-arc AC. Hence, as $B \neq D$, then either B is on the K-arc AD or B is on the K-arc DC. Hence, by Lemma J, either ABD or DBC.

It can easily be shown that the following Lemmas O–S are logical consequences of Lemmas F, H, G, K, M, and N.

LEMMA O. *If ABC and ADC and $B \neq D$, then ABD or ADB.*

LEMMA P. *If ABC and BCD, then ABD.*

LEMMA Q. *If ABC and BDC, then ABD.*

LEMMA R. *If ABC and DBC and $A \neq D$, then ADB or DAB.*

LEMMA S. *If ABC and DBC, and $A \neq D$, then ADC or DAC.*

LEMMA T. *If P is a point of K, which is contained in some segment of K and M is a subset of K, then P is a limit point of M, if, and only if, every segment of K, containing P contains at least one point of M distinct from P.*

Proof. Suppose P is a limit point of M. Let $\overline{APB}$ denote any segment of K that contains P. Construct a simple closed curve J such that (1) A and B are on J and (2) $J - A - B = \overline{AF_1B} + \overline{AF_2B}$, where $\overline{AF_1B}$ is a subset of S_1 while $\overline{AF_2B}$ is a subset of S_2. All points of the segment $\overline{APB}$ lie within J. As P is a limit point of M, the interior of J must contain at least one point $\overline{P}$ of M, different from P. All points of M are points of K. Hence, as $\overline{APB}$ contains all points of K in the interior of J, $\overline{APB}$ contains a point of M distinct from P.

* For a proof of this statement see R. L. Moore, loc. cit., Theorem 27, pp. 144–5.

† See R. L. Moore, loc. cit., Theorem 25, pp. 141–2.

Suppose every segment containing P contains at least one point of M distinct from P. Let APB be any segment of K containing P. Let us assume that P is not a limit point of M. Then there exists a region R containing P and containing neither A, B, nor any point of M other than P. Let X denote the first point of the K-arc PA on R', the boundary of R. Let Y denote the first point of the K-arc PB on R'. The K-segment XPY is within R. Hence the supposition that R contains no point of M different from P leads to a contradiction.

LEMMA U. *If A and B $(A \neq B)$ are points of K, then B is not a limit point of the set of all points $[\bar{C}]$ such that $BA\bar{C}$.*

Proof. Suppose B is a limit point of the set of all points $[\bar{C}]$ such that $BA\bar{C}$. Put about B a region R. Infinitely many points of $[\bar{C}]$ must lie within R. Of these points we can select a sequence $\bar{\bar{C}}_1$, $\bar{\bar{C}}_2$, $\cdots$, having B as their only limit point and lying in R.* Call $\bar{\bar{C}}_1$, C_1. The K-arc AC_1 cannot contain all the points $\bar{\bar{C}}_2$, $\bar{\bar{C}}_3$, $\bar{\bar{C}}_4$, $\cdots$. For suppose it did. Then as B is a limit point of $\bar{\bar{C}}_2$, $\bar{\bar{C}}_3$, $\bar{\bar{C}}_4$, $\cdots$, B is a limit point of the point set AC_1. But the set AC_1 is closed. Hence B must belong to the point set AC_1. By Lemma G and Definition 1, $A \neq B \neq C_1$. Hence ABC_1. But by Lemma H, if BAC_1, then not ABC_1.

For no values of i and j $(i \neq j)$ can B lie on the K-arc $\bar{\bar{C}}_i\,\bar{\bar{C}}_j$. For suppose for $i = k$ and $j = n$, B is on the K-arc $\bar{\bar{C}}_k\,\bar{\bar{C}}_n$. As $\bar{\bar{C}}_k \neq B \neq \bar{\bar{C}}_n$, then by Lemma J, $\bar{\bar{C}}_k B \bar{\bar{C}}_n$. By Lemma Q, $\bar{\bar{C}}_k B \bar{\bar{C}}_n$ and $BA\bar{\bar{C}}_n$ imply that $\bar{\bar{C}}_k BA$ and hence, by Lemma F, $AB\bar{\bar{C}}_k$. But, by Lemma H, $BA\bar{\bar{C}}_k$ implies not $AB\bar{\bar{C}}_k$.

Let C_2 denote that point of the set $\bar{\bar{C}}_2$, $\bar{\bar{C}}_3$, $\bar{\bar{C}}_4$, $\cdots$ of least subscript which does not lie on the K-arc AC_1. Let C_3 denote that point of least subscript of the set $\bar{\bar{C}}_3$, $\bar{\bar{C}}_4$, $\bar{\bar{C}}_5$, $\cdots$, which does not lie on the K-arc AC_2. Continue this process and obtain an infinite sequence of points C_1, C_2, C_3, $\cdots$, and an infinite sequence of K-arcs AC_1, C_1C_2, $\cdots$. Let M denote the set of all points of the arcs of this sequence. The point B is the only limit point of the set C_1, C_2, C_3, $\cdots$. In view of the above lemmas concerning order on K, it is clear that the points B, A, C_1, C_2, $\cdots$, are in the order $BAC_1C_2 \cdots C_n C_{n+1} \cdots$.

Either the point B is the only limit point of M, not contained in M, or there is at least one other limit point of M, not in M.

Case I. Suppose some point E, not in M, and different from B, is a limit point of M.

As M is a subset of K and as K is closed, E is a point of K. Hence, as $A \neq B$, $B \neq E$, $E \neq A$, it follows, by Lemma K, that either EAB, ABE, or BEA.

* See R. L. Moore, loc. cit., Theorems 8 and 9, p. 134.

Since the K-segment AB contains no points of M, it follows, by Lemma T, that the order BEA does not hold.

Suppose EAB. Then since A is a limit point of M and M is connected, it easily follows, with the help of Lemma T, that M is a subset of the K-arc EB. No point P of M is in the order APB. Hence M is a subset of the K-arc EA. The K-arc EA is closed. Hence B, which is not on this arc, cannot be a limit point of M. Thus the supposition that E, A, and B are in the order EAB leads to a contradiction.

Suppose ABE. Then, since every point P of M is in the order BAP, it follows that no point of M is on the K-segment EA. Hence, by Lemma T, B is not a limit point of M. Hence the supposition that ABE leads to a contradiction.

Case II. Suppose B is the only limit point of M, not in M. In Case II it can easily be shown that the point set $M + B$ is a simple continuous arc from A to B having no points except A and B in common with the K-arc AXB. But this is contrary to Lemma C, as $M + AXB$ is a simple closed curve which is a subset of K.

Hence the supposition that B is a limit point of $[\bar{C}]$ leads to a contradiction.

Lemma V. *If A and B $[A \neq B]$ are points of K, then there exists a point E of K such that ABE.*

Proof. Construct a simple closed curve J such that (1) A and B are on J and (2) $J - A - B = \underline{AF_1 B} + \underline{AF_2 B}$, where $\underline{AF_1 B}$ is a subset of S_1 while $\underline{AF_2 B}$ is a subset of S_2. By Lemma U, B is not a limit point of the set of all points $[\bar{C}]$ such that $BA\bar{C}$. Hence we can put about B a region R, containing no point $\bar{C}$ such that $BA\bar{C}$ and containing no point of the arc $F_1 AF_2$. There exists a segment $\underline{MXN}$ such that (1) M and N are on J, (2) the segment $\underline{MBN}$ of the curve J is within R and (3) $\underline{MXN}$ and the interior of $MXNBM$ are in R and without J.* The points M and N cannot both be points of S_1. For if M and N are both points of S_1, then that arc of J from M to N which contains B, will also contain A. But this is contrary to our choice of R. In like manner M and N are not both points of S_2.

As the arc MXN joins a point of S_1 to a point of S_2 it must contain at least one point E of K. As E is without J, the order AEB does not hold. As E is within R, the order EAB does not hold.

Hence, by Lemma K, ABE.

We are now in a position to prove our theorem. That K is connected follows at once from Lemmas A and B.

Let P be any point of K. That $K - P = K_1 + K_2$, where K_1 and K_2 are two mutually exclusive connected point sets neither of which contains a limit point of the other may be proved as follows. Take A any point of K,

* See R. L. Moore, loc. cit., Theorem 45, p. 157.

different from P. Let K_1 denote the set of all points $[X]$ of K such that $X = A$, PXA or PAX, while K_2 denotes the set of all points $[Y]$ such that APY. That K_2 is not vacuous is a consequence of Lemma V. That K_1 and K_2 are connected follows with the aid of Lemma B, while it may be proved, with the aid of Lemma U, that neither of these sets contains a limit point of the other.

UNIVERSITY OF PENNSYLVANIA