A THEOREM FOR SPACE ANALOGOUS TO CESÀRO'S THEOREM FOR PLANE ISOGONAL SYSTEMS*

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Investigations† in the geometry of ordinary differential equations of the first order have yielded many interesting results, especially in the properties of the osculating circles of the integral curves of such equations. In particular, Scheffers (l. c.) has introduced two simple lineal element transformations, which have been well described by Kasner‡ as "turns" and "slides," respectively. A *turn* converts a lineal element into one having the same point and a direction making a given angle with the original direction. Under a *slide* an element is displaced a given distance along the line of the element. Applied to the integral curves of

$$F(x, y, p) = 0, \tag{1}$$

a turn of each lineal element satisfying the equation leads to a second equation whose integral curves form a system of isogonal trajectories of the integral curves of (1). A slide leads to equitangential trajectories. Cesàro§ has shown that the osculating circles of the curves of an isogonal system at any given point will pass through a second common point. Scheffers (l. c.) has demonstrated a companion theorem for an equitangential system of curves, namely, the osculating circles of the curves of such a system which touch a given line will also touch a second line.

In approaching questions of this nature for space, it is clear that an understanding must be reached as to the direction the investigation should take.||

* Presented to the Society, December 2, 1916.

† The following papers may be referred to in this connection: Scheffers, *Isogonalkurven, Äquitangentialkurven und complexe Zahlen*, Mathematische Annalen, vol. 60 (1905), p. 491; Gundelfinger, *On the geometry of line elements in the plane with reference to osculating circles*, American Journal of Mathematics, vol. 33 (1910), p. 153.

‡ Kasner, *The group of turns and slides*, etc., American Journal of Mathematics, vol. 33 (1910), p. 193.

§ *Geometria intrinseca*, 1896.

|| Kasner, for example, has proposed and solved a problem for space, different from that discussed here, in his paper *Equitangential congruences of curves in space*, Rendiconti del Circolo Matematico di Palermo, vol. 35 (1913), p. 283.

To consider partial differential equations of the first order and their integral surfaces is a first thought, but to consider osculating spheres of such surfaces as analogous to the osculating circles of plane curves is wide of the mark, since at a general point on a surface there is no osculating sphere. Furthermore, a turn, or a slide, of a lineal element is definite in so far as the point, or the line, of the element is unique. It is natural to define a *turn of a surface element as a rotation of the element through a given angle about any axis through the point of the element and lying on its plane.* Likewise by a *slide* we may understand *a translation of the element in its plane a given distance in any direction.* But in the first instance, the axis of the turn, and in the second, the direction of the slide, are not definite. A study of the geometry of partial differential equations of the first order, to which this paper is devoted, clears up the vagueness referred to above, and leads to theorems which appear to be as fundamental in connection with such differential equations as those cited for ordinary differential equations.

1. Preliminary formulas

Consider a surface element (x, y, z, p, q), and a consecutive element $(x + dx, y + dy, z + dz, p + dp, q + dq)$ *united* with the first, that is, such that

$$dz = p\,dx + q\,dy, \tag{1}$$

the ratios of the infinitesimals* satisfying

$$\frac{dx}{\alpha} = \frac{dy}{\beta} = \frac{dz}{p\alpha + q\beta} = \frac{dp}{\delta} = \frac{dq}{\epsilon}. \tag{2}$$

The pair of united elements will establish ∞^1 curvature elements (x, y, z, p, q, r, s, t), where the last three coördinates satisfy

$$\delta = r\alpha + s\beta, \qquad \epsilon = s\alpha + t\beta. \tag{3}$$

For any surface to which the pair of united elements belong, the curvature of normal sections at (x, y, z) containing the direction $\delta x : \delta y : \delta z$ is

$$\frac{1}{R} = \frac{1}{\sqrt{1 + p^2 + q^2}} \frac{r\,\delta x^2 + 2s\,\delta x\,\delta y + t\,\delta y^2}{\delta s^2}. \tag{4}$$

Eliminating r and s by using (3), this becomes

$$\frac{1}{R} = \frac{1}{\sqrt{1 + p^2 + q^2}} \frac{\beta\delta\,\delta x^2 + \alpha\epsilon\,\delta y^2 - s(\beta\,\delta x - \alpha\,\delta y)^2}{\alpha\beta\,\delta s^2}. \tag{5}$$

* Cf. a paper by the author *On osculating element-bands associated with loci of surface elements,* these Transactions, vol. 11 (1910), p. 302.

The pair of surface elements under discussion present two directions (usually distinct) which possess obvious geometrical interest and simplicity. These directions establish a line-pair lying in the plane E of the first element (x, y, z, p, q) and passing through its point $P(x, y, z)$. The first direction is that of the line joining the points of the united elements. Here

$$(6)\qquad \delta x : \delta y : \delta z :: \alpha : \beta : p\alpha + q\beta .$$

The second direction is that of the line of intersection of the planes of the elements. For this line

$$(7)\qquad \delta x : \delta y : \delta z = \epsilon : - \delta : p\epsilon - q\delta .$$

For the first direction, equation (5) becomes

$$(8)\qquad \frac{1}{R_1} = \frac{1}{\sqrt{1 + p^2 + q^2}} \frac{\alpha\delta + \beta\epsilon}{\alpha^2 + \beta^2 + (p\alpha + q\beta)^2} .$$

The disappearance of s indicates that the normal curvature of every surface containing the pair of elements is the same in the direction in which these elements are united. The normal sections of these surfaces in this direction have therefore a common osculating circle. Hence we may associate with a pair of united surface elements a *first osculating circle* whose radius is given by (8) and which has the property described in the preceding sentences, namely, *it is the osculating circle of the normal section of any surface containing the element-pair taken in the direction in which the elements are united.*

Reserving for a later section (§ 3) a similar discussion for the direction (7), pass now to the application of the preceding exposition to the partial differential equation (assumed not linear)

$$(9)\qquad F(x, y, z, p, q) = 0,$$

in which, as usual, $p = \partial z/\partial x$, $q = \partial z/\partial y$. Let S be an integral surface of this differential equation, $P(x, y, z)$ a point on S, and C the characteristic on S through P. Infinitely many integral surfaces of (9) contain the characteristic C and all touch S along C. On all of these surfaces the direction conjugate to that of the characteristic at P is common.* That is, in each surface element satisfying (9) there are two significant directions, namely, (1) *the direction of the characteristic determined by this element,* and (2), *the common conjugate direction* above described. If we consider a pair of united elements satisfying (9) whose points are on the characteristic determined by the first element, these two directions are obviously precisely those dis-

* Carathéodory, *Zur geometrischen Deutung der Charakteristiken einer partiellen Differentialgleichung erster Ordnung,* Mathematische Annalen, vol. 95 (1904), p. 377. An excellent exposition of the geometry involved here is given by Goursat in his *Leçons sur l'intégration des équations aux derivées partielles du premier ordre* (Paris, 1891), pp. 181–88.

cussed above (equations (6) and (7)). For this pair of elements, the infinitesimals dx, dy, etc., satisfy the well-known differential equations of the characteristics (or characteristic bands)

$$(10) \qquad \frac{dx}{F_p} = \frac{dy}{F_q} = \frac{dz}{pF_p + qF_q} = \frac{dp}{-F_x - pF_z} = \frac{dq}{-F_y - qF_z},$$

where subscripts indicate partial differentiation with respect to the variables expressed.

To obtain a *first osculating circle* for any surface element satisfying (9), we may use formula (8) in which α, β, etc., of equations (2) are to be replaced by the corresponding denominators in (10). The property possessed by this circle is clear. *The normal section of any integral surface containing this surface element taken in the direction of the characteristic will be osculated by this circle.* In the following discussion the phrase "first osculating circle for the surface element" will be employed to designate this circle.

To derive a convenient expression for the radius of the first osculating circle, it is desirable to introduce "homogeneous element coördinates" by setting

$$(11) \qquad p = -\frac{p_1}{p_3}, \qquad q = -\frac{p_2}{p_3}.$$

The given differential equation (9) now becomes

$$(12) \qquad F(x, y, z, p, q) = G(x, y, z, p_1, p_2, p_3) = 0,$$

where G is homogeneous and of degree zero in p_1, p_2, p_3. Hence, if we set $G_i = \partial G/\partial p_i \ (i = 1, 2, 3)$,

$$(13) \qquad p_1 G_1 + p_2 G_2 + p_3 G_3 = 0.$$

Furthermore, $F_p = -p_3 G_1$, $F_q = -p_3 G_2$, $pF_p + qF_q = -p_3 G_3$. Then we easily obtain from (8), (2), and (10) the symmetrical formula* for the *curvature of the first osculating circle*

$$(14) \qquad \frac{1}{R_1} = \frac{G_1 G_x + G_2 G_y + G_3 G_z}{\omega (G_1^2 + G_2^2 + G_3^2)},$$

where $\omega^2 = p_1^2 + p_2^2 + p_3^2$.

This convenient formula being secured we proceed in the following section

* It is not without interest to derive an expression for the curvature of the osculating circle at (x, y) of the integral curve of the ordinary differential equation $F(x, y, p) = 0$ determined by a lineal element (x, y, p) satisfying this equation. In fact, if we place $p = -p_1/p_2$, and proceed from the usual expression for curvature, it is easy to show that the resulting formula is precisely (14) written for two dimensions instead of three. In other words, *the first osculating circle here introduced is a literal generalization to three dimensions of the osculating circle of the plane.*

to apply to each of the surface elements of equation (12) the transformation designated a *turn.*

In the analytical developments which follow it will be necessary to refer to a result which may be assumed from differential geometry.

If $(\alpha_1, \alpha_2, \alpha_3)$, (l_1, l_2, l_3), and $(\lambda_1, \lambda_2, \lambda_3)$ are the direction cosines of three lines mutually at right angles, and if such directions are chosen on these lines that

$$(14') \qquad \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ l_1 & l_2 & l_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{vmatrix} = +1$$

(rather than the alternative value -1), then, indicating by primes differentiation with respect to a common parameter, relations will hold of which the following is the type:

$$(15) \qquad \alpha_i' = l_i \begin{vmatrix} l_1 & l_2 & l_3 \\ l_1' & l_2' & l_3' \\ \lambda_1 & \lambda_2 & \lambda_3 \end{vmatrix} - \lambda_i \begin{vmatrix} l_1 & l_2 & l_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1' & \lambda_2' & \lambda_3' \end{vmatrix}.$$

2. Theorems for an isogonal system

Turn now to the differential equation

$$(16) \qquad G(x, y, z, p_1, p_2, p_3) = 0,$$

and consider one of its surface elements (x, y, z, p_1, p_2, p_3). We give to the normal (λ) of this element a definite direction by choosing its direction cosines $\lambda_1, \lambda_2, \lambda_3$ such that

$$(17) \quad \lambda_1 = p_1/\omega, \qquad \lambda_2 = p_2/\omega, \qquad \lambda_3 = p_3/\omega, \qquad \omega = +\sqrt{p_1^2 + p_2^2 + p_3^2}.$$

To the tangent (l) of the characteristic through (x, y, z) in this element we assign as positive direction that whose direction cosines are (see (10))

$$(18) \quad l_1 = G_1/G, \qquad l_2 = G_2/G, \qquad l_3 = G_3/G, \qquad G = +\sqrt{G_1^2 + G_2^2 + G_3^2}.$$

For $F_p : F_q : pF_p + qF_q = G_1 : G_2 : G_3$.

We next establish an *axis* (α) through (x, y, z) in the plane of the element at right angles to (l) with direction cosines $(\alpha_1, \alpha_2, \alpha_3)$ and such that (14′) holds. A rotation of the element about the axis (α) is now entirely definite, and we may set down the

Definition. *A turn of a surface element satisfying a partial differential equation of the first order is a rotation through a given angle about the axis* (α).

The equations for a turn T_θ through an angle θ are obviously,

$$(19) \qquad \lambda_i' = \lambda_i \cos\theta + l_i \sin\theta \qquad (i = 1, 2, 3),$$

in which the direction cosines of the normal to the element in its new position are $(\lambda_1', \lambda_2', \lambda_3')$. The turned elements will also satisfy a partial differential equation of the first order. To obtain this equation we replace λ_i and l_i in (19) by their values as defined in (17) and (18), and from the three equations thus obtained and (16) eliminate p_1, p_2, and p_3. Then placing

$$\lambda_i' = p_i'/\omega', \qquad \omega' = +\sqrt{p_1'^2 + p_2'^2 + p_3'^2} \qquad (i = 1, 2, 3), \tag{20}$$

the result of the elimination may be indicated by

$$G'(x, y, z, p_1', p_2', p_3') = 0, \tag{21}$$

in which the parameter θ is implicitly involved. Thus equation (21) represents a system of differential equations, which will be called an *isogonal system*. Justification for this terminology will appear presently.

The capital fact in connection with the turn T_θ given by (19) is this. *For a fixed point* (x, y, z) *the turns form a group.* In other words, *the direction of the characteristic in a turned element is perpendicular to the axis* (α). To establish the truth of this statement the following considerations may suffice. The planes of the surface elements of the differential equation (16) with a common point P envelop a cone,*—the *elementary cone* of Monge. The elements of this cone are the tangents of the characteristics through P. A turn converts the tangent planes of this cone into those of the elementary cone of the transformed equation. *The elements of the original cone also "turn" into those of the transformed cone.* This simple geometric fact is not common knowledge but may readily be established.

An important consequence of the preceding result is the following. If (l_1', l_2', l_3') are the direction cosines of the characteristic of $G' = 0$ in the turned element $(x, y, z, p_1', p_2', p_3')$, then

$$l_i' = -\lambda_i \sin\theta + l_i \cos\theta \qquad (i = 1, 2, 3), \tag{22}$$

since this characteristic is at right angles to the axis. Here $l_i' = G_i'/G'$, and $G'^2 = \sum_{i=1}^{i=3} G_i'^2$.

It will now be possible to prove for the isogonal system (21) a theorem analogous to that of Cesàro for plane isogonal systems. The preceding presentation should have made clear the following situation.

(I) Turning an element of the original partial differential equation $G = 0$ gives rise in the isogonal system $G' = 0$ to a pencil of elements whose axis is the axis of the turn.

(II) The characteristics in this pencil of elements are at right angles to the axis.

(III) The first osculating circles for the elements of this pencil are coplanar,

* For simplicity of statement, the equation is assumed not linear. The conclusion, however, is true also for this case.

their common plane being at right angles to the axis. Moreover, these circles have by definition one common point,—the common point of the pencil of elements.

To show that these osculating circles have a *second* common point is the next step. For this purpose, apply formula (14) to the curvature of the first osculating circle in the isogonal system (21). This leads to

$$\frac{1}{R_1'} = \frac{G_1' G_x' + G_2' G_y' + G_3' G_z'}{\omega' (G_1'^2 + G_2'^2 + G_3'^2)}. \tag{23}$$

Making the convenient and permissible assumption,

$$\omega = \omega', \tag{24}$$

the transformation T_θ (19) becomes

$$p_i' = p_i \cos\theta + l_i \omega \sin\theta. \tag{25}$$

The partial derivatives of $G = 0$ and the transformed equation $G' = 0$ are related as follows. First,

$$G_x = G_x' + \sin\theta\omega \sum_{i=1}^{3} G_i' \frac{\partial l_i}{\partial x}. \tag{26}$$

Now, by (22),

$$\frac{G_i'}{G'} = -\frac{p_i}{\omega}\sin\theta + l_i \cos\theta. \tag{27}$$

In consequence, the sum in (26) is

$$G'\left(-\frac{1}{\omega}\sin\theta \sum_i p_i \frac{\partial l_i}{\partial x} + \cos\theta \sum_i l_i \frac{\partial l_i}{\partial x}\right). \tag{28}$$

From $\sum p_i l_i = 0$, and $\sum l_i^2 = 1$, this expression vanishes. Hence

$$G_x = G_x', \qquad G_y = G_y', \qquad G_z = G_z'. \tag{29}$$

Again,

$$\begin{aligned} G_1 &= G_1' \cos\theta + \sin\theta \sum_{i=1}^{3} G_i' \frac{\partial}{\partial p_1}(\omega l_i), \\ &= G_1' \cos\theta + \frac{p_1}{\omega}\sin\theta \sum G_i' l_i + \omega \sin\theta \sum G_i' \frac{\partial l_i}{\partial p_1}, \\ &= G_1' \cos\theta + \frac{G' p_1}{\omega}\sin\theta\cos\theta + G' l_1 \sin^2\theta, \end{aligned}$$

the last term arising from the fact that, since $\sum p_i l_i = 0$, then

$$\sum p_i \frac{\partial l_i}{\partial p_1} = -l_1.$$

Hence, referring to (25), the final result is

$$G_1 = G_1' \cos \theta + G' \frac{p_1'}{\omega} \sin \theta, \tag{29'}$$

with similar expressions for G_2 and G_3.

From these equations we find at once

$$\sum_i G_i^2 = \sum_i G_i'^2. \tag{30}$$

Thus the relations established in (27), (29), and (30), make it possible to write formula (23) in the form

$$\frac{1}{R_1'} = \frac{(G_1 G_x)}{\omega G^2} \cos \theta - \frac{(p_1 G_x)}{\omega^2 G} \sin \theta, \tag{31}$$

where

$$(G_1 G_x) = G_1 G_x + G_2 G_y + G_3 G_z, \qquad (p_1 G_x) = p_1 G_x + p_2 G_y + p_3 G_z.$$

From (31) follows the fact that the centers of the first osculating circles lie on a line. To see this it suffices to take as axes of rectangular coördinates the normals to the elements for $\theta = 0$, and $\theta = 90°$, and to remember that $R_1' \cos \theta$, $R_1' \sin \theta$ are the rectangular coördinates of the center of the osculating circle with respect to these axes. Hence is established

THEOREM 1. *When a surface element of a partial differential equation of the first order is turned about an axis in its plane drawn at right angles to the characteristic, the first osculating circles for the transformed elements form a coaxal system.*

The system of transformed equations

$$G'(x, y, z, p_1', p_2', p_3') = 0 \tag{21}$$

may properly be called as *isogonal system* from the following considerations. Assume S to be an integral surface of the original equation $G = 0$ (i. e., $\theta = 0$). Let us direct our attention upon an orthogonal trajectory D of the characteristics upon the surface S. Under T_θ each surface element of S along the curve D is turned through the angle θ about an axis tangent to D. Assume a fixed value for θ. The turned elements satisfy an equation of the isogonal system and form a surface band intersecting S along D under the angle θ. By the Cauchy existence theorem this band will lie on a unique integral surface of the isogonal system. Then it is clear that any given integral surface of $G = 0$ is cut under a given arbitrary constant angle θ by a one-parameter family of integral surfaces of the isogonal system (21), the curves of intersection on the given surface being the orthogonal trajectories of the characteristics on that surface. This result may be expressed in

THEOREM 2. *Any integral surface of an isogonal system will be cut by a* *one-parameter family of integral surfaces of that system under a given arbitrary constant angle along orthogonal trajectories of the characteristics.*

Consider the special case when the characteristics of the original equation $G = 0$ are *lines of curvature on all integral surfaces.* The condition therefor is found by expressing that the direction on an integral surface conjugate to that of the characteristic is at right angles to the characteristic. This conjugate direction (common to all surfaces containing the characteristic*) is given by (7) when δ and ϵ are the denominators of dp and dq, respectively, in (10). In homogeneous coördinates it is readily found that, for this common conjugate direction,

$$\delta x : \delta y : \delta z = p_2 G_z - p_3 G_y : p_3 G_x - p_1 G_z : p_1 G_y - p_2 G_x . \tag{32}$$

The condition sought, namely, $G_1 \delta x + G_2 \delta y + G_3 \delta z = 0$, reduces to

$$\alpha_1 G_x + \alpha_2 G_y + \alpha_3 G_z = 0 . \tag{33}$$

But each term in (33) is unchanged by T_θ. Hence follows

THEOREM 3. *In the isogonal system $G' = 0$ if the characteristics are lines of curvature on all integral surfaces for any value of θ, then this is true for all values.*

The class of partial differential equations of the first order for which the characteristics are lines of curvature on all integral surfaces forms therefore an invariant system under T_θ, and has been studied from this point of view by Liebmann.† Theorem 1 takes a special form in this case (as proved by Liebmann), since the first osculating circles are now great circles on the principal spheres of one system. The paper of Liebmann deals with these spheres only.

In a plane isogonal system the integral curves may be arranged in isogonal nets, or two one-parameter systems such that each curve of one system will intersect all those of the other under the same angle θ. We inquire if corresponding double families of surfaces exist among the integral surfaces of the isogonal system $G' = 0$. Let S be an integral surface of $G = 0$. For convenience, denote the family (Theorem 2) cutting S by S'_θ. Consider those characteristics on the surfaces S'_θ which intersect one of the characteristics on S. The locus of these ∞^1 characteristics is a surface orthogonal to S, and hence arises a system of surfaces T orthogonal to S intersecting it along the characteristics, and also containing the characteristics on the S'_θ. If now S belongs to a one-parameter family S_0 of integral surfaces of $G = 0$ each of which stands in the same relation to the system S'_θ as S, then the T and S'_θ surfaces must intersect everywhere orthogonally. Furthermore, this

* See p. 524.

† *Aequitangential- und Isogonaltransformationen der partiellen Differentialgleichungen* D_{12}, Rendiconti del Circolo Matematico di Palermo, vol. 29 (1910), p. 139.

fact is to be independent of θ. Consider, in particular, $\theta = 90°$. The T, the S'_θ, and the S_0 surfaces now form a triple orthogonal system and hence intersect along lines of curvature. That is, a *necessary condition* is that the characteristics shall be lines of curvature on the S_0 surfaces. But in this case the characteristics are lines of curvature on the S'_θ for all θ (Theorem 3). Since the T and S'_θ surfaces for every θ are orthogonal, it appears that the characteristics on the S'_θ must be lines of curvature on the T surfaces *for every* θ, that is, the T surfaces must be plane or spherical. A necessary condition for the aforesaid arrangement is thus readily seen to be that all integral surfaces shall be developable or canal surfaces. The isogonal net of plane curves is, therefore, not duplicated for the case under discussion in any general sense.

Certain points in the preceding argument have excluded the linear equation

$$Xp + Yq - Z = 0, \tag{34}$$

in which X, Y, Z are functions of x, y, z, only. The isogonal system arising here is

$$Xp + Yq - Z + \sin\theta\,\sqrt{1 + p^2 + q^2}\,\sqrt{X^2 + Y^2 + Z^2} = 0, \tag{35}$$

in which the primes on p and q have been dropped. The elementary cones in (35) for a given value of θ are congruent cones of revolution whose axes are the tangents of the characteristics of (34). Equation (35) may be written in the alternative form

$$\left\|\begin{matrix} X & Y & Z \\ p & q & -1 \end{matrix}\right\|^2 = \cos^2\theta, \tag{36}$$

the left-hand member being the sum of the squares of the three determinants of the matrix. For $\theta = 90°$, real integral surfaces exist only if the characteristics of (34) form a normal congruence, and in that case, there are a simple infinity of such surfaces only.

3. Theorems for an equitangential system

In the differential equation

$$F(x, y, z, p, q) = 0 \tag{9}$$

we may inquire as to the locus of the points of the surfaces elements having a common plane

$$z = px + qy + c. \tag{37}$$

This locus is plainly the curve* arising by regarding (9) and (37) simultaneous equations with p and q constant. We may call this curve the *ele-*

* Caratheodory (l. c.) calls this curve "die Elementplankurve."

mentary curve of the plane. What is the direction of the tangent of this curve? The infinitesimals dx, dy, and dz at a point on the curve satisfy

$$(38) \qquad F_x\,dx + F_y\,dy + F_z\,dz = 0, \qquad dz = p\,dx + q\,dy,$$

from which the ratios $dx : dy : dz$ are found to be

$$(39) \qquad \frac{dx}{-F_y - qF_z} = \frac{dy}{F_x + pF_z} = \frac{dz}{qF_x - pF_y}.$$

This direction in the element (x, y, z, p, q) is precisely the second of the two directions referred to in § 1, namely, *that direction on all integral surfaces containing the element which is conjugate to the direction of the characteristic,*— the *common conjugate direction,* as we shall call it.

It is well now to place in juxtaposition the following statements which appear dual.

(*a*) The planes of the elements satisfying $F = 0$ and having a common point envelop a cone, *the elementary cone of the point.* The elements of the cone are tangent to the characteristics through the point.

(*b*) The points of the elements satisfying $F = 0$ and having a common plane lie on a curve, *the elementary curve of the plane.* The tangents of this curve are the common conjugate directions in the plane.

These facts justify us in assigning to the tangents of the elementary curve of a plane a rôle in the geometry of partial differential equations equal in importance to that of the tangents of the characteristics. This we do in the following developments.

Let (c_1, c_2, c_3) be the direction cosines of the common conjugate direction. Then by (32),

$$(40) \qquad c_1 : c_2 : c_3 = \left\| \begin{matrix} p_1 & p_2 & p_3 \\ G_x & G_y & G_z \end{matrix} \right\|,$$

the notation meaning that the c's are proportional to the corresponding second order determinants of the matrix of the right-hand member.

In § 2 we defined a turn of a surface element of a partial differential equation of the first order. We now define a slide.

Definition. *A slide of a surface element satisfying a partial differential equation of the first order is a translation through a given distance in its plane in a direction perpendicular to the common conjugate direction.*

Applied to a given equation

$$G(x, y, z, p_1, p_2, p_3) = 0,$$

a slide S_a through the distance a transforms the elements into those satisfying a new equation

$$(41) \qquad G'(x', y', z', p_1, p_2, p_3) = 0.$$

The transformed elementary curve in any plane is obviously a *parallel curve* of the original elementary curve. In fact, it is precisely this phase of the matter which suggested the direction of the slide. An immediate result is expressed in

THEOREM 4. *Under a slide the common conjugate direction in an element remains parallel to itself.*

If $(\sigma_1, \sigma_2, \sigma_3)$ are the direction cosines of the line along which the point of the element moves, then the equations of the slide S_a are

$$x' = x + a\sigma_1, \qquad y' = y + a\sigma_2, \qquad z' = z + a\sigma_3. \tag{42}$$

For a given equation $G = 0$, the slides S_a form a group.

It is our purpose to prove a theorem for space analogous to Scheffers' result already quoted for plane equitangential systems. To do this a second osculating circle must be associated with a given pair of united surface elements. Consider a developable surface upon which these elements lie. The direction of the generator of this developable is the common conjugate direction (c). From equations (3), § 1, we find the relation*

$$\alpha\beta(rt - s^2) = \delta\epsilon - (\alpha\delta + \beta\epsilon)s. \tag{43}$$

For a developable surface, $rt - s^2 = 0$, or†

$$s = \frac{\delta\epsilon}{\alpha\beta + \delta\epsilon}. \tag{44}$$

Putting this value of s in equation (5), we find for the curvature of normal sections of the developable,

$$\frac{1}{R_2} = \frac{1}{\sqrt{1 + p^2 + q^2}} \frac{(\delta\,\delta x + \epsilon\,\delta y)^2}{(\alpha\delta + \beta\epsilon)\,\delta s^2}. \tag{45}$$

Apply this to the direction (σ). For this purpose we merely have to set $\sigma_1 = \delta x/\delta s$, $\sigma_2 = \delta y/\delta s$. Replacing in (45) the values of $\alpha, \beta, \delta, \epsilon$, etc., in homogeneous coördinates as in § 1, simple reductions lead to the result

$$\frac{1}{R_2} = \frac{(\sigma_1 G_x + \sigma_2 G_y + \sigma_3 G_z)^2}{\omega(G_1 G_x + G_2 G_y + G_3 G_z)}, \tag{46}$$

giving the radius R_2 of the *second osculating circle*.

It will be well to describe this *second osculating circle*. The integral surfaces along a characteristic have in common a *characteristic surface band* (in Lie's terminology). This characteristic band will lie on a developable, the *characteristic developable*. The second osculating circle for an element $(x, y, z,$

* See the paper by the author already cited.

† We assume $\alpha\beta + \delta\epsilon \neq 0$, the meaning of which is that the characteristic and common conjugate directions in the element (x, y, z, p, q) are distinct.

p, q) of the characteristic band is the circle osculating the normal section of the characteristic developable which is at right angles to the generator. The first osculating circle is, moreover, the osculating circle of this developable in a normal section along the characteristic. In fact (45) reduces to (8) when $\delta x : \delta y = \alpha : \beta$. The two osculating circles are therefore associated with this developable. Moreover, *when the characteristics are lines of curvature, the two osculating circles are identical.* For the generator of the developable is now at right angles to the characteristic. If ϕ is the angle between the characteristic and common conjugate directions, then, by Euler's Theorem, $R_2 = R_1 \sin^2 \phi$.

Return now to an element (x, y, z, p_1, p_2, p_3) of the differential equation $G = 0$. Consider the second osculating circles of the ∞^1 transformed elements under S_a. These circles lie in a plane (perpendicular to the conjugate direction (c)), and touch the line which is the path of the point of the transformed element. But these ∞^1 circles also *touch a second line,* as we now prove.

Under the transformation

$$x' = x + a\sigma_1, \qquad y' = y + a\sigma_2, \qquad z' = z + a\sigma_3, \tag{42}$$

with $p_1 : p_2 : p_3$ remaining unchanged, the given equation $G = 0$ is transformed into the system

$$G'(x', y', z', p_1, p_2, p_3) = 0. \tag{41}$$

For this equation the curvature of the second osculating circle is, by (46),

$$\frac{1}{R'_2} = \frac{(\sigma_1 G'_{x'} + \sigma_2 G'_{y'} + \sigma_3 G'_{z'})^2}{\omega(G'_1 G'_{x'} + G'_2 G'_{y'} + G'_3 G'_{z'})}. \tag{46'}$$

The partial derivatives of $G = 0$ and $G' = 0$ under (42) are related as follows:

$$G_x = G'_{x'} + a\left(G'_{x'}\frac{\partial \sigma_1}{\partial x} + G'_{y'}\frac{\partial \sigma_2}{\partial x} + G'_{z'}\frac{\partial \sigma_3}{\partial x}\right). \tag{47}$$

The directions (c), (σ), and (λ) (equation (17)), are mutually at right angles. Assume

$$\begin{vmatrix} \sigma_1 & \sigma_2 & \sigma_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = +1. \tag{48}$$

Then the application of (15) gives

$$\frac{\partial \sigma_i}{\partial x} = -c_i \begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ c_1 & c_2 & c_3 \\ \dfrac{\partial c_1}{\partial x} & \dfrac{\partial c_2}{\partial x} & \dfrac{\partial c_3}{\partial x} \end{vmatrix} \qquad (i = 1, 2, 3). \tag{49}$$

Denoting the determinant in the right-hand member by Δ_x, equation (47) becomes

$$G_x = G'_{x'} - a\Delta_x (G'_{x'} c_1 + G'_{y'} c_2 + G'_{z'} c_3). \tag{50}$$

Similar equations hold for G_y and G_z.

Now $c_1 G_x + c_2 G_y + c_3 G_z = 0$ by (40). Hence from (50) will follow the equation

$$(G'_{x'} c_1 + G_{y'} c_2 + G_{z'} c_3)(1 - a(c_1 \Delta_x + c_2 \Delta_y + c_3 \Delta_z)) = 0, \tag{51}$$

which must hold for *all values of* a. In consequence, the first factor must be zero, and hence G_x, G_y, and G_z *remain unaltered by a slide.* The same fact was observed for the turn S_θ (equations (29)). Next,

$$G_1 = G'_1 + a\left(G'_{x'} \frac{\partial \sigma_1}{\partial p_1} + G'_{y'} \frac{\partial \sigma_2}{\partial p_1} + G'_{z'} \frac{\partial \sigma_3}{\partial p_1} \right). \tag{52}$$

Referring again to (15), we may write

$$\frac{\partial \sigma_i}{\partial p_1} = A\lambda_i + Bc_i, \tag{53}$$

in which A and B are determinants. Moreover, $A = -\sigma_1/\omega$, when equations (17) are used. We thus find the final form

$$G'_1 = G_1 + \frac{\sigma_1}{\omega^2}(p_1 G_x) a, \tag{54}$$

and similar equations for G'_2 and G'_3, the same abbreviation $(p_1 G_x)$ being used as in (31).

Substitution in (46′) gives

$$\frac{1}{R'_2} = \frac{(\sigma_1 G_x)^2}{\omega(G_1 G_x) + a(p_1 G_x)(\sigma_1 G_x)/\omega}, \tag{55}$$

the parentheses standing for sums of three terms as before. But this equation shows that the centers of the second osculating circles *lie on a line.* For the coördinates of their centers referred to the line (σ) and the diameter of the circle for $a = 0$ as axes, are R'_2 and a, and, in these coördinates, (55) is the equation of a straight line. Hence

THEOREM 5. *When a surface element of a partial differential equation of the first order is slid in a direction at right angles to the common conjugate direction in the element the center of the second osculating circle describes a line.*

The system of transformed equations

$$G'(x', y', z', p_1, p_2, p_3) = 0 \tag{41}$$

may aptly be called an *equitangential system.* For let S be an integral surface for $a = 0$. Under S_a each surface element of S is displaced a distance a in a direction at right angles to the common conjugate direction. Consider therefore on S the system of curves (Σ) which are the orthogonal trajectories of the system conjugate to the characteristics.* Then each element of S slides along a tangent of a (Σ) curve. Draw the system on S conjugate to the (Σ) curves, and consider one of these curves,—call it γ. Circumscribe to S along γ the tangent developable Γ. The surface elements of S along γ become under S_a for any arbitrary given a a surface band along a curve γ_a upon the tangent developable Γ such that the corresponding surface elements of S and of this band have a common distance of length a. This band along γ_a determines a unique integral surface S_a of the system $G' = 0$. This surface and S have a common tangent developable and the distance on any generator of this developable between the points of contact with S and S_a is constant and equal to a. That is, S and S_a are *equitangential surfaces.* Accordingly we have

THEOREM 6. *Associated with any integral surface in an equitangential system is a one-parameter family of integral surfaces each of which is an equitangential surface of the original surface. The curves of contact of the common tangent developables are conjugate to the orthogonal trajectories of the system conjugate to the characteristics.*

In the preceding section it appeared that the partial differential equations of the first order for which the characteristics are lines of curvature on all integral surfaces form an invariant class under T_θ. A corresponding theorem is true when turns are replaced by slides. For the condition (33) applied to (41) is

$$\alpha_1' G_x + \alpha_2' G_y + \alpha_3' G_z = 0.$$

Now $\alpha_1' = l_2' \lambda_3' - l_3' \lambda_2' = l_2' \lambda_3 - l_3' \lambda_2$, etc., and the l_i' are proportional to the G_i' of equation (54), say $l_i' = \rho G_i'$. Then

$$\alpha_1' = \rho (G_2 \lambda_3 - G_3 \lambda_2) + \frac{a\rho}{\omega^2} (p_1 G_x)(\sigma_2 \lambda_3 - \sigma_3 \lambda_2).$$

Hence

$$\alpha_1' G_x + \alpha_2' G_y + \alpha_3' G_z = \rho' (\alpha_1 G_x + \alpha_2 G_y + \alpha_3 G_z), \tag{56}$$

where $\rho' = \sqrt{G_1^2 + G_2^2 + G_3^2}/\sqrt{G_1'^2 + G_2'^2 + G_3'^2}$. We may therefore state a companion theorem to Theorem 3 as

THEOREM 6. *In the equitangential system $G' = 0$ if the characteristics are lines of curvature on all integral surfaces for any value of a, then this is true for all values of a.*

This result was established by Liebmann in the paper cited above.

From formula (55) it appears that R_2' vanishes for one value of a, that is,

* The point is here that the system conjugate to the characteristics takes the rôle assumed by the characteristics in § 2.

$(G_1' \, G_x) = 0$ for the corresponding surface element. In deriving (45) exception was made of the case $\alpha\delta + \beta\epsilon = 0$, namely, when the common conjugate direction and the characteristic direction coincide. This exceptional case occurs, however, normally for an element of the transformed system, the vanishing of $(G_1' \, G_x)$ being precisely this case. To include all cases, therefore, we may assume in (46) that $(G_1 \, G_x)$ may vanish, and no flaw in the argument developed will be occasioned thereby.

4. General theorems

The result established in § 2 as stated in Theorem 1 suggests an investigation along the following lines.

In the differential equation

$$G(x, y, z, p_1, p_2, p_3) = 0, \tag{12}$$

the first osculating circles for an element (x, y, z, p_1, p_2, p_3) and the elements derived by turning it form a coaxal system passing through a common second point $\bar{P}(\bar{x}, \bar{y}, \bar{z})$. Are these circles also first osculating circles at $\bar{P}$ for a second isogonal system of differential equations? This query is the more natural since Scheffers has shown, in the memoir cited above, that the corresponding theorem holds for a plane isogonal system.

Begin by determining the direction of the axis of the coaxal system. Let C be the angle between this axis and the normal to the surface element $E : (x, y, z, p_1, p_2, p_3)$ at P. Obviously C is the value of θ in (31) when R_1' becomes infinite, and hence

$$\tan C = \frac{\omega \, (l_1 \, G_x)}{(p_1 \, G_x)}. \tag{57}$$

For convenience, denote the first osculating circle for the element E by c_0. The radius R_1 of this circle is given by

$$\frac{1}{R_1} = \frac{(l_1 \, G_x)}{\omega \, (G)}.$$

Since the circle c_0 is to be the first osculating circle at $\bar{P}$ for an element $\bar{E} : (\bar{x}, \bar{y}, \bar{z}, \bar{p}_1, \bar{p}_2, \bar{p}_3)$ of an equation

$$\bar{G}(\bar{x}, \bar{y}, \bar{z}, \bar{p}_1, \bar{p}_2, \bar{p}_3) = 0, \tag{58}$$

simple considerations show that the relations between the coördinates of E and $\bar{E}$ are

$$\bar{p}_i = p_i \cos 2C + \omega \, l_i \sin 2C \qquad (i = 1, 2, 3),$$

$$\bar{x} = x + \frac{R_1}{\omega}(\bar{p}_1 - p_1), \qquad \bar{y} = y + \frac{R_1}{\omega}(\bar{p}_2 - p_2), \tag{59}$$

$$\bar{z} = z + \frac{R_1}{\omega}(\bar{p}_3 - p_3).$$

In other words, equation (57) results from the given equation $G = 0$ by the transformation (59).

Express now the partial derivatives of the original equation in terms of those of the transformed equation (58). Omitting details, the relations are as follows.

$$G_x = \bar{G}_x + \frac{1}{\omega} R_{1x} H + 2C_x (I_1 g_1) - \sin 2C (p_1 \alpha_2 \alpha_{3x}) K, \tag{60}$$

and similar equations for G_y and G_z resulting when the subscript x has been replaced by y, etc. Moreover,

$$H = (\bar{p}_1 \bar{G}_{\bar{x}}) - (p_1 \bar{G}_{\bar{x}}) = 2 \sin C (\omega \cos C (l_1 \bar{G}_{\bar{x}}) - \sin C (p_1 \bar{G}_{\bar{x}})),$$

$$I_1 = \frac{R_1}{\omega} \bar{G}_{\bar{x}} + \bar{G}_1, \qquad I_2 = \frac{R_1}{\omega} \bar{G}_{\bar{y}} + \bar{G}_2, \qquad I_3 = \frac{R_1}{\omega} \bar{G}_{\bar{z}} + \bar{G}_3, \tag{61}$$

$$g_i = - p_i \sin 2C + \omega l_i \cos 2C \qquad (i = 1, 2, 3),$$

$$K = (\alpha_1 I_1) = \frac{R_1}{\omega} (\alpha_1 \bar{G}_{\bar{x}}) + (\alpha_1 \bar{G}_1),$$

and $(p_1 \alpha_2 \alpha_{3x})$ is a determinant with the usual notation. The partial derivatives with respect to p_1, p_2, and p_3 are expressed by equations of which the following is typical.

$$\begin{aligned} G_1 = \bar{G}_1 + \left(\frac{R_0}{\omega}\right)_1 H + 2C_1 (I_1 g_1) - 2 \sin^2 C\, I_1 \\ + \sin 2C \left(\frac{p_1}{\omega} (l_1 I_1) - \frac{l_1}{\omega} (p_1 I_1) - (p_1 \alpha_2 \alpha_{31}) K \right). \end{aligned} \tag{62}$$

Consider now the *necessary* conditions attached to the assumption that c_0 is the first osculating circle of the element $\bar{E}$ of the differential equation $\bar{G} = 0$. Using, as above, a superior bar on symbols associated with $\bar{G} = 0$, then, since the characteristic in $\bar{E}$ must be tangent to c_0 at $\bar{P}$, we must have

$$\frac{\bar{G}_i}{(\bar{G})} = \bar{l}_i = \frac{p_i}{\omega} \sin 2C - l_i \cos 2C = - g_i/\omega \qquad (i = 1, 2, 3), \tag{63}$$

in which $(\bar{G}) = + \sqrt{\bar{G}_1^2 + \bar{G}_2^2 + \bar{G}_3^2}$. Then from (61),

$$\begin{aligned} (I_1 g_1) = \frac{-\omega}{(\bar{G})} (I_1 \bar{G}_1) = \frac{-\omega}{(\bar{G})} \left[\frac{R_1}{\omega} (\bar{G}_1 \bar{G}_{\bar{x}}) + (\bar{G}_1^2) \right] \\ = - \frac{\omega (\bar{G}_1 \bar{G}_{\bar{x}})}{(\bar{G})} \left(\frac{R_1}{\omega} - \frac{\bar{R}_1}{\omega} \right). \end{aligned} \tag{64}$$

Hence the necessary condition $(I_1 g_1) = 0$, since R_1 must equal $\bar{R}_1$.

Recognizing next that the angle C must be the same whether associated with $G = 0$, or $\bar{G} = 0$, we have, comparing with (57),

$$\tan \bar{C} = \frac{\bar{\omega}(\bar{l}_1 \bar{G}_{\bar{x}})}{(\bar{p}_1 \bar{G}_{\bar{x}})} = \tan C.$$

This condition leads easily to $H = 0$. To complete the discussion, multiply equation (62) by α_1, and form the sum $(\alpha_1 G_1)$. Using the conditions $H = (I_1 g_1) = 0$ already found, and remembering that $(\alpha_1 G_1) = (\alpha_1 \bar{G}_1) = 0$, we find that $K = 0$. Then from equations (61) we see that $(\alpha_1 \bar{G}_{\bar{x}}) = 0$, and since, from (59), $G_x = \bar{G}_{\bar{x}}$, also $(\alpha_1 G_x) = 0$. But this condition holds when and only when the characteristics are lines of curvature on all integral surfaces, as shown in (33), Conversely, assuming $(\alpha_1 G_x) = 0$, we may readily show that $(\alpha_1 \bar{G}_1) = H = (I_1 g_1) = K = 0$, and hence this is a sufficient condition. The details in this demonstration may be omitted since the sufficiency condition is implicity demonstrated in the paper of Liebmann referred to above. Thus we have

THEOREM 7. *The first osculating circles for a pencil of corresponding surface elements of an isogonal system $G' = 0$ have the same relation to a second isogonal system at their second common point of intersection when and only when the characteristics are lines of curvature on all integral surfaces in both systems.*

For an equitangential system the discussion proceeds along similar lines with a similar result. Recalling that the configuration arising by continuously sliding an element is a flat straight surface band, we may state

THEOREM 8. *The second osculating circles for a flat straight band of corresponding elements of an equitangential system have the same relation to a second equitangential system along their second common tangent when and only when the characteristics are lines of curvature on all integral surfaces in both systems.*

When an element of $G = 0$ is turned, a relation between the radii of the first and second osculating circles may be derived in the form

$$\frac{1}{R_2} = \frac{1}{R_1} + lR_1, \qquad l = \frac{(\alpha_1 G_x)^2}{\omega^2 (G_1^2)}, \tag{65}$$

and l is an invariant of the turn T_θ. Similarly, when the element is slid,

$$R_1 = R_2 + \frac{L}{R_2}, \qquad L = \frac{(\alpha_1 G_x)^2 \omega^2 (G_1^2)}{(\sigma_1 G_x)^4}, \tag{66}$$

and L is invariant under S_a. The conclusion may be drawn at once that the center of the second osculating circle describes a conic when the element is turned, a similar property holding for the center of the first osculating circle when the element is slid.

In the preceding discussion the derivatives G_x, G_y, and G_z remained unchanged under both transformations T_θ and S_a. The interpretation of this fact possesses geometric interest. When an element is turned, the common conjugate direction moves in a plane whose normal has the direction established by the ratios $G_x : G_y : G_z$. When an element is transformed by S_a, the line of centers of the second osculating circles has this direction.

One final detail may be mentioned. When an element is slid, the characteristic direction turns about a fixed point in the plane of the element. That is, the tangents to the characteristics form a pencil. Denote the vertex of this pencil by V. Direct the attention upon an element at the point P. When this element is turned, the line of centers of the first osculating circles will pierce the plane of the element in a point W. It may be shown without difficulty that P is the mid-point of V and W.

SHEFFIELD SCIENTIFIC SCHOOL
May, 1917

ERRATA, VOLUME 18

Page 73. T. H. HILDEBRANDT. *On a theory of linear differential equations in general analysis.*

Page 79, line 25, the expression

$$\sum_{k=1}^{n} c_{ik}\, y_{0kj}(x)$$

should be replaced by

$$\sum_{k=1}^{n} y_{0ik}(x)\, c_{kj}\,.$$