SETS OF INDEPENDENT POSTULATES FOR BETWEENNESS*

BY

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Introduction

The "universe of discourse" of the present paper is the class of all well-defined systems (K, R) where K is any class of elements $A, B, C, \cdots$, and R is any triadic relation. The notation $\mathrm{R}[ABC]$, or simply ABC, indicates that three given elements A, B, C, in the order stated, satisfy the relation R.

Examples of such systems (K, R) are the following, of which example (*a*) is the most important:

(*a*) K is the class of points on a line; AXB means that the point X lies between the points A and B.

(*b*) K is the class of natural numbers; AXB means that the number X is the product of the numbers A and B.

(*c*) K is the class of human beings; AXB means that X is a descendant of A and an ancestor of B.

(*d*) K is the class of points on the circumference of a circle; AXB means that the arc A-X-B is less than 180°.

(*e*) K is a class comprising four elements, namely, the numbers $2, 6, -6$, and 648; AXB means $X^4 = A \times B$.

It is obvious that these systems, and others like them, will possess a great variety of properties expressible in terms of the fundamental variables K and R. The object of this paper is to state clearly the characteristic properties of the type of system represented by example (*a*) above, by which this type of system is distinguished from all other possible systems (K, R).

In Section 1, we give a basic list of twelve postulates, due essentially to Pasch,† from which various sets of *independent* postulates will later be selected.

* Presented to the Society, September 5, 1916. The parts of the paper which do not involve the postulates here numbered 5 and 8 were presented by Professor Huntington at the meetings of December 31, 1912, and April 26, 1913. The necessity of adding postulates 5 and 8 was kindly pointed out by Professor R. L. Moore, and all the theorems and examples which involve these two postulates are due to Dr. Kline.

† M. Pasch, *Vorlesungen über neuere Geometrie*, Leipzig, 1882; G. Peano, *Sui fondamenti della geometria*, Rivista di Matematica, vol. 4 (1894), pp. 51–90; F. Schur, *Grundlagen der Geometrie*, Leipzig, 1909. Other sets of postulates for betweenness have been given

These postulates are all "general laws" as distinguished from "existence postulates," and include, in fact, all the possible general laws of linear order concerning not more than four elements.

In Sections 2 and 3 we give an exhaustive discussion of all the possible ways in which any one of these basic postulates can be deduced from any others of the list, and in Section 4, we give an exhaustive list of all the distinct sets of independent postulates (eleven in number) which can be selected from the basic list.*

Any one of these sets of independent postulates may be used, as in Section 5, to define the type of system (K, R) which we are considering—that is, to define the relation of betweenness.

The existence postulates which might be imposed, in addition to the general laws, would serve to distinguish the various sub-types which are included within the general type of system (K, R) here considered. These existence postulates, such as the postulates of discreteness, density, continuity, etc., are already well known, and will not be discussed further in the present paper.†

1. Basic list of twelve postulates

In this section we give the basic list of twelve postulates from which various sets of independent postulates will later be selected.

The first four postulates, A–D, concern three elements.

POSTULATE A. $AXB . \supset . BXA .$

That is, if AXB is true, then BXA is true. In other words, in the notation ABC, an interchange of the terminal elements is always allowable.

POSTULATE B.

$A \neq B . B \neq C . C \neq A :\supset: BAC \smile CAB \smile ABC \smile CBA \smile ACB \smile BCA .$

That is, if A, B, C, are distinct, then *at least one* of the three elements will occupy the middle position in a true triad.

POSTULATE C. $A \neq X . X \neq Y . Y \neq A :\supset: AXY . AYX . = . 0 .$

by D. Hilbert; *Grundlagen der Geometrie*, 1899, third edition 1909; and by O. Veblen: *A system of axioms for geometry*, these Transactions, vol. 5 (1904), pp. 343–384, or *The foundations of geometry*, in the volume called *Monographs on Topics of Modern Mathematics*, edited by J. W. A. Young, 1911, pp. 1–51.

* The postulates of each of these sets are independent of each other in the ordinary sense of the term "independence"; that is, no postulate of any one set can be deduced from the remaining postulates of that set. It is probable that the postulates of each set are also "completely independent" in the sense suggested by E. H. Moore in his *Introduction to a Form of General Analysis* (*New Haven Colloquium*, 1906, published by the Yale University Press, New Haven, 1910, p. 82); but no attempt to discuss the "complete existential theory" of the postulates, in the sense there defined, has here been made.

† See, for example, E. V. Huntington, *The continuum as a type of order*, reprinted from the Annals of Mathematics, 1905 (Publication Office of Harvard University); second edition, Harvard University Press, 1917.

That is, if A, X, Y, are distinct, we cannot have AXY and AYX both true at the same time.

From Postulates A and C it follows that if A, B, C are distinct elements, then *not more than one* of the three elements can occupy the middle position in a true triad.

From Postulates A, B, and C, together, it follows that if A, B, and C are distinct elements, then one and only one of the triads ABC, BCA, CAB will be true.

Postulate D. $ABC :\supset: A \neq B \,.\, B \neq C \,.\, C \neq A$.

That is, if ABC is true, then the elements A, B, and C, are distinct.

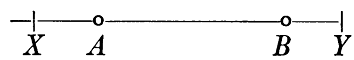
The remaining eight postulates are concerned with four distinct elements.

Postulates 1–8. If

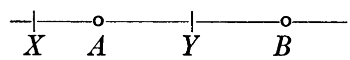
$$A \neq B \,.\, A \neq X \,.\, A \neq Y \,.\, B \neq X \,.\, B \neq Y \,.\, X \neq Y,$$

then

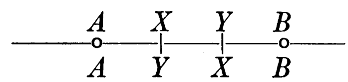
1. $XAB \,.\, ABY \,.\supset.\, XAY$;



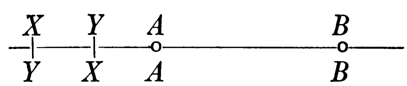
2. $XAB \,.\, AYB \,.\supset.\, XAY$;
3. $XAB \,.\, AYB \,.\supset.\, XYB$;



4. $AXB \,.\, AYB \,.\supset.\, AXY \smile AYX$;
5. $AXB \,.\, AYB \,.\supset.\, AXY \smile YXB$;



6. $XAB \,.\, YAB \,.\supset.\, XYB \smile YXB$;
7. $XAB \,.\, YAB \,.\supset.\, XYA \smile YXA$;
8. $XAB \,.\, YAB \,.\supset.\, XYA \smile YXB$.



The eight Postulates 1–8 (together with the analogous postulates obtained from these by the aid of Postulate A alone) include *all the possible "general laws" of betweenness concerning four distinct elements.* For, if we think of A and B as two given points on a line, the hypotheses of these postulates state all the possible relations in which two other distinct points X and Y of the line can stand in regard to A and B. (See, however, the Appendix.)

We shall see later that no further general laws—that is, no general laws concerning more than four distinct elements—need be assumed as fundamental. (Existence postulates, which play a very different rôle from the general laws, are not here considered.)

2. Theorems on deducibility

In this section we take up all the cases in which the following question is to be answered in the affirmative:

Given, any subset S of the twelve postulates of our basic list, and any postulate P of the list, not belonging to S; is P deducible from S?

The answers are comprised in the following 71 theorems. In the proofs of these theorems, all the steps are given explicitly, except those depending only on Postulate A. Moreover, in case any postulate (except Postulate A) is used more than once in a proof, the frequency of its use is indicated by an exponent; this latter information, however, is added merely as a matter of possible interest to the reader, and the omission of the exponents would not affect the conclusions of the paper in any way.*

A summary of the theorems will be found at the end of § 2.

Proofs of postulate 1

Theorem 1a. *Proof of* 1 *from* A, B, C^3, 2, 4.

To prove: $XAB \,.\, ABY \,.\supset.\, XAY$. By B, $XAY \smile AYX \smile AXY$. Suppose AXY. Then by 4, $ABY \,.\, AXY \,.\supset.\, ABX \smile AXB$, contrary to XAB, by C. Suppose AYX. Then by 2, $BAX \,.\, AYX \,.\supset.\, BAY$, contrary to ABY, by C. Therefore XAY.

Theorem 1b. *Proof of* 1 *from* A, B, C^2, 3, 4.

To prove: $XAB \,.\, ABY \,.\supset.\, XAY$. By C, AXB and ABX are false, since XAB is true. By B, $XYA \smile AXY \smile XAY$.

Case 1. Suppose XYA. Then by 3, $XYA \,.\, YBA \,.\supset.\, XBA$, which is false.

Case 2. Suppose AXY. Then by 4, $AXY \,.\, ABY \,.\supset.\, AXB \smile ABX$, which are both false. Therefore XAY.

Theorem 1c. *Proof of* 1 *from* A, B, C^3, 2^2, 5.

To prove: $XAB \,.\, ABY \,.\supset.\, XAY$. By B, $XAY \smile AYX \smile AXY$.
Suppose AXY. Then by 5, $AXY \,.\, ABY \,.\supset.\, AXB \smile BXY$. But AXB is contrary to XAB, by C; while if BXY, then by 2, $ABY \,.\, BXY \,.\supset.\, ABX$, contrary to XAB, by C.
Suppose AYX. Then by 2, $BAX \,.\, AYX \,.\supset.\, BAY$, contrary to ABY, by C. Therefore XAY.

Theorem 1d. *Proof of* 1 *from* A, B, C, 3^2, 5.

To prove: $XAB \,.\, ABY \,.\supset.\, XAY$. By C, XBA is false, since XAB is true. By B, $XYA \smile AXY \smile XAY$.

Case 1. Suppose XYA. Then by 3, $XYA \,.\, YBA \,.\supset.\, XBA$, which is false.

* We are indebted to Mr. R. M. Foster, of Harvard University, for reductions in the "frequency exponents" (chiefly in regard to Postulate C) in the following theorems: 1b, 1d; 2c, 2g; 3a, 3b; 4b; 5f; 6j; 7b, 7c, 7j; 8b, 8c, 8d, 8f, 8j. A notion similar to that of "frequency exponents" was introduced by H. Brandes in his Halle Dissertation, 1908, *Über die axiomatische Einfachheit, mit besonderer Berücksichtigung der auf Addition beruhenden Zerlegungsbeweise des Pythagoräischen Lehrsatzes*. Compare F. Bernstein, *Ueber die axiomatische Einfachheit von Beweisen*, Atti del IV Congresso Internazionale dei Matematici, Roma, 1908, vol. 3 (1909), pp. 391–392, and also E. Lemoine's *Géométrographie* of 1902.

Case 2. If AXY, then by 5, $ABY \,.\, AXY \,.\supset.\, ABX \smile XBY$; but ABX is false, and if XBY, then by 3, $YBX \,.\, BAX \,.\supset.\, YAX$. Therefore XAY.

Proofs of postulate 2

Theorem 2a. *Proof of* 2 *from* A, B, C^3, 1^3, 7.

To prove: $XAB \,.\, AYB \,.\supset.\, XAY$. By B, $XAY \smile AYX \smile AXY$. Suppose AYX. Then by 7, $BYA \,.\, XYA \,.\supset.\, BXY \smile XBY$. But if BXY, then by 1, $BXY \,.\, XYA \,.\supset.\, BXA$, contrary to XAB, by C; and if XBY, then by 1, $XBY \,.\, BYA \,.\supset.\, XBA$, contrary to XAB, by C. Suppose AXY. Then by 1, $BAX \,.\, AXY \,.\supset.\, BAY$, contrary to AYB, by C. Therefore XAY.

Theorem 2b. *Proof of* 2 *from* A, B, C^3, 1, 6.

To prove: $XAB \,.\, AYB \,.\supset.\, XAY$. By B, $XAY \smile AYX \smile YXA$. Suppose YXA. Then by 1, $BAX \,.\, AXY \,.\supset.\, BAY$, contrary to AYB, by C. Suppose AYX. Then by 6, $XYA \,.\, BYA \,.\supset.\, XBA \smile BXA$, contrary to XAB, by C. Therefore XAY.

Theorem 2c. *Proof of* 2 *from* A, B, C^2, 3, 6.

To prove: $XAB \,.\, AYB \,.\supset.\, XAY$. By C, BXA and XBA are false, since XAB is true. By B, $YXA \smile XYA \smile XAY$.

Case 1. Suppose YXA. Then by 3, $BYA \,.\, YXA \,.\supset.\, BXA$, which is false.

Case 2. Suppose XYA. Then by 6, $BYA \,.\, XYA \,.\supset.\, BXA \smile XBA$, which are both false. Therefore XAY.

Theorem 2d. *Proof of* 2 *from* A, C, 3^2, 7.

To prove: $XAB \,.\, AYB \,.\supset.\, XAY$. By 3, $XAB \,.\, AYB \,.\supset.\, XYB$. Hence by 7, $XYB \,.\, AYB \,.\supset.\, XAY \smile AXY$. But if AXY, then by 3, $BYA \,.\, YXA \,.\supset.\, BXA$, contrary to XAB, by C. Therefore XAY.

Theorem 2e. *Proof of* 2 *from* A, C^2, 3, 4, 6.

To prove: $XAB \,.\, AYB \,.\supset.\, XAY$. By 3, $XAB \,.\, AYB \,.\supset.\, XYB$. Hence by 4, $XYB \,.\, XAB \,.\supset.\, XYA \smile XAY$. But if XYA, then by 6, $XYA \,.\, BYA \,.\supset.\, XBA \smile BXA$, contrary to XAB, by C. Therefore XAY.

Theorem 2f. *Proof of* 2 *from* A, B, C^3, 1^2, 8.

To prove: $XAB \,.\, AYB \,.\supset.\, XAY$. By B, $XAY \smile AYX \smile YXA$. Suppose YXA. Then by 1, $BAX \,.\, AXY \,.\supset.\, BAY$, contrary to AYB, by C. Suppose AYX. Then by 8, $XYA \,.\, BYA \,.\supset.\, XBY \smile BXA$. But BXA is contrary to XAB, by C; while if XBY, then by 1, $XBY \,.\, BYA \,.\supset.\, XBA$, contrary to XAB, by C. Therefore XAY.

Theorem 2g. *Proof of* 2 *from* A, B^2, C^3, 1^3, 5.

To prove: $XAB \,.\, AYB \,.\supset.\, XAY$. By C, BAY and XBA and BXA

are false, since AYB and XAB are true. By B, $AXY \smile XYA \smile XAY$; and by B, $XBY \smile BXY \smile BYX$.

Case 1. Suppose AXY. Then by 1, $BAX . AXY . \supset . BAY$, which is false.

Case 2. Suppose XBY. Then by 1, $XBY . BYA . \supset . XBA$, which is false.

Case 3. Suppose XYA and BXY. Then by 1, $BXY . XYA . \supset . BXA$, which is false.

Case 4. Suppose BYX. Then by 5, $BAX . BYX . \supset . BAY \smile YAX$, where BAY is false. Therefore XAY.

THEOREM 2h. *Proof of* 2 *from A, C,* 3, 8.

To prove: $XAB . AYB . \supset . XAY$. By 3, $XAB . AYB . \supset . XYB$. Hence by 8, $XYB . AYB . \supset . XAY \smile AXB$. But AXB is contrary to XAB, by C. Therefore XAY.

THEOREM 2i. *Proof of* 2 *from A, C,* 3, 5.

To prove: $XAB . AYB . \supset . XAY$. By 3, $XAB . AYB . \supset . XYB$. Hence by 5, $XAB . XYB . \supset . XAY \smile YAB$. But YAB is contrary to AYB, by C. Therefore XAY.

PROOFS OF POSTULATE 3

THEOREM 3a. *Proof of* 3 *from A,* B^2, C^3, 1^4.

To prove: $XAB . AYB . \supset . XYB$. By C, XBA and BAY and BXA are false, since XAB and AYB are true. By B, $XBY \smile BXY \smile XYB$; and by B, $AXY \smile XYA \smile YAX$.

Case 1. Suppose XBY. Then by 1, $XBY . BYA . \supset . XBA$, which is false.

Case 2. Suppose AXY. Then by 1, $BAX . AXY . \supset . BAY$, which is false.

Case 3. Suppose BXY and XYA. Then by 1, $BXY . XYA . \supset . BXA$, which is false.

Case 4. Suppose YAX. Then by 1, $BYA . YAX . \supset . BYX$. Therefore XYB.

THEOREM 3b. *Proof of* 3 *from A, B, C,* 2^3.

To prove: $XAB . AYB . \supset . XYB$. By B, $YBX \smile YXB \smile XYB$.

Case 1. Suppose YBX. Then by 2, $YBX . BAX . \supset . YBA$, contrary to AYB, by C.

Case 2. If YXB, then by 2, $YXB . XAB . \supset . YXA$, whence, by 2, $BYA . YXA . \supset . BYX$. Therefore XYB.

THEOREM 3c. *Proof of* 3 *from A, C,* 2^2, 6.

To prove: $XAB . AYB . \supset . XYB$. By 2, $XAB . AYB . \supset . XAY$.

Hence by 6, $BAX \,.\, YAX \,.\!\supset\!.\, BYX \smile YBX$. But if YBX, then by 2, $YBX \,.\, BAX \,.\!\supset\!.\, YBA$, contrary to AYB, by C. Therefore XYB.

THEOREM 3d. *Proof of* 3 *from* A, 1, 2.

To prove: $XAB \,.\, AYB \,.\!\supset\!.\, XYB$. By 2, $XAB \,.\, AYB \,.\!\supset\!.\, XAY$. Hence by 1, $BYA \,.\, YAX \,.\!\supset\!.\, BYX$. Therefore XYB.

THEOREM 3e. *Proof of* 3 *from* A, C, 2, 8.

To prove: $XAB \,.\, AYB \,.\!\supset\!.\, XYB$. By 2, $XAB \,.\, AYB \,.\!\supset\!.\, XAY$. Hence by 8, $YAX \,.\, BAX \,.\!\supset\!.\, YBA \smile BYX$. But YBA is contrary to AYB, by C. Therefore XYB.

PROOFS OF POSTULATE 4

THEOREM 4a. *Proof of* 4 *from* A, B, C, 1.

To prove: $AXB \,.\, AYB \,.\!\supset\!.\, AXY \smile AYX$. By B, $AXY \smile AYX \smile XAY$. Suppose XAY. Then by 1, $XAY \,.\, AYB \,.\!\supset\!.\, XAB$, contrary to AXB, by C. Therefore $AXY \smile AYX$.

THEOREM 4b. *Proof of* 4 *from* A, B, 1, 2.

To prove: $AXB \,.\, AYB \,.\!\supset\!.\, AXY \smile AYX$. By B, $AXY \smile AYX \smile XAY$.
But if XAY, then by 1, $BYA \,.\, YAX \,.\!\supset\!.\, BYX$,
whence, by 2, $AXB \,.\, XYB \,.\!\supset\!.\, AXY$.
Therefore $AXY \smile AYX$.

THEOREM 4c. *Proof of* 4 *from* A, B, 1^2, 7.

To prove: $AXB \,.\, AYB \,.\!\supset\!.\, AXY \smile AYX$. By B, $AXY \smile AYX \smile YAX$.
But if XAY, then by 1, $XAY \,.\, AYB \,.\!\supset\!.\, XAB$;
and by 1, $YAX \,.\, AXB \,.\!\supset\!.\, YAB$;
whence by 7, $XAB \,.\, YAB \,.\!\supset\!.\, XYA \smile YXA$. Therefore $AXY \smile AYX$.

THEOREM 4d. *Proof of* 4 *from* A, C, 5^2.

To prove: $AXB \,.\, AYB \,.\!\supset\!.\, AXY \smile AYX$.
By 5, $AXB \,.\, AYB \,.\!\supset\!.\, AXY \smile YXB$;
and by 5, $AYB \,.\, AXB \,.\!\supset\!.\, AYX \smile XYB$.
Suppose AXY and AYX are both false. Then YXB and XYB, contrary to C.
Therefore $AXY \smile AYX$.

THEOREM 4e. *Proof of* 4 *from* A, 3^2, 5^2, 7^2.

To prove: $AXB \,.\, AYB \,.\!\supset\!.\, AXY \smile AYX$.
By 5, $AXB \,.\, AYB \,.\!\supset\!.\, AXY \smile YXB$,
and by 5, $AYB \,.\, AXB \,.\!\supset\!.\, AYX \smile XYB$.
Suppose AXY and AYX are both false.
Then YXB and XYB, whence by 7, $XYB \,.\, AYB \,.\!\supset\!.\, AXY \smile XAY$.
But if XAY, then by 3, $BXY \,.\, XAY \,.\!\supset\!.\, BAY$,
and by 3, $BYX \,.\, YAX \,.\!\supset\!.\, BAX$;
whence by 7, $XAB \,.\, YAB \,.\!\supset\!.\, XYA \smile YXA$. Therefore $AXY \smile AYX$.

THEOREM 4f. *Proof of* 4 *from* A, 5^2, 7, 8^2.

To prove: $AXB \,.\, AYB \,.\supset.\, AXY \smile AYX$.

By 5, $AXB \,.\, AYB \,.\supset.\, AXY \smile YXB$,

and by 5, $AYB \,.\, AXB \,.\supset.\, AYX \smile XYB$.

Suppose AXY and AYX are both false.

Then YXB and XYB, whence by 8, $AXB \,.\, YXB.\supset.\, AYX \smile YAB$,

and by 8, $AYB \,.\, XYB \,.\supset.\, AXY \smile XAB$.

But if YAB and XAB, then by 7, $XAB \,.\, YAB \,.\supset.\, XYA \smile YXA$.

Therefore $AXY \smile AYX$.

THEOREM 4g. *Proof of* 4 *from* 2, 5.

To prove: $AXB \,.\, AYB \,.\supset.\, AXY \smile AYX$.

By 5, $AXB \,.\, AYB \,.\supset.\, AXY \smile YXB$.

But if YXB, then by 2, $AYB \,.\, YXB \,.\supset.\, AYX$. Therefore $AXY \smile AYX$.

THEOREM 4h. *Proof of* 4 *from* A, 1^2, 5, 7^2.

To prove: $AXB \,.\, AYB \,.\supset.\, AXY \smile AYX$.

By 5, $AXB \,.\, AYB \,.\supset.\, AXY \smile YXB$.

Suppose AXY false.

Then YXB, whence, by 7, $YXB \,.\, AXB \,.\supset.\, YAX \smile AYX$.

But if YAX, then by 1, $YAX \,.\, AXB \,.\supset.\, YAB$,

and by 1, $XAY \,.\, AYB \,.\supset.\, XAB$,

whence by 7, $XAB \,.\, YAB \,.\supset.\, XYA \smile YXA$. Therefore $AXY \smile AYX$.

The following two theorems, 4i and 4j, are the only ones in which Postulate C is used without Postulate A. (It will be noted that there are no cases in which Postulate B is used without Postulate A.)

THEOREM 4i. *Proof of* 4 *from* C^2, 5^2, 7^3, 8^2.

To prove: $AXB \,.\, AYB \,.\supset.\, AXY \smile AYX$.

By 5, $AXB \,.\, AYB \,.\supset.\, AXY \smile YXB$,

and by 5, $AYB \,.\, AXB \,.\supset.\, AYX \smile XYB$.

Suppose AXY and AYX are both false. Then YXB and XYB,

whence by 7, $YXB \,.\, AXB \,.\supset.\, YAX \smile AYX$,

and by 7, $XYB \,.\, AYB \,.\supset.\, XAY \smile AXY$; whence $YAX$and XAY.

Again, by 8, $AXB \,.\, YXB \,.\supset.\, AYX \smile YAB$,

and by 8, $AYB \,.\, XYB \,.\supset.\, AXY \smile XAB$, whence YAB and XAB;

whence, by 7, $YAB \,.\, XAB \,.\supset.\, YXA \smile XYA$, contrary to YAX and XAY, respectively, by C. Therefore $AXY \smile AYX$.

THEOREM 4j. *Proof of* 4 *from* C^2, 1^2, 5^2, 7^3.

To prove: $AXB \,.\, AYB \,.\supset.\, AXY \smile AYX$.

By 5, $AXB \,.\, AYB \,.\supset.\, AXY \smile YXB$,

and by 5, $AYB \,.\, AXB \,.\supset.\, AYX \smile XYB$.

Suppose AXY and AYX are both false. Then YXB and XYB,

whence by 7, $AXB \,.\, YXB \,.\supset.\, AYX \smile YAX$,
and by 7, $XYB \,.\, AYB \,.\supset.\, XAY \smile AXY$.
But if YAX and XAY, then by 1, $YAX \,.\, AXB \,.\supset.\, YAB$,
and by 1, $XAY \,.\, AYB \,.\supset.\, XAB$,
whence by 7, $YAB \,.\, XAB \,.\supset.\, YXA \smile XYA$, contrary to YAX and XAY, respectively, by C. Therefore $AXY \smile AYX$.

Proofs of postulate 5

Theorem 5a. *Proof of* 5 *from A, B,* 1, 2.

To prove: $AXB \,.\, AYB \,.\supset.\, AXY \smile YXB$. By B, $AXY \smile XYA \smile XAY$. But if XYA, then by 2, $BXA \,.\, XYA \,.\supset.\, BXY$; and if XAY, then by 1, $BXA \,.\, XAY \,.\supset.\, BXY$. Therefore $AXY \smile YXB$.

Theorem 5b. *Proof of* 5 *from A, B,* 1^2, 7.

To prove: $AXB \,.\, AYB \,.\supset.\, AXY \smile YXB$. By B, $AXY \smile XAY \smile XYA$.

Case 1. If XAY, then by 1, $BXA \,.\, XAY \,.\supset.\, BXY$.

Case 2. If XYA, then by 7, $XYA \,.\, BYA \,.\supset.\, XBY \smile BXY$. But if XBY, then by 1, $AXB \,.\, XBY \,.\supset.\, AXY$. Therefore $AXY \smile YXB$.

Theorem 5c. *Proof of* 5 *from A, B, C,* 1, 8.

To prove: $AXB \,.\, AYB \,.\supset.\, AXY \smile YXB$. By B, $AXY \smile XAY \smile XYA$.

Case 1. Suppose XAY; then by 1, $BXA \,.\, XAY \,.\supset.\, BXY$.

Case 2. Suppose XYA; then by 8, $BYA \,.\, XYA \,.\supset.\, BXY \smile XBA$, where XBA is contrary to AXB, by C. Therefore $AXY \smile YXB$.

Theorem 5d. *Proof of* 5 *by A,* B^2, C^3, 1^2, 6.

To prove: $AXB \,.\, AYB \,.\supset.\, AXY \smile YXB$. By B, $AXY \smile XAY \smile XYA$, and by B, $YXB \smile BYX \smile YBX$.

Case 1. If XAY, then by 1, $BXA \,.\, XAY \,.\supset.\, BXY$.

Case 2. If XYA and BYX, then by 6, $AYX \,.\, BYX \,.\supset.\, ABX \smile BAX$, contrary to AXB, by C.

Case 3. If XYA and YBX, then by 1, $YBX \,.\, BXA \,.\supset.\, YBA$, contrary to AYB, by C. Therefore $AXY \smile YXB$.

Theorem 5e. *Proof of* 5 *from A,* 2, 4.

To prove: $AXB \,.\, AYB \,.\supset.\, AXY \smile YXB$.
By 4, $AXB \,.\, AYB \,.\supset.\, AXY \smile AYX$.
But if AYX, then by 2, $BXA \,.\, XYA \,.\supset.\, BXY$. Therefore $AXY \smile YXB$.

Theorem 5f. *Proof of* 5 *from A, C,* 4^2, 7.

To prove: $AXB \,.\, AYB \,.\supset.\, AXY \smile YXB$.
By 4, $AXB \,.\, AYB \,.\supset.\, AXY \smile AYX$;
and by 4, $BXA \,.\, BYA \,.\supset.\, BXY \smile BYX$.
Suppose AYX and BYX. Then by 7, $XYA \,.\, BYA \,.\supset.\, XBY \smile BXY$, where XBY is contrary to BYX, by C. Therefore $AXY \smile YXB$.

THEOREM 5g. *Proof of* 5 *from* A, C^2, 4^2, 6.

To prove: $AXB \,.\, AYB \,.\supset.\, AXY \smile YXB$.
By 4, $AXB \,.\, AYB \,.\supset.\, AXY \smile AYX$;
and by 4, $BXA \,.\, BYA \,.\supset.\, BXY \smile BYX$.
Suppose AXY and YXB are both false.
Then AYX and BYX, whence by 6, $AYX \,.\, BYX \,.\supset.\, ABX \smile BAX$, contrary to AXB, by C. Therefore $AXY \smile YXB$.

THEOREM 5h. *Proof of* 5 *from* A, 1, 4, 7.

To prove: $AXB \,.\, AYB \,.\supset.\, AXY \smile YXB$.
By 4, $AXB \,.\, AYB \,.\supset.\, AXY \smile AYX$.
If AYX, then by 7, $XYA \,.\, BYA \,.\supset.\, XBY \smile BXY$. But if XBY, then by 1, $AXB \,.\, XBY \,.\supset.\, AXY$. Therefore $AXY \smile YXB$.

THEOREM 5i. *Proof of* 5 *from* A, C, 4, 8.

To prove: $AXB \,.\, AYB \,.\supset.\, AXY \smile YXB$.
By 4, $AXB \,.\, AYB \,.\supset.\, AXY \smile AYX$.
But if AYX, then by 8, $BYA \,.\, XYA \,.\supset.\, BXY \smile XBA$, where XBA is contrary to AXB, by C. Therefore $AXY \smile YXB$.

THEOREM 5j. *Proof of* 5 *from* A, 3^2, 4^2, 7.

To prove: $AXB \,.\, AYB \,.\supset.\, AXY \smile YXB$.
By 4, $AXB \,.\, AYB \,.\supset.\, AXY \smile AYX$,
and by 4, $BXA \,.\, BYA \,.\supset.\, BXY \smile BYX$.
Suppose AXY and YXB are both false.
Then AYX and BYX, whence by 7, $AYX \,.\, BYX \,.\supset.\, ABY \smile BAY$.
But if ABY, then by 3, $YBA \,.\, BXA \,.\supset.\, YXA$; and if BAY, then by 3, $YAB \,.\, AXB \,.\supset.\, YXB$. Therefore $AXY \smile YXB$.

PROOFS OF POSTULATE 6

THEOREM 6a. *Proof of* 6 *from* A, B, C, 2.

To prove: $XAB \,.\, YAB \,.\supset.\, XYB \smile YXB$. By B, $XYB \smile YBX \smile BXY$. Suppose XBY. Then by 2, $XBY \,.\, BAY \,.\supset.\, XBA$, contrary to XAB, by C. Therefore $XYB \smile YXB$.

THEOREM 6b. *Proof of* 6 *from* A, B, 2^2, 7.

To prove: $XAB.\, YAB \,.\supset.\, XYB \smile YXB$. By B, $XYB \smile YBX \smile BXY$. But if XBY, then by 2, $XBY \,.\, BAY \,.\supset.\, XBA$;
and by 2, $YBX \,.\, BAX \,.\supset.\, YBA$;
whence by 7, $XBA \,.\, YBA \,.\supset.\, XYB \smile YXB$. Therefore $XYB \smile YXB$.

THEOREM 6c. *Proof of* 6 *from* 1^2, 7.

To prove: $XAB \,.\, YAB \,.\supset.\, XYB \smile YXB$.
By 7, $XAB \,.\, YAB \,.\supset.\, XYA \smile YXA$.

Case 1. If XYA, then by 1, $XYA \,.\, YAB \,.\supset.\, XYB$.

Case 2. If YXA, then by 1, $YXA \, . \, XAB \, . \supset . \, YXB$.
Therefore $XYB \smile YXB$.

THEOREM 6d. *Proof of* 6 *from* A, 3^2, 7.

To prove: $XAB \, . \, YAB \, . \supset . \, XYB \smile YXB$.
By 7, $XAB \, . \, YAB \, . \supset . \, XYA \smile YXA$.

Case 1. If XYA, then by 3, $BAX \, . \, AYX \, . \supset . \, BYX$.

Case 2. If YXA, then by 3, $BAY \, . \, AXY \, . \supset . \, BXY$.
Therefore $XYB \smile YXB$.

THEOREM 6e. *Proof of* 6 *from* 1, 8.

To prove: $XAB \, . \, YAB \, . \supset . \, XYB \smile YXB$.
By 8, $XAB \, . \, YAB \, . \supset . \, XYA \smile YXB$.
But if XYA, then by 1, $XYA \, . \, YAB \, . \supset . \, XYB$.
Therefore $XYB \smile YXB$.

THEOREM 6f. *Proof of* 6 *from* A, 3, 8.

To prove: $XAB \, . \, YAB \, . \supset . \, XYB \smile YXB$.
By 8, $XAB \, . \, YAB \, . \supset . \, XYA \smile YXB$.
But if XYA, then by 3, $BAX \, . \, AYX \, . \supset . \, BYX$.
Therefore $XYB \smile YXB$.

THEOREM 6g. *Proof of* 6 *from* A, B, 2^2, 8.

To prove: $XAB \, . \, YAB \, . \supset . \, XYB \smile YXB$. By B, $XYB \smile YXB \smile YBX$, and by 8, $XAB \, . \, YAB \, . \supset . \, XYA \smile YXB$. But if YBX and XYA, then by 2, $YBX \, . \, BAX \, . \supset . \, YBA$, whence by 2, $XYA \, . \, YBA \, . \supset . \, XYB$. Therefore $XYB \smile YXB$.

THEOREM 6h. *Proof of* 6 *from* A, B, C^2, 3, 5.

To prove: $XAB \, . \, YAB \, . \supset . \, XYB \smile YXB$. By B, $XYB \smile YBX \smile BXY$. Suppose YBX. Then by 3, $YBX \, . \, BAX \, . \supset . \, YAX$, whence by 5,
$YBX \, . \, YAX \, . \supset . \, YBA \smile ABX$, contrary to YAB and XAB, respectively, by C. Therefore $XYB \smile YXB$.

THEOREM 6i. *Proof of* 6 *from* A, C, 8^2.

To prove: $XAB \, . \, YAB \, . \supset . \, XYB \smile YXB$.
By 8, $XAB \, . \, YAB \, . \supset . \, XYA \smile YXB$;
and by 8, $YAB \, . \, XAB \, . \supset . \, YXA \smile XYB$.
Suppose YXB and XYB are both false. Then XYA and YXA, contrary to C. Therefore $XYB \smile YXB$.

THEOREM 6j. *Proof of* 6 *from* A, B^2, C^2, 1^2, 5.

To prove: $XAB \, . \, YAB \, . \supset . \, XYB \smile YXB$. By C, XBA and ABY are false, since XAB and YAB are true. By B, $XAY \smile XYA \smile YXA$; and by B, $XBY \smile XYB \smile YXB$.

Case 1. Suppose XAY and XBY.
Then by 5, $XBY \, . \, XAY \, . \supset . \, XBA \smile ABY$, which are both false.

Case 2. If XYA, then by 1, $XYA \,.\, YAB \,.\!\supset\!.\, XYB$.

Case 3. If YXA, then by 1, $YXA \,.\, XAB \,.\!\supset\!.\, YXB$.

Therefore $XYB \smile YXB$.

PROOFS OF POSTULATE 7

THEOREM 7a. *Proof of* 7 *from A, B, C,* 2^3.

To prove: $XAB \,.\, YAB \,.\!\supset\!.\, XYA \smile YXA$. By B, $XYB \smile YXB \smile XBY$.

Case 1. If XYB, then by 2, $XYB \,.\, YAB \,.\!\supset\!.\, XYA$.

Case 2. If YXB, then by 2, $YXB \,.\, XAB \,.\!\supset\!.\, YXA$.

Case 3. If XBY, then by 2, $XBY \,.\, BAY \,.\!\supset\!.\, XBA$, contrary to XAB, by C. Therefore $XYA \smile YXA$.

THEOREM 7b. *Proof of* 7 *from A, B,* C^3, 6^3.

To prove: $XAB \,.\, YAB \,.\!\supset\!.\, XYA \smile YXA$.

By 6, $XAB \,.\, YAB \,.\!\supset\!.\, XYB \smile YXB$. Hence by C, YBX is false.

By B, $XAY \smile XYA \smile YXA$. Suppose XAY.

Then by 6, $YAX \,.\, BAX \,.\!\supset\!.\, YBX \smile BYX$;

and by 6, $XAY \,.\, BAY \,.\!\supset\!.\, XBY \smile BXY$. But YBX is false; hence both BYX and BXY must be true, which is impossible by C.

Therefore $XYA \smile YXA$.

THEOREM 7c. *Proof of* 7 *from A,* C^3, 4^2, 6^3.

To prove: $XAB \,.\, YAB \,.\!\supset\!.\, XYA \smile YXA$.

By 6, $XAB \,.\, YAB \,.\!\supset\!.\, XYB \smile YXB$.

Case 1. Suppose XYB true. Then by 4, $XAB \,.\, XYB \,.\!\supset\!.\, XAY \smile XYA$. But if XAY, then by 6, $XAY \,.\, BAY \,.\!\supset\!.\, XBY \smile BXY$, contrary to XYB by C. Hence in Case 1, XYA.

Case 2. Suppose XYB false; then YXB.

Then by 4, $YAB \,.\, YXB \,.\!\supset\!.\, YAX \smile YXA$.

But if YAX, then by 6, $YAX \,.\, BAX \,.\!\supset\!.\, YBX \smile BYX$, where BYX is false by hypothesis, and YBX is contrary to YXB by C. Hence in Case 2, YXA.

Therefore $XYA \smile YXA$.

THEOREM 7d. *Proof of* 7 *from* 2^2, 6.

To prove: $XAB \,.\, YAB \,.\!\supset\!.\, XYA \smile YXA$.

By 6, $XAB \,.\, YAB \,.\!\supset\!.\, XYB \smile YXB$.

Case 1. If XYB, then by 2, $XYB \,.\, YAB \,.\!\supset\!.\, XYA$.

Case 2. If YXB, then by 2, $YXB \,.\, XAB \,.\!\supset\!.\, YXA$.

Hence in either case, $XYA \smile YXA$.

THEOREM 7e. *Proof of* 7 *from* 2, 8.

To prove: $XAB \,.\, YAB \,.\!\supset\!.\, XYA \smile YXA$.

By 8, $XAB \,.\, YAB \,.\!\supset\!.\, XYA \smile YXB$.

If YXB, then by 2, $YXB \,.\, XAB \,.\!\supset\!.\, YXA$.

Therefore $XYA \smile YXA$.

THEOREM 7f. *Proof of* 7 *from* A, C, 8^2.

To prove: $XAB \,.\, YAB \,.\!\supset\!.\, XYA \smile YXA$.
By 8, $XAB \,.\, YAB \,.\!\supset\!.\, XYA \smile YXB$;
and by 8, $YAB \,.\, XAB \,.\!\supset\!.\, YXA \smile XYB$.
Suppose XYA and YXA are both false. Then YXB and XYB, contrary to C. Therefore $XYA \smile YXA$.

THEOREM 7g. *Proof of* 7 *from* A, B^2, C^4, 5^3.

To prove: $XAB \,.\, YAB \,.\!\supset\!.\, XYA \smile YXA$. By B, $XYB \smile YXB \smile XBY$, and by B, $XAY \smile XYA \smile YXA$.

Case 1. If XYB, then by 5, $XYB \,.\, XAB \,.\!\supset\!.\, XYA \smile AYB$, where AYB is contrary to YAB, by C. Hence in Case 1, XYA.

Case 2. If YXB, then by 5, $YXB \,.\, YAB \,.\!\supset\!.\, YXA \smile AXB$, where AXB is contrary to XAB, by C. Hence in Case 2, YXA.

Case 3. If XBY and XAY, then by 5, $XBY \,.\, XAY \,.\!\supset\!.\, XBA \smile ABY$, contrary to XAB and YAB, by C. Therefore $XYA \smile YXA$.

THEOREM 7h. *Proof of* 7 *from* A, C^2, 5^2, 6.

To prove: $XAB \,.\, YAB \,.\!\supset\!.\, XYA \smile YXA$.
By 6, $XAB \,.\, YAB \,.\!\supset\!.\, XYB \smile YXB$.

Case 1. If XYB, then by 5, $XYB \,.\, XAB \,.\!\supset\!.\, XYA \smile AYB$, where AYB is contrary to YAB, by C. Hence in Case 1, XYA.

Case 2. If YXB, then by 5, $YXB \,.\, YAB \,.\!\supset\!.\, YXA \smile AXB$, where AXB is contrary to XAB, by C. Hence in Case 2, YXA. Therefore $XYA \smile YXA$.

THEOREM 7i. *Proof of* 7 *from* A, 4, 5^2, 8^2.

To prove: $XAB \,.\, YAB \,.\!\supset\!.\, XYA \smile YXA$.
By 8, $XAB \,.\, YAB \,.\!\supset\!.\, XYA \smile YXB$;
and by 8, $YAB \,.\, XAB \,.\!\supset\!.\, YXA \smile XYB$.
Suppose XYA and YXA are both false; then YXB and XYB, whence by 5, $YXB \,.\, YAB \,.\!\supset\!.\, YXA \smile AXB$, and by 5, $XYB \,.\, XAB \,.\!\supset\!.\, XYA \smile AYB$.
But if AXB and AYB, then by 4, $AXB \,.\, AYB \,.\!\supset\!.\, AXY \smile AYX$.
Therefore $XYA \smile YXA$.

THEOREM 7j. *Proof of* 7 *from* A, 1^2, 4^3, 5^2, 6^3.

To prove: $XAB \,.\, YAB \,.\!\supset\!.\, XYA \smile YXA$.
By 6, $XAB \,.\, YAB \,.\!\supset\!.\, XYB \smile YXB$.
If XYB is true, then by 4, $XYB \,.\, XAB \,.\!\supset\!.\, XYA \smile XAY$;
and by 5, $XYB \,.\, XAB \,.\!\supset\!.\, XYA \smile AYB$.
If YXB is true, then by 4, $YXB \,.\, YAB \,.\!\supset\!.\, YXA \smile YAX$,
and by 5, $YXB \,.\, YAB \,.\!\supset\!.\, YXA \smile AXB$. Suppose that XYA and YXA are both false. Then there are three cases to consider.

Case 1. Suppose XYB true and YXB false. Then XAY and AYB.

Then by 6, $XAY \,.\, BAY \,.\!\supset\!.\, XBY \smile BXY$,
whence by 1, $AYB \,.\, YBX \,.\!\supset\!.\, AYX$.

Case 2. Suppose XYB false and YXB true. Then YAX and AXB. Then by 6, $YAX \,.\, BAX \,.\!\supset\!.\, YBX \smile BYX$,
whence by 1, $AXB \,.\, XBY \,.\!\supset\!.\, AXY$.

Case 3. Suppose XYB true and YXB true. Then AYB and AXB. Then by 4, $AXB \,.\, AYB \,.\!\supset\!.\, AXY \smile AYX$. Therefore $XYA \smile YXA$.

Proofs of postulate 8

Theorem 8a. *Proof of* 8 *from* A, B, C, 2^2.

To prove: $XAB \,.\, YAB \,.\!\supset\!.\, XYA \smile YXB$. By B, $XBY \smile XYB \smile YXB$.

Case 1. If XBY, then by 2, $XBY \,.\, BAY \,.\!\supset\!.\, XBA$, contrary to XAB, by C.

Case 2. If XYB, then by 2, $XYB \,.\, YAB \,.\!\supset\!.\, XYA$.
Therefore $XYA \smile YXB$.

Theorem 8b. *Proof of* 8 *from* A, B^2, C^3, 1, 5^2.

To prove: $XAB \,.\, YAB \,.\!\supset\!.\, XYA \smile YXB$. By C, XBA and ABY and AYB are false, since XAB and YAB are true. By B, $XAY \smile YXA \smile XYA$; and by B, $XBY \smile XYB \smile YXB$.

Case 1. Suppose XAY and XBY.
Then by 5, $XBY \,.\, XAY \,.\!\supset\!.\, XBA \smile ABY$, which are both false.

Case 2. If XYB, then by 5, $XYB \,.\, XAB \,.\!\supset\!.\, XYA \smile AYB$, where AYB is false.

Case 3. If YXA, then by 1, $YXA \,.\, XAB \,.\!\supset\!.\, YXB$.
Therefore $XYA \smile YXB$.

Theorem 8c. *Proof of* 8 *from* A, B^2, C^3, 3, 5^2.

To prove: $XAB \,.\, YAB \,.\!\supset\!.\, XYA \smile YXB$. By C, XBA and ABY and AYB are false, since XAB and YAB are true. By B, $XAY \smile YXA \smile XYA$; and by B, $XBY \smile XYB \smile YXB$.

Case 1. Suppose XAY and XBY.
Then by 5, $XBY \,.\, XAY \,.\!\supset\!.\, XBA \smile ABY$, which are both false.

Case 2. If XYB, then by 5, $XYB \,.\, XAB \,.\!\supset\!.\, XYA \smile AYB$, where AYB is false.

Case 3. If YXA, then by 3, $BAY \,.\, AXY \,.\!\supset\!.\, BXY$.
Therefore $XYA \smile YXB$.

Theorem 8d. *Proof of* 8 *from* A, B, C, 3, 6^2.

To prove: $XAB \,.\, YAB \,.\!\supset\!.\, XYA \smile YXB$.
By 6, $XAB \,.\, YAB \,.\!\supset\!.\, XYB \smile YXB$; and by B, $AXY \smile XAY \smile XYA$.

Case 1. If AXY, then by 3, $BAY \,.\, AXY \,.\!\supset\!.\, BXY$. Hence in Case 1, YXB.

Case 2. If XAY and XYB, then by 6, $XAY \,.\, BAY \,.\supset.\, XBY \smile BXY$, where XBY is contrary to XYB, by C. Hence in Case 2, YXB. Therefore $XYA \smile YXB$.

THEOREM 8e. *Proof of* 8 *from* 1, 7.

To prove: $XAB \,.\, YAB \,.\supset.\, XYA \smile YXB$.
By 7, $XAB \,.\, YAB \,.\supset.\, XYA \smile YXA$.
But if YXA, then by 1, $YXA \,.\, XAB \,.\supset.\, YXB$.
Therefore $XYA \smile YXB$.

THEOREM 8f. *Proof of* 8 *from* A, B, C, 1, 6^2.

To prove: $XAB \,.\, YAB \,.\supset.\, XYA \smile YXB$.
By 6, $XAB \,.\, YAB \,.\supset.\, XYB \smile YXB$; and by B, $YXA \smile XAY \smile XYA$.

Case 1. If YXA, then by 1, $YXA \,.\, XAB \,.\supset.\, YXB$. Hence in Case 1, YXB.

Case 2. If XAY and XYB, then by 6, $XAY \,.\, BAY \,.\supset.\, XBY \smile BXY$, where XBY is contrary to XYB, by C. Hence in Case 2, YXB. Therefore $XYA \smile YXB$.

THEOREM 8g. *Proof of* 8 *from* A, 3, 7.

To prove: $XAB \,.\, YAB \,.\supset.\, XYA \smile YXB$.
By 7, $XAB \,.\, YAB \,.\supset.\, XYA \smile YXA$.
But if YXA, then by 3, $BAY \,.\, AXY \,.\supset.\, BXY$.
Therefore $XYA \smile YXB$.

THEOREM 8h. *Proof of* 8 *from* 2, 6.

To prove: $XAB \,.\, YAB \,.\supset.\, XYA \smile YXB$.
By 6, $XAB \,.\, YAB \,.\supset.\, XYB \smile YXB$.
But if XYB, then by 2, $XYB \,.\, YAB \,.\supset.\, XYA$.
Therefore $XYA \smile YXB$.

THEOREM 8i. *Proof of* 8 *from* A, C, 5, 6.

To prove: $XAB \,.\, YAB \,.\supset.\, XYA \smile YXB$.
By 6, $XAB \,.\, YAB \,.\supset.\, XYB \smile YXB$.
But if XYB, then by 5, $XYB \,.\, XAB \,.\supset.\, XYA \smile AYB$, where AYB is contrary to YAB, by C. Therefore $XYA \smile YXB$.

THEOREM 8j. *Proof of* 8 *from* A, C, 4, 6^2.

To prove: $XAB \,.\, YAB \,.\supset.\, XYA \smile YXB$.
By 6, $XAB \,.\, YAB \,.\supset.\, XYB \smile YXB$. Suppose XYB.
Then by 4, $XAB \,.\, XYB \,.\supset.\, XAY \smile XYA$. But if XAY, then by 6, $XAY \,.\, BAY \,.\supset.\, XBY \smile BXY$, where XBY is contrary to XYB, by C.
Therefore $XYA \smile YXB$.

THEOREM 8k. *Proof of* 8 *from* A, B, 2^3, 7.

To prove: $XAB \,.\, YAB \,.\supset.\, XYA \smile YXB$. By B, $XBY \smile XYB \smile YXB$.

Case 1. If XBY, then by 2, $YBX \,.\, BAX \,.\supset.\, YBA$,

and by 2, $XBY \,.\, BAY \,.\!\supset\!.\, XBA$,
whence by 7, $XBA \,.\, YBA \,.\!\supset\!.\, XYB \smile YXB$.

Case 2. If XYB, then by 2, $XYB \,.\, YAB \,.\!\supset\!.\, XYA$.
Therefore $XYA \smile YXB$.

THEOREM 8l. *Proof of* 8 *from* A, B, 1^2, 5, 6^2.

To prove: $XAB \,.\, YAB \,.\!\supset\!.\, XYA \smile YXB$. By B, $YXA \smile XAY \smile XYA$, and by 6, $XAB \,.\, YAB \,.\!\supset\!.\, XYB \smile YXB$.

Case 1. If YXA, then by 1, $YXA \,.\, XAB \,.\!\supset\!.\, YXB$.

Case 2. If XAY and XYB, then by 5, $XYB \,.\, XAB \,.\!\supset\!.\, XYA \smile AYB$, and by 6, $XAY \,.\, BAY \,.\!\supset\!.\, XBY \smile BXY$. But if AYB and XBY, then by 1, $AYB \,.\, YBX \,.\!\supset\!.\, AYX$. Therefore $XYA \smile YXB$.

THEOREM 8m. *Proof of* 8 *from* A, 1, 4, 5, 6^2.

To prove: $XAB \,.\, YAB \,.\!\supset\!.\, XYA \smile YXB$. Suppose XYA and YXB are both false. Now by 6, $XAB \,.\, YAB \,.\!\supset\!.\, XYB \smile YXB$. As YXB is false, XYB. By 4, $XYB \,.\, XAB \,.\!\supset\!.\, XYA \smile XAY$. As XYA is false, XAY. By 5, $XYB \,.\, XAB \,.\!\supset\!.\, XYA \smile AYB$. As XYA is false, AYB. By 6, $BAY \,.\, XAY \,.\!\supset\!.\, BXY \smile XBY$. As BXY is false, then XBY. By 1, $AYB \,.\, YBX \,.\!\supset\!.\, AYX$, contrary to supposition. Therefore $XYA \smile YXB$.

The results of these 71 theorems may be conveniently summarized in the following table. In this table, the numbers in the last column indicate the sets of independent postulates, if any (see § 4), in connection with which each theorem is available.

TABLE I. THEOREMS ON DEDUCIBILITY

Theorem	Postulate	follows from											Set.
1a	1	A	B	C		2		4					6
1b	1	A	B	C			3	4					9, 10, 11
1c	1	A	B	C		2			5				7
1d	1	A	B	C			3		5				8
2a	2	A	B	C	1						7		4
2b	2	A	B	C	1					6			3
2c	2	A	B	C			3			6			9
2d	2	A		C			3				7		10
2e	2	A		C			3	4		6			9
2f	2	A	B	C	1							8	5
2g	2	A	B	C	1				5				2
2h	2	A		C			3					8	11
2i	2	A		C			3		5				8
3a	3	A	B	C	1								1, 2, 3, 4, 5
3b	3	A	B	C		2							1, 6, 7
3c	3	A		C		2				6			
3d	3	A			1	2							1
3e	3	A		C		2						8	
4a	4	A	B	C	1								1, 2, 3, 4, 5
4b	4	A	B		1	2							1
4c	4	A	B		1						7		4
4d	4	A		C					5				2, 7, 8
4e	4	A					3		5		7		

TABLE I—*Continued*

Theorem	Postulate	follows from											Set.
4f	4	A							5		7	8	
4g	4					2			5				7
4h	4	A			1				5		7		
4i	4			C					5		7	8	
4j	4			C	1				5		7		
5a	5	A	B		1	2							1
5b	5	A	B		1						7		4
5c	5	A	B	C	1							8	5
5d	5	A	B	C	1					6			3
5e	5	A				2		4					6
5f	5	A		C				4			7		10
5g	5	A		C				4		6			9
5h	5	A			1			4			7		
5i	5	A		C				4				8	11
5j	5	A					3	4			7		10
6a	6	A	B	C		2							1, 6, 7
6b	6	A	B			2					7		
6c	6				1						7		4
6d	6	A					3				7		10
6e	6				1							8	5
6f	6	A					3					8	11
6g	6	A	B			2						8	
6h	6	A	B	C			3		5				8
6i	6	A		C								8	5, 11
6j	6	A	B	C	1				5				2
7a	7	A	B	C		2							1, 6, 7
7b	7	A	B	C						6			3, 9
7c	7	A		C				4		6			9
7d	7					2				6			
7e	7					2						8	
7f	7	A		C								8	5, 11
7g	7	A	B	C					5				2, 7, 8
7h	7	A		C					5	6			
7i	7	A						4	5			8	
7j	7	A			1			4	5	6			
8a	8	A	B	C		2							1, 6, 7
8b	8	A	B	C	1				5				2
8c	8	A	B	C			3		5				8
8d	8	A	B	C			3			6			9
8e	8				1						7		4
8f	8	A	B	C	1					6			3
8g	8	A					3				7		10
8h	8					2				6			
8i	8	A		C					5	6			
8j	8	A		C				4		6			9
8k	8	A	B			2					7		
8l	8	A	B		1				5	6			
8m	8	A			1			4	5	6			

3. Theorems on non-deducibility and examples of pseudo-betweenness

In this section we show that the question proposed at the beginning of § 2 must always be answered in the negative, except in the cases covered by the 71 theorems just established. That is, we show that no one of the twelve postulates of our basic list is deducible from any others of the list, except in the cases covered by our 71 theorems.

In order to prove this statement, we first construct 44 *examples of pseudo-betweenness*, that is, 44 examples of systems (K, R) which satisfy some but not all of the twelve postulates.

In the first four examples, A–D, the class K consists of three elements.

EXAMPLE A. Let K = a class of three numbers, say 1, 2, 3, and let XYZ be true in the cases 123, 231, and false in all other cases.

Here 123 is true, while 321 is false, so that Postulate A is not satisfied. C is satisfied vacuously, since the conditions mentioned in the hypothesis do not occur. B and D hold. Postulates 1–8 are satisfied vacuously, since the class contains only three elements.

EXAMPLE B. Let K = a class of three numbers, say 1, 2, 3, and let XYZ be false for all values of X, Y, Z.

Here B is clearly not satisfied. A, C, D, and 1–8 are satisfied vacuously.

EXAMPLE C. Let K = a class of three numbers, say 1, 2, 3, and let XYZ mean that X, Y, Z are distinct.

Here A, B, and D are satisfied, while C is not. Postulates 1–8 are satisfied vacuously.

EXAMPLE D. Let K = a class of any three numbers; and let XYZ mean that Y belongs to the interval from X to Z inclusive, when X, Y, and Z are arranged in order of magnitude.

Here A, B, and C are satisfied, while D is not. Postulates 1–8 are satisfied vacuously.

In the remaining examples, 1–40, the class K consists of four numbers, 1, 2, 3, 4, and the meaning of R $[XYZ]$ is defined by simply giving a catalog of the ordered triads of elements for which the relation is true.

In all these examples, Postulate D is satisfied.* Which of the other postulates is satisfied in each case may be ascertained from Table II.

In examples 1–5, all four of the Postulates A, B, C, D are satisfied.

EXAMPLE 1. 124, 134, 213, 243, 312, 342, 421, 431.

EXAMPLE 2. 123, 142, 234, 241, 314, 321, 413, 432.

EXAMPLE 3. 123, 142, 143, 241, 243, 321, 341, 342.

EXAMPLE 4. 142, 213, 234, 241, 312, 314, 413, 432.

EXAMPLE 5. 123, 143, 214, 321, 324, 341, 412, 423.

In Examples 6–11, Postulate B fails, while A, C, and D hold.

EXAMPLE 6. 123, 234, 321, 432.

EXAMPLE 7. 123, 143, 243, 321, 341, 342.

EXAMPLE 8. 123, 243, 321, 342.

EXAMPLE 9. 123, 124, 243, 321, 342, 421.

* In verifying these examples with respect to Postulates 5 and 8, the following peculiarity of these two postulates should be borne in mind: as to 5, for example, it is not sufficient to test for $AXB \cdot AYB$; it is necessary also to test for $AYB \cdot AXB$; and similarly as to 8. For illustrations of the importance of this precaution, see, for instance, Examples 33 and 34.

Example 10. 123, 143, 321, 341.

Example 11. 213, 214, 312, 412.

In Examples 12–24, Postulate C fails, while A, B, and D hold.

Example 12. 123, 132, 214, 231, 234, 314, 321, 324, 412, 413, 423, 432.

Example 13. 124, 132, 134, 213, 214, 231, 234, 312, 314, 324, 412, 413, 421, 423, 431, 432.

Example 14. 124, 213, 234, 312, 314, 324, 413, 421, 423, 432.

Example 15. 123, 124, 143, 234, 243, 321, 341, 342, 421, 432.

Example 16. 123, 143, 213, 214, 243, 312, 314, 321, 324, 341, 342, 412, 413, 423.

Example 17. 123, 143, 214, 243, 321, 324, 341, 342, 412, 423.

Example 18. 213, 214, 234, 243, 312, 314, 324, 342, 412, 413, 423, 432.

Example 19. 123, 124, 132, 134, 231, 234, 321, 324, 421, 423, 431, 432.

Example 20. 123, 124, 134, 143, 234, 243, 321, 324, 341, 342, 421, 423, 431, 432.

Example 21. 123, 132, 134, 142, 143, 213, 214, 231, 234, 241, 243, 312, 321, 341, 342, 412, 431, 432.

Example 22. 123, 124, 132, 134, 143, 214, 231, 243, 321, 324, 341, 342, 412, 421, 423, 431.

Example 23. 123, 124, 134, 143, 321, 324, 341, 421, 423, 431.

Example 24. 123, 134, 142, 214, 241, 321, 324, 412, 423, 431.

In Examples 25–37, Postulate A fails, while B, C, and D hold.

Example 25. 123, 142, 234, 341.

Example 26. 123, 142, 143, 213, 214, 243, 413, 423, 421.

Example 27. 123, 124, 243, 341.

Example 28. 123, 143, 324, 421.

Example 29. 123, 143, 213, 243, 412, 413, 423.

Example 30. 123, 124, 143, 213, 214, 243, 413, 421, 423.

Example 31. 123, 214, 341, 423.

Example 32. 123, 142, 314, 412, 423.

Example 33. 123, 142, 143, 324.

Example 34. 123, 314, 412, 423.

Example 35. 123, 124, 143, 243, 412, 423.

Example 36. 123, 124, 143, 243, 423.

Example 37. 123, 143, 214, 243, 412, 423.

In Example 38, Postulates A and C fail, while B and D hold.

Example 38. 123, 143, 213, 214, 243, 412, 413, 421, 423.

In Examples 39 and 40, Postulates B and C fail, while A and D hold.

EXAMPLE 39. 123, 142, 214, 241, 321, 324, 412, 423.

EXAMPLE 40. 123, 143, 243, 321, 324, 341, 342, 423.

The properties possessed by these 44 systems are conveniently exhibited in Table II, in which a dot (.) indicates that a postulate is satisfied, while a cross (×) indicates that it is not satisfied.

TABLE II. EXAMPLES OF PSEUDO-BETWEENNESS

Ex.	A	B	C	D	1	2	3	4	5	6	7	8	Lemma in which used.
A	×	.	.	.	.	.	.	.	.	.	.	.	A′
B	.	×	.	.	.	.	.	.	.	.	.	.	B′
C	.	.	×	.	.	.	.	.	.	.	.	.	C′
D	.	.	.	×	.	.	.	.	.	.	.	.	D′
1	.	.	.	.	×	.	.	×	×	.	.	.	1′a, 4′a, 5′b.
2	.	.	.	.	×	×	×	.	.	.	.	.	1′b, 2′b, 3′a.
3	.	.	.	.	.	×	.	.	×	×	×	×	2′a, 5′a, 6′a, 7′a, 8′a.
4	.	.	.	.	×	×	×	.	.	×	.	×	6′b, 8′c.
5	.	.	.	.	×	×	×	×	×	.	.	×	8′b.
6	.	×	.	.	×	.	.	.	.	.	.	.	1′c.
7	.	×	.	.	.	×	.	×	×	.	×	×	2′c, 7′b, 8′f.
8	.	×	.	.	.	×	×	.	.	.	.	.	2′d, 3′b.
9	.	×	.	.	×	.	×	.	.	×	.	×	3′c, 6′d, 8′e.
10	.	×	.	.	.	.	.	×	×	.	.	.	4′b, 5′c.
11	.	×	.	.	.	.	.	.	.	×	×	×	6′c, 7′c, 8′d.
12	.	.	×	.	×	.	.	.	.	.	.	.	1′d.
13	.	.	×	.	.	×	.	.	.	.	.	.	2′e.
14	.	.	×	.	.	×	×	.	.	.	.	.	3′d.
15	.	.	×	.	×	.	×	.	.	.	.	.	3′e.
16	.	.	×	.	.	×	.	×	.	.	×	.	4′c, 7′d.
17	.	.	×	.	×	×	×	×	.	.	.	×	4′d.
18	.	.	×	.	.	×	.	.	×	.	×	.	5′d, 7′e.
19	.	.	×	.	.	.	.	.	.	×	×	×	6′e, 7′f, 8′g.
20	.	.	×	.	×	×	.	.	.	.	×	×	7′g, 8′h.
21	.	.	×	.	×	×	×	.	×	.	.	.	5′e.
22	.	.	×	.	×	×	×	.	.	.	.	×	8′i.
23	.	.	×	.	.	×	.	.	×	.	×	×	8′j.
24	.	.	×	.	×	×	×	.	.	×	.	.	6′f.
25	×	.	.	.	×	.	.	.	.	.	.	.	1′e.
26	×	.	.	.	.	×	.	.	.	.	.	.	2′f.
27	×	.	.	.	.	.	×	.	.	.	.	.	3′f.
28	×	.	.	.	.	.	.	×	×	.	.	.	4′e.
29	×	.	.	.	.	×	.	×	.	.	×	.	4′f.
30	×	.	.	.	.	×	.	.	.	.	×	.	7′h.
31	×	.	.	.	.	.	.	.	.	×	×	×	6′h, 7′i, 8′o.
32	×	.	.	.	×	.	.	.	.	×	.	.	6′i.
33	×	.	.	.	.	.	.	.	×	.	.	.	5′f.
34	×	.	.	.	×	.	.	.	.	×	.	×	8′l.
35	×	.	.	.	×	×	.	.	.	.	.	×	8′m.
36	×	.	.	.	.	×	.	.	.	.	×	×	8′n.
37	×	.	.	.	×	×	.	×	.	.	.	×	4′g.
38	×	.	×	.	.	×	.	×	.	.	.	.	4′h.
39	.	×	×	.	×	.	×	.	.	×	.	.	6′g.
40	.	×	×	.	.	×	.	×	.	.	×	×	8′k.

By inspection of these 44 examples, we have at once 68 lemmas on non-deducibility, as exhibited in Table III.

TABLE III. LEMMAS ON NON-DEDUCIBILITY

Lemma	Postulate	is not deducible from postulates												Proof by example
A′	A		B	C	D	1	2	3	4	5	6	7	8	A
B′	B	A		C	D	1	2	3	4	5	6	7	8	B
C′	C	A	B		D	1	2	3	4	5	6	7	8	C
D′	D	A	B	C		1	2	3	4	5	6	7	8	D
1′a	1	A	B	C	D		2	3			6	7	8	1
1′b	1	A	B	C	D				4	5	6	7	8	2
1′c	1	A		C	D		2	3	4	5	6	7	8	6
1′d	1	A	B		D		2	3	4	5	6	7	8	12
1′e	1		B	C	D		2	3	4	5	6	7	8	25
2′a	2	A	B	C	D	1		3	4					3
2′b	2	A	B	C	D				4	5	6	7	8	2
2′c	2	A		C	D	1		3			6			7
2′d	2	A		C	D	1			4	5	6	7	8	8
2′e	2	A	B		D	1		3	4	5	6	7	8	13
2′f	2		B	C	D	1		3	4	5	6	7	8	26
3′a	3	A	B	C	D				4	5	6	7	8	2
3′b	3	A		C	D	1			4	5	6	7	8	8
3′c	3	A		C	D		2		4	5		7		9
3′d	3	A	B		D	1			4	5	6	7	8	14
3′e	3	A	B		D		2		4	5	6	7	8	15
3′f	3		B	C	D	1	2		4	5	6	7	8	27
4′a	4	A	B	C	D		2	3			6	7	8	1
4′b	4	A		C	D	1	2	3			6	7	8	10
4′c	4	A	B		D	1		3		5	6		8	16
4′d	4	A	B		D					5	6	7		17
4′e	4		B	C	D	1	2	3			6	7	8	28
4′f	4		B	C	D	1		3		5	6		8	29
4′g	4		B	C	D			3		5	6	7		37
4′h	4		B		D	1		3		5	6	7	8	38
5′a	5	A	B	C	D	1		3	4					3
5′b	5	A	B	C	D		2	3			6	7	8	1
5′c	5	A		C	D	1	2	3			6	7	8	10
5′d	5	A	B		D	1		3	4		6		8	18
5′e	5	A	B		D				4		6	7	8	21
5′f	5		B	C	D	1	2	3	4		6	7	8	33
6′a	6	A	B	C	D	1		3	4					3
6′b	6	A	B	C	D				4	5		7		4
6′c	6	A		C	D	1	2	3	4	5				11
6′d	6	A		C	D		2		4	5		7		9
6′e	6	A	B		D	1	2	3	4	5				19
6′f	6	A	B		D				4	5		7	8	24
6′g	6	A			D		2		4	5		7	8	39
6′h	6		B	C	D	1	2	3	4	5				31
6′i	6		B	C	D		2	3	4	5		7	8	32
7′a	7	A	B	C	D	1		3	4					3
7′b	7	A		C	D	1		3			6			7
7′c	7	A		C	D	1	2	3	4	5				11
7′d	7	A	B		D	1		3		5	6		8	16
7′e	7	A	B		D	1		3	4		6		8	18
7′f	7	A	B		D	1	2	3	4	5				19
7′g	7	A	B		D			3	4	5	6			20
7′h	7		B	C	D	1		3	4	5	6		8	30
7′i	7		B	C	D	1	2	3	4	5				31

TABLE III—*Continued.*

Lemma	Postulate	is not deducible from postulates											Proof by example
8′a	8	A	B	C	D	1		3	4				3
8′b	8	A	B	C	D						6	7	5
8′c	8	A	B	C	D				4	5		7	4
8′d	8	A		C	D	1	2	3	4	5			11
8′e	8	A		C	D		2		4	5		7	9
8′f	8	A		C	D	1		3			6		7
8′g	8	A	B		D	1	2	3	4	5			19
8′h	8	A	B		D			3	4	5	6		20
8′i	8	A	B		D				4	5	6	7	22
8′j	8	A	B		D	1		3	4		6		23
8′k	8	A			D	1		3		5	6		40
8′l	8		B	C	D		2	3	4	5		7	34
8′m	8		B	C	D			3	4	5	6	7	35
8′n	8		B	C	D	1		3	4	5	6		36
8′o	8		B	C	D	1	2	3	4	5			31

By comparing these lemmas with the theorems in § 2, we can now establish the following theorem of non-deducibility:

THEOREM. *No one of the twelve postulates of our basic list is deducible from any others of the list, except in the cases covered by our* 71 *theorems.*

For example, consider the case of Postulate 3. By Lemmas 3′a–3′f, we see that Postulate 3 can certainly not be proved without the use of at least one postulate from each of the following groups: 1, 2; B, 2; B, 1, 6, 8; C, 2; C, 1; A; hence the following combinations are the only ones which need to be investigated: A, 1, 2; A, B, C, 1; A, B, C, 2; A, C, 2, 6; A, C, 2, 8. But by reference to Theorems 3a–3e we see that each one of these combinations is in fact sufficient to prove Postulate 3.

The truth of the theorem for each of the other cases is established in a similar way.

4. Eleven sets of independent postulates

We are now in position to select from our basic list of twelve postulates, several smaller lists which are free from redundancies.

An examination of our results in regard to deducibility shows that this selection can be made in precisely eleven ways; that is, there are precisely eleven sets of independent postulates which can be selected from our basic list.

The eleven sets are as follows:

(1) A, B, C, D, 1, 2.
(2) A, B, C, D, 1, 5.
(3) A, B, C, D, 1, 6.
(4) A, B, C, D, 1, 7.
(5) A, B, C, D, 1, 8.
(6) A, B, C, D, 2, 4.
(7) A, B, C, D, 2, 5.
(8) A, B, C, D, 3, 5.
(9) A, B, C, D, 3, 4, 6.
(10) A, B, C, D, 3, 4, 7.
(11) A, B, C, D, 3, 4, 8.

That the postulates of each set are independent of one another is proved by the existence of Examples A–D, 1, 2, 3 above. (See Lemmas A–D, 1′a, 2′a, 3′a, 4′a, 5′a, 6′a, 7′a, 8′a, 1′b, 2′b, and 5′b.)

That the postulates of each set are sufficient to establish the entire list of twelve postulates is proved by our theorems on deducibility; in fact, in several cases the missing postulates can be deduced from the given postulates in more than one way. Table I, at the end of § 2, will show clearly all the possible ways in which the missing postulates in each set can be deduced from the given postulates of that set.

It will be noticed that certain theorems are not directly available in any of the eleven sets, since no one of the sets contains explicitly the postulates used in the proof of these theorems.

A comparison of the merits of the eleven sets of postulates by the aid of Table I, while perhaps not convincing in the present state of our knowledge of the standards to which such sets of postulates should conform, would at any rate be of some interest.

For example, if our aim is to find the set which shall be the most condensed, and from which the remaining postulates can be most readily deduced, we should select Set 1. If, on the other hand, our aim is to analyze the postulates down to their lowest terms, that is, to find a set from which the necessary deductions can just barely be made, we should then probably select Set 10. It is quite possible, of course, that some other considerations (not now clear) might lead us to select some other of the eleven sets as preferable for some purpose then in view.

In any case, it is satisfactory to know that these eleven sets are the only sets of independent postulates which can be selected from the basic list of twelve postulates from which we started.

Moreover, this basic list of twelve postulates must always occupy a central place in any theory of betweenness. For, as we have already pointed out, this set contains all the general laws concerning the betweenness relations among three or four elements; and even if further propositions concerning five or more distinct elements should be added to the list, no one of the basic list of twelve could thereby be made redundant. To prove this fact, we have merely to notice that the system exhibited in each of the examples used above in proving independence contains at most four elements, and would therefore satisfy vacuously any proposition involving five or more distinct elements.

5. Definition of betweenness

The following definition of betweenness may now be formulated:

Definition. Any system (K, R) in which the class K and the triadic relation R are found to possess all the properties demanded by any one of our sets of independent postulates (see § 4) may be called an *ordered class* or *series,* and the relation R itself may then be called the relation of *betweenness.*

The most familiar example of such an ordered class or series is the system (K, R) in which K is the class of points on a line, and AXB means that the point X belongs to the interior of the segment AB.

Another example is the system (K, R) in which K is the class of natural numbers and AXB means that the number X is larger than the smaller of the two numbers A and B, and smaller than the larger one.

In each of these examples we say that X is "between" A and B.

The relation between the theory of betweenness and the theory of serial order may be expressed as follows.

Let A and B be any two distinct elements of a "betweenness" system, and let X and Y be any other distinct elements of the system. Then we say that X *precedes* Y, *in the order* AB, if any one of the following conditions is true: (1) XAB and either XYA or $Y = A$ or AYB or $Y = B$ or ABY; (2) $X = A$ and either AYB or $Y = B$ or ABY; (3) AXB and either XYB or $Y = B$ or ABY; (4) $X = B$ and ABY; (5) ABX and BXY. From this definition, and the properties of betweenness, it is easy to derive the usual properties of the dyadic relation of serial order, always, however, with respect to the fixed base AB.

On the relation between the theory of betweenness and the theory of cyclic order, see a paper by E. V. Huntington: *A set of independent postulates for cyclic order,* Proceedings of the National Academy of Sciences, vol. 2 (1916), p. 630.

Appendix

Remark Concerning Postulate A.—In regard to the general laws of betweenness concerning four elements A, B, X, Y on a line, if we agree to read always in the direction from A towards B, the total number of these general laws appears at first sight to be twenty-four, which group themselves into nine groups, as follows:

$-X$——A———B———Y——	$-Y$——A———$B-X$———
1. $XAB \,.\, ABY \,.\supset.\, XAY$	1a. $YAB \,.\, ABX \,.\supset.\, YAX$
1c. $XAB \,.\, ABY \,.\supset.\, XBY$	1b. $YAB \,.\, ABX \,.\supset.\, YBX$
$-X$——$A-Y$——B—————	$-Y$——$A-X$——B—————
2. $XAB \,.\, AYB \,.\supset.\, XAY$	2a. $YAB \,.\, AXB \,.\supset.\, YAX$
3. $XAB \,.\, AYB \,.\supset.\, XYB$	3a. $YAB \,.\, AXB \,.\supset.\, YXB$
————$A-X$——$B-Y$———	————$A-Y$——$B-X$———
2c. $AXB \,.\, ABY \,.\supset.\, XBY$	2b. $AYB \,.\, ABX \,.\supset.\, YBX$
3c. $AXB \,.\, ABY \,.\supset.\, AXY$	3b. $AYB \,.\, ABX \,.\supset.\, AYX$

4. $AXB \,.\, AYB \,.\!\supset\!.\, AXY \smile AYX$	$\text{———}A\text{—}\genfrac{}{}{0pt}{}{X}{Y}\text{—}\genfrac{}{}{0pt}{}{Y}{X}\text{—}B\text{———}$
4b. $AXB \,.\, AYB \,.\!\supset\!.\, XYB \smile YXB$	
5. $AXB \,.\, AYB \,.\!\supset\!.\, AXY \smile YXB$	5a. $AXB \,.\, AYB \,.\!\supset\!.\, AYX \smile XYB$
6. $XAB \,.\, YAB \,.\!\supset\!.\, XYB \smile YXB$	$\text{—}\genfrac{}{}{0pt}{}{X}{Y}\text{—}\genfrac{}{}{0pt}{}{Y}{X}\text{—}A\text{———}B\text{———}$
7. $XAB \,.\, YAB \,.\!\supset\!.\, XYA \smile YXA$	
8. $XAB \,.\, YAB \,.\!\supset\!.\, XYA \smile YXB$	8a. $XAB \,.\, YAB \,.\!\supset\!.\, YXA \smile XYB$
6b. $ABX \,.\, ABY \,.\!\supset\!.\, AXY \smile AYX$	$\text{———}A\text{———}B\text{—}\genfrac{}{}{0pt}{}{X}{Y}\text{—}\genfrac{}{}{0pt}{}{Y}{X}\text{—}$
7b. $ABX \,.\, ABY \,.\!\supset\!.\, BXY \smile BYX$	
8b. $ABX \,.\, ABY \,.\!\supset\!.\, AXY \smile BYX$	8c. $ABX \,.\, ABY \,.\!\supset\!.\, AYX \smile BXY$

But each of the postulates in the second column is immediately obtainable from the postulate standing opposite it in the first column, without the use of any other postulate, so that the list of 24 is at once reducible to 15.

Furthermore, any two of the 24 postulates which bear the same number are deducible from each other by the aid of Postulate A alone. Hence the list of 24 reduces to 8, which may be selected in various ways; all these selections are equivalent in view of Postulate A; the Postulates 1–8 of the text represent one such selection.

On the other hand, if we agree to read either forward or backward along the line, the list of 24 would have to be greatly enlarged, so as to include, for example, such postulates as $XAB \,.\, YBA \,.\!\supset\!.\, YAX$. All such postulates are immediately deducible from Postulates 1–8 by the aid of Postulate A, and are not here considered. It should be noted, however, that if it were desired to give a complete discussion of what could be proved without the aid of Postulate A, it would be necessary to consider the whole of the enlarged list, and also to modify slightly the wording of Postulate C.

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