HASKINS'S MOMENTAL THEOREM AND ITS CONNECTION WITH STIELTJES'S PROBLEM OF MOMENTS*

BY

EDWARD B. VAN VLECK

In the last volume of the Transactions † a simple and elegant proof of Haskins's Momental Theorem was given by Dunham Jackson through use of polynomial approximation. This interesting theorem is as follows: If for two bounded functions $f(x)$ and $\phi(x)$ integrable in the sense of Lebesgue for $a \leqq x \leqq b$ we have

$$(1) \qquad \int_a^b [f(x)]^n \, dx = \int_a^b [\phi(x)]^n \, dx \qquad (n = 1, 2, \cdots),$$

then the measures of the sets of points $E_{\alpha\beta}(f)$ and $E_{\alpha\beta}(\phi)$ for which the values of $f(x)$ and $\phi(x)$ respectively lie between α and β inclusive will be equal for all pairs of numbers (α, β) included between the upper and lower bounds of the functions; i. e.,

$$(2) \qquad mE_{\alpha\beta}(f) = mE_{\alpha\beta}(\phi).$$

A proof (unpublished) of the theorem with very "considerable restrictions" on the functions had been previously obtained by Haskins‡ "by reduction to the theorem of Stieltjes and Lebesgue concerning the moments." That theorem asserts that if for two bounded functions we have

$$(3) \qquad \int_a^b x^n f(x) \, dx = \int_a^b x^n \phi(x) \, dx \qquad (n = 0, 1, 2, \cdots),$$

then $f(x)$ and $\phi(x)$ are equal "almost everywhere" in the interval of integration, or, in other words, are equal except for at most a set of points of measure 0.§

In the following pages it is shown that a general proof of Haskins's theorem

* Presented to the Society, December 22, 1916.

† Vol. 17 (1916), p. 178.

‡ Cf. Haskins, these Transactions, vol. 17 (1916), p. 185, footnote.

§ The identity of $f(x)$ and $\phi(x)$, when continuous, was pointed out by Stieltjes in correspondence with Hermite in 1893, and the above extension was given by Lebesgue in 1909. For literature references see Haskins, p. 185.

can be obtained by reduction to Stieltjes's Theorem of Moments when the latter is expressed more comprehensively in terms of Stieltjes integrals. Thus in place of (3) we have as our basal set of equations one of the form

$$\int_a^b x^n \, df(x) = \int_a^b x^n \, d\phi(x) \qquad (n = 0, 1, 2, \cdots), \tag{4}$$

where the two functions are of limited variations and vanish at a. A very simple proof, closely resembling Jackson's proof of (2), is then given that $f(x)$ and $\phi(x)$ coincide throughout (a, b) except possibly at their points of discontinuity. It is also shown how this result may be obtained, though not so directly, from work of Stieltjes.

Incidentally it appears that every Lebesgue integral can be thrown into the form of a Stieltjes integral. It seems altogether likely that this may be known to some, but I have not found it stated. Lebesgue* has noted that every Stieltjes integral $\int f(x) \, d\alpha(x)$ containing a continuous $f(x)$ and an $\alpha(x)$ of limited variation can be transformed into a Lebesgue integral and, in fact, into a Riemannian integral.

By hypothesis let $f(x)$ and $\phi(x)$ be two functions bounded in the interval (a, b) of integration so that we have

$$h \leqq f(x) \leqq H, \qquad h \leqq \phi(x) \leqq H. \tag{5}$$

Place

$$\psi_f(y) = mE_{hy}(f) \equiv mE(h \leqq f(x) \leqq y),$$

where y is any value in the function range (h, H). This function† may be called the measure function of $f(x)$. It is obviously a monotone, non-decreasing function, continuous on the right-hand side. Now by the definition of the Lebesgue integral we have

$$\int_a^b [f(x)]^n \, dx = \lim \sum_h^H y_i^n \cdot mE(y_{i-1} < f(x) \leqq y_i) + h^n \, mE(f(x) = h). \tag{6}$$

But $mE(y_{i-1} < f(x) \leqq y_i)$ is the increment of ψ_f when the argument increases from y_{i-1} to y_i. Consequently we have the identity

$$\int_a^b [f(x)]^n \, dx = \int_h^H y^n \, d\psi_f(y), \tag{7}$$

provided we suppose the value of $\psi_f(y)$ to be altered at h and made there equal to 0, if necessary, so as to include in this Stieltjes integral the last member on the right of (6).

* Comptes Rendus, vol. 150 (1910), p. 86.

† Compare with Haskins's function M_y, loc. cit., p. 184.

For $n = 1$ the last equation converts a Lebesgue integral into a Stieltjes integral. The conversion is also applicable when $f(x)$ is an unlimited integrable function if we put $H = +\infty$, $h = -\infty$.

By virtue of (7) our hypothesis (1) now takes the form

$$\int_h^H y^n \, d\psi_f(y) = \int_h^H y^n \, d\psi_\phi(y) \qquad (n = 0, 1, 2, \cdots). \tag{8}$$

Consider any two limited, non-decreasing functions $\psi_1(x)$ and $\psi_2(x)$, for which we have

$$\psi_1(h) = \psi_2(h) = 0, \quad \int_h^H x^n \, d\psi_1(x) = \int_h^H x^n \, d\psi_2(x) \qquad (n = 0, 1, 2, \cdots). \tag{9}$$

The points of discontinuity of either function are countable, and the value of the function at each such point (h and H excepted) may be changed to the right-hand limiting value without affecting the value of the integral. We suppose for the moment this change made for both functions. If $P_n(x)$ is any polynomial, we have by (9)

$$\int_h^H P_n(x) \, d\psi_1(x) = \int_h^H P_n(x) \, d\psi_2(x). \tag{10}$$

Take any fixed point x' in the interval of integration and construct a function $F(x)$ equal to 1 for $h \leqq x \leqq x'$, equal between x' and $x' + \zeta$ to a linear function which decreases from 1 to 0, and equal to 0 for $x' + \zeta \leqq x \leqq H$. Since $F(x)$ is continuous, a polynomial exists such that throughout the interval (h, H) we have $|F(x) - P_n(x)| \leqq \epsilon$. Then the left-hand member of (10) will differ from

$$\int_h^H F(x) \, d\psi_1(x)$$

by not more than $\epsilon \, \psi_1(H)$. From the definition of $F(x)$ we have

$$\int_h^H F(x) \, d\psi_1(x) = \psi_1(x') + \epsilon',$$

where ϵ' is a non-negative quantity not exceeding $\psi_1(x' + \zeta) - \psi'(x')$. Hence by choosing ϵ and ζ sufficiently small we can make the first member of (10) to differ from $\psi_1(x')$ as little as we please, and similarly for its second member. Consequently we must have by (10) $\psi_1(x') = \psi_2(x')$. Application of this result to (8) makes $\psi_f(y) \equiv \psi_\phi(y)$ inasmuch as both of these functions possess right-handed continuity. Thus Haskins's theorem is established.

In the preceding paragraph the values of $\psi_1(x)$ and $\psi_2(x)$ were altered only in their points of discontinuity. They are therefore identical excepting possibly at these points. This conclusion can be extended at once to two functions $\psi_1(x)$, $\psi_2(x)$ of limited variation. For if in (10) each function is expressed as the difference of two non-decreasing functions and the negative integral on each side of the equation is transposed to the other, the preceding result becomes applicable.

The identity of $\psi_1(x)$ and $\psi_2(x)$ except at points of discontinuity can also be deduced from work of Stieltjes. In his remarkable 1894 memoir* he supposed given the values of the constants

$$c_n = \int_0^\infty x^n \, d\psi(x) \qquad (n = 0, 1, 2, \cdots), \tag{11}$$

and considered $\psi(x)$ as a monotone, non-decreasing function to be determined, for which $\psi(0) = 0$. We have then his "Problem of Moments" for which he determines two cases, a determinate and an indeterminate one. To bring the integrals in (9) into the form (11), it will be necessary to change the origin when h is negative. For this purpose place $x = x' + h$ so that (9) becomes

$$\int_0^{H-h} (x' + h)^n \, d\bar{\psi}_1(x') = \int_0^{H-h} (x' + h)^n \, d\bar{\psi}_2(x'),$$

where $\bar{\psi}_1(x')$, $\bar{\psi}_2(x')$ are the transformed functions. On putting in succession $n = 0, 1, 2, \cdots$, we obtain

$$\int_0^{H-h} x'^n \, d\bar{\psi}_1(x') = \int_0^{H-h} x'^n \, d\bar{\psi}_2(x').$$

Consequently we may suppose $h = 0$ without restricting the problem.

In solving the problem of moments Stieltjes connects the formal expansion

$$\int_0^\infty \frac{d\psi(x)}{z + x} = \frac{c_0}{z} - \frac{c_1}{z^2} + \frac{c_2}{z^3} - \cdots$$

with a corresponding continued fraction

$$\frac{1}{a_1 z +} \, \frac{1}{a_2 +} \, \frac{1}{a_3 z +} \, \frac{1}{a_4 +} \cdots$$

in which a_n is positive. The constants c_n can not be arbitrarily chosen. One of the conditions to which they are subject is $c_{n+1}/c_n > c_n/c_{n-1}$, a condition which is also noted by Haskins. Supposing these conditions fulfilled, the necessary and sufficient condition for the existence of the determinate case is the divergence of $\sum a_n$; and the monotone function $\psi(x)$ is then completely determined by (11) except at a countable set of points which are points of discontinuity.

* Annales de la Faculté des Sciences de Toulouse, vol. 8, 1894.

Now the ratio c_{n+1}/c_n increases with n, and Stieltjes notes* that when it does not increase without limit, the numbers $b_n = 1/a_n a_{n+1}$ are limited. Consequently $\sum a_n$ is then divergent. If the interval of integration is finite, c_{n+1}/c_n can not increase without limit, for since $\psi(x)$ is monotone we have by the first mean value theorem

$$c_{n+1} \equiv \int_0^H x^{n+1}\, d\psi(x) = \xi \int_0^H x^n\, d\psi(x) = \xi c_n,$$

in which $0 < \xi \leqq H$. Hence the integrals (9) fall under the determinate case of (11), and the two functions $\psi_1(x)$, $\psi_2(x)$ are accordingly identical except at their points of discontinuity.

In conclusion, it may be remarked that the problem of moments presented under (3) may be reformulated so as to be embraced under (9). For this purpose put

$$F(x) = \int_a^x f(x)\,dx, \qquad \Phi(x) = \int_a^x \phi(x)\,dx.$$

Suppose first that $f(x) \geqq 0$ between a and b. If we then partition the interval (ab) (including 0 as a point of division in case it lies within the interval), and apply the mean value theorem for Lebesgue integrals, to each subinterval, we obtain

$$(12) \quad \int_a^b x^n f(x)\,dx = \lim_{m=\infty} \sum_{i=1}^{m} \xi_i^n \int_{x_i}^{x_{i+1}} f(x)\,dx = \lim \sum \xi_i^n [F(x_{i+1}) - F(x_i)],$$

where the ξ_i denote appropriately chosen points in the subintervals (x_i, x_{i+1}). As $F(x)$ is of limited variation, the last limit in (12) is a Stieltjes integral

$$\int_a^b x^n\, dF(x).$$

If $f(x)$ is not of constant sign in the interval, put $f(x) = f_1(x) - f_2(x)$, where $f_1(x) = 0$ when $f(x)$ is negative and $f_2(x) = 0$ when $f(x)$ is positive. By consideration of $f_1(x)$ and $f_2(x)$ separately we get finally the same Stieltjes integral. Hence our equations (3) may be replaced by the equations

$$\int_a^b x^n\, dF(x) = \int_a^b x^n\, d\Phi(x) \qquad (n = 0, 1, 2, \cdots).$$

As $F(x)$ and $\Phi(x)$ are of limited variation, they may each be expressed as the difference of two non-decreasing functions, so that by transposition we get an equation of form (9) with monotone integrands.

University of Wisconsin

* Loc. cit., p. 23.