ON BOUNDARY VALUE PROBLEMS IN LINEAR DIFFERENTIAL EQUATIONS IN GENERAL ANALYSIS*

BY

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In a paper *On a theory of linear differential equations in general analysis*,† we considered the solution of the general linear differential equations

$$(A) \qquad M_1(\eta) = D\eta - \alpha - J\alpha\eta = 0,$$

$$(B) \qquad M_2(\eta) = D\eta - J\alpha\eta = 0,$$

$$(C) \qquad M_2(\eta) - \alpha_0 = D\eta - J\alpha\eta - \alpha_0 = 0,$$

and their adjoints

$$(A') \qquad N_1(\hat{\eta}) = D\hat{\eta} + \alpha + J\hat{\eta}\alpha = 0,$$

$$(B') \qquad N_2(\hat{\eta}) = D\hat{\eta} + J\hat{\eta}\alpha = 0,$$

$$(C') \qquad N_2(\hat{\eta}) - \alpha_0 = D\hat{\eta} + J\hat{\eta}\alpha - \alpha_0 = 0.$$

In the case of equations (A), (B), and (C) we found‡ that the general solution of class $\mathfrak{H}'$ is expressible in the form

$$\eta = \kappa + J\eta_0\kappa + \eta_1,$$

in which η_1 is a particular solution of the equation in question, η_0 is a solution of equation (A) whose Fredholm determinant is not zero, and κ is any function of the class $\mathfrak{K}$. Similarly, the general solution of equations (A'), (B'), and (C') has the form

$$\hat{\eta} = \kappa + J\kappa\hat{\eta}_0 + \hat{\eta}_1.$$

We found further that there is a unique solution of each of these equations which satisfies an initial condition of the form

$$\eta(x_1) = \kappa_0,$$

where x_1 is any element of $\mathfrak{X}$, and κ_0 any function of $\mathfrak{K}$.

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* Presented to the Society, April 6, 1917.

† These Transactions, vol. 18 (1917), pp. 73–96. This paper will be referred to as I in the sequel. With the exception of an additional condition on J in §§ 2 and 3, the postulates and properties of the classes $\mathfrak{M}'$, $\mathfrak{M}''$, $\mathfrak{H}$, $\mathfrak{H}'$, and $\mathfrak{K}$, and the functions α, η, and κ, and the operators J and D in the present paper are the same as in I.

‡ Cf. I, loc. cit., pp. 84, 86, 87.

In Section 1 of this paper we consider the solutions of the equations (A), (B), and (C), and (A'), (B'), and (C'), whose values at two or more elements of $\mathfrak{X}$ satisfy a linear relation, i. e., a linear boundary condition. Section 2 is devoted to the definition of adjoint systems of boundary conditions. Section 3 derives the usual theorems concerning the interrelations of solutions of adjoint systems. They are in the main similar to those derived in the first three sections of the paper by Bôcher on *Applications and generalizations of the concept of adjoint systems** and include them as special cases. We have not given particular instances of the theory developed in the following pages. It is an easy matter to construct some of the more important ones along the lines outlined in §§ 12–14 of I.† We might remark, however, that if we let $\mathfrak{P}' \equiv \mathfrak{P}'' \equiv [0 \leqq p \leqq 1]$ and $\mathfrak{M}' \equiv \mathfrak{M}'' \equiv$ the class of all continuous functions on the interval $(0, 1)$ and the operator J be the definite integral $\int_0^1 dp$, we have below a theory of linear boundary value problems for linear integro-differential equations.

1. **Boundary conditions.** The boundary conditions which we consider are linear in the values of the function η at two points x_1 and x_2 of the class $\mathfrak{X}$. The relations are further chosen so that the substitution of the general solution of the differential equations (A), (B), or (C) reduces the determination of the function κ to the solution of a linear general integral equation of the second or Fredholm type, i. e., one to which the Fredholm-Moore‡ theory is applicable. Such boundary conditions have the form§

$$S(\eta) = c\eta(x_1) + J\sigma_1 \eta(x_1) + d\eta(x_2) + J\sigma_2 \eta(x_2) = 0,$$

$$S(\eta) = \sigma_0,$$

in which σ_1, σ_2, and σ_0 are functions of the class $\mathfrak{K}$, and c and d are constants for which $c + d \neq 0$. By dividing by $c + d$, we get equivalent conditions with $c + d = 1$. We assume in the sequel that c and d satisfy this relation. In so far as the expression $S(\eta) + \sigma_1 + \sigma_2$ is of frequent occurrence, we denote it by $S_0(\eta)$.

In a similar way, we consider relative to the adjoint differential equations (A'), (B'), and (C') boundary conditions of the form:

$$T(\hat{\eta}) = c\hat{\eta}(x_1) + J\hat{\eta}(x_1)\tau_1 + d\hat{\eta}(x_2) + J\hat{\eta}(x_2)\tau_2 = 0,$$

$$T(\hat{\eta}) = \tau_0,$$

* These Transactions, vol. 14 (1913), pp. 403–420.

† Loc. cit., pp. 87–96.

‡ Cf. E. H. Moore: Bulletin of the American Mathematical Society, vol. 18 (1912), pp. 351–361.

§ If the classes $\mathfrak{P}' = \mathfrak{P}'' = [1, 2, \cdots, n]$ and $J = \Sigma_1^n$, these conditions reduce to the boundary conditions considered by Bounitzky: Journal de Mathematiques, ser. 6, vol. 5 (1909), p. 68.

where τ_1, τ_2, and τ_0 are functions of the class $\mathfrak{K}$, and $c + d = 1$. We set

$$T_0(\hat{\eta}) = T(\hat{\eta}) + \tau_1 + \tau_2.$$

The following propositions are easily obtained by making the required substitutions and rearranging the terms properly.

$$(1) \qquad S(\eta_1 + \eta_2) = S(\eta_1) + S(\eta_2),$$

$$(2) \qquad S(J\eta\kappa) = JS(\eta)\kappa,$$

$$(3) \qquad S(\kappa + J\eta\kappa) = \kappa + JS_0(\eta)\kappa,$$

$$(4) \qquad S_0(\eta_1 + \eta_2) = S_0(\eta_1) + S(\eta_2),$$

$$(5) \qquad S_0(\kappa + \eta + J\eta\kappa) = \kappa + S_0(\eta) + JS_0(\eta)\kappa,$$

$$(1') \qquad T(\hat{\eta}_1 + \hat{\eta}_2) = T(\hat{\eta}_1) + T(\hat{\eta}_2),$$

$$(2') \qquad T(J\kappa\hat{\eta}) = J\kappa T(\hat{\eta}),$$

$$(3') \qquad T(\kappa + J\kappa\hat{\eta}) = \kappa + J\kappa T_0(\hat{\eta}),$$

$$(4') \qquad T_0(\hat{\eta}_1 + \hat{\eta}_2) = T_0(\hat{\eta}_1) + T(\hat{\eta}_2),$$

$$(5') \qquad T_0(\kappa + \hat{\eta} + J\kappa\hat{\eta}) = \kappa + T_0(\hat{\eta}) + J\kappa T_0(\hat{\eta}).$$

These propositions immediately give rise to the following theorems relating to the systems

$$(1) \begin{cases} M_1(\eta) = D\eta - \alpha - J\alpha\eta, \\ S_0(\eta) = 0, \end{cases} \qquad (1') \begin{cases} N_1(\hat{\eta}) = D\hat{\eta} + \alpha + J\hat{\eta}\alpha, \\ T_0(\hat{\eta}) = 0, \end{cases}$$

$$(2) \begin{cases} M_2(\eta) = D\eta - J\alpha\eta = 0, \\ S(\eta) = 0, \end{cases} \qquad (2') \begin{cases} N_2(\hat{\eta}) = D\hat{\eta} + J\hat{\eta}\alpha = 0, \\ T(\hat{\eta}) = 0, \end{cases}$$

$$(3) \begin{cases} M_2(\eta) = \alpha_0, \\ S(\eta) = \sigma_0, \end{cases} \qquad (3') \begin{cases} N_2(\hat{\eta}) = \alpha_0, \\ T(\hat{\eta}) = \tau_0. \end{cases}$$

THEOREM I. *A necessary and sufficient condition that the system* (1) [(1′)] *has a solution is that the Fredholm determinant of* $S_0(\eta_0)$ [$T_0(\hat{\eta}_0)$] *be different from zero,* η_0 [$\hat{\eta}_0$] *being a solution of equation* (*A*) [(*A′*)] *whose Fredholm determinant is not zero.*

For if η_0 is a particular solution of $M_1(\eta) = 0$, whose Fredholm determinant is not zero, the general solution can be written

$$\eta = \kappa + \eta_0 + J\eta_0\kappa,$$

and so κ must satisfy the equation

$$S_0(\eta) = \kappa + S_0(\eta_0) + JS_0(\eta_0)\kappa = 0.$$

This has the form of a reciprocal relation. It has a solution and one only if the Fredholm determinant of $S_0(\eta_0)$ is not zero.*

If $S^{-1}(\eta_0)$ is the reciprocal of $S_0(\eta_0)$, then we have the

COROLLARY. *If a solution* η *of class* $\mathfrak{H}'$ *of the system* (1) *exists, it has the form*

$$\eta = S^{-1}(\eta_0) + \eta_0 + J\eta_0 S^{-1}(\eta_0).$$

Denoting, for convenience, by η_0 [$\hat{\eta}_0$] a solution of equation (A) [(A')] whose Fredholm determinant is not zero, we get in a similar way by using Propositions (3) [(3′)]:

THEOREM II. *A necessary and sufficient condition that there exist a solution of the system* (2) [(2′)] *which is not identically zero, is that the Fredholm determinant of* $S_0(\eta_0)$ [$T_0(\hat{\eta}_0)$] *be zero. If* $\mu_1', \cdots, \mu_n'$ [$\mu_1'', \cdots, \mu_n''$] *are a complete set of linearly independent solutions of*

$$\mu' + JS_0(\eta_0)\mu' = 0, \qquad [\mu'' + J\mu''\, T_0(\hat{\eta}_0) = 0],$$

then the general solution of the system (2) [(2′)] *can be written*

$$\eta = \sum_{m=1}^{n} (\mu_m' + J\eta_0\, \mu_m')\, \mu_m'', \qquad \left[\hat{\eta} = \sum_{m=1}^{n} \mu_m' (\mu_m'' + J\mu_m''\, \hat{\eta}_0)\right],$$

where μ_m'' [μ_m'] *are any n functions of the class* $\mathfrak{M}''$ [$\mathfrak{M}'$].

In such a case, the system (2) [(2′)] is said to have *n-fold compatibility.*

Using propositions (1) and (3) [(1′) and (3′)], and denoting by η_1 [$\hat{\eta}_1$] a particular solution of equation (C) [(C')], we have

THEOREM III. *The system* (3) [(3′)] *has a unique solution if the Fredholm determinant of* $S_0(\eta_0)$ [$T_0(\hat{\eta}_0)$] *is not zero. If this determinant is zero, then a necessary and sufficient condition for the existence of a solution of this system is that*

$$J\kappa(S(\eta_1) - \sigma_0) = 0, \qquad [J(T(\hat{\eta}_1) - \tau_0)\kappa = 0],$$

for every solution κ *of the homogeneous equation*

$$\kappa + J\kappa S_0(\eta_0) = 0, \qquad [\kappa + JT_0(\hat{\eta}_0)\kappa = 0].$$

A similar set of propositions and theorems can be derived if the boundary conditions are linear relations in the values of the solution of the differential equations at n points of $\mathfrak{X} : x_1, \cdots, x_n$. For the equation ($B$) the conditions take the form

$$S(\eta) = \sum_{m=1}^{n} ((c_m\, \eta(x_m) + J\sigma_m\, \eta(x_m)) = 0, \qquad S(\eta) = \sigma_0,$$

where $\sigma_1, \cdots, \sigma_n$, and σ_0 are any functions of $\mathfrak{K}$ and $c_1, \cdots, c_n$ a set of constants for which

$$c_1 + c_2 + \cdots + c_n = 1.$$

* Cf. E. H. Moore: loc. cit., p. 354.

2. **On adjoint systems.** Before taking up the definition of adjoint systems, it will be necessary to add a further postulate on the operator J. In this and the succeeding paragraphs, we shall assume that J has the following property:

$$\text{If } J\mu_0''\mu' = 0 \text{ for every } \mu' \text{ of the class } \mathfrak{M}', \text{ then } \mu_0'' \equiv 0,$$

and

$$\text{if } J\mu''\mu_0' = 0 \text{ for every } \mu'' \text{ of the class } \mathfrak{M}'', \text{ then } \mu_0' \equiv 0.$$

This property corresponds to the definite property P_0 of Moore* when $\mathfrak{P}' = \mathfrak{P}'' = \mathfrak{P}$, and $\mathfrak{M}' = \mathfrak{M}'' = \mathfrak{M}$. We shall therefore call it the *definite property* P_0. It is equivalent to the following property, which is the form in which we shall have occasion to apply it.

If $J\kappa\kappa_0 = 0$ or $J\kappa_0\kappa = 0$ for every κ of the class $\mathfrak{K}$, then $\kappa_0 \equiv 0$.

We observe that J has this definite property in the finite case, and also the instances (a), (b), (c) but not (d) of I.†

Consider now the system

$$(1)\qquad \begin{aligned} M_2(\eta) &= D\eta - J\alpha\eta, \\ S_1(\eta) &= c_1\,\eta(x_1) + J\sigma_{11}\,\eta(x_1) + d_1\,\eta(x_2) + J\sigma_{12}\,\eta(x_2). \end{aligned}$$

We assume that the expression $S_1(\eta)$ is what we shall call *linearly independent*, i. e., there exists another linear combination of $\eta(x_1)$ and $\eta(x_2)$

$$S_2(\eta) = c_2\,\eta(x_1) + J\sigma_{21}\,\eta(x_1) + d_2\,\eta(x_2) + J\sigma_{22}\,\eta(x_2),$$

with $c_1 d_2 - c_2 d_1 \neq 0$, such that the equations $S_1(\eta) = 0$ and $S_2(\eta) = 0$ have the unique solution

$$\eta(x_1) = 0 \qquad \text{and} \qquad \eta(x_2) = 0.\ddagger$$

We can always determine the constants a and b with $b \neq 0$ so that in

$$S_2' = aS_1 + bS_2,$$

$$c_2' = c_1 - 1, \qquad d_2' = 2 - c_1,$$

i. e.,

$$c_2' + d_2' = 1, \qquad c_1 d_2' - c_2' d_1 = 1.$$

Evidently S_1 and S_2' will be completely equivalent to S_1 and S_2, and we shall assume in the sequel that in the S_2 chosen, c_2 and d_2 have the character of c_2' and d_2'.

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* Loc. cit., p. 361.

† Loc. cit., pp. 89–92.

‡ If $\mathfrak{P}' = \mathfrak{P}'' = (1, \cdots, n)$, and $J = \Sigma_{m=1}^{m=n}$ then this condition actually reduces to the linear independence of the boundary conditions S_1. A discussion of the above definition for the case of linear integral expressions by L. J. Rouse will appear shortly.

On account of the assumption relative to S_1 and S_2, it is possible to solve the equations

$$c_1 \eta(x_1) + J\sigma_{11} \eta(x_1) + d_1 \eta(x_2) + J\sigma_{12} \eta(x_2) = S_1,$$

$$c_2 \eta(x_1) + J\sigma_{21} \eta(x_1) + d_2 \eta(x_2) + J\sigma_{22} \eta(x_2) = S_2,$$

for $\eta(x_1)$ and $\eta(x_2)$ in terms of S_1 and S_2. If we substitute in the expression

$$J\hat{\eta}(x_2)\eta(x_2) - J\hat{\eta}(x_1)\eta(x_1)$$

and collect the coefficients of S_1 and S_2, these coefficients will be linear expressions in $\hat{\eta}(x_1)$ and $\hat{\eta}(x_2)$ of the form

$$c_1' \hat{\eta}(x_1) + J\hat{\eta}(x_1)\tau_{11} + d_1' \hat{\eta}(x_2) + J\hat{\eta}(x_2)\tau_{12} = T_1(\hat{\eta}),$$

$$c_2' \hat{\eta}(x_1) + J\hat{\eta}(x_1)\tau_{21} + d_2' \hat{\eta}(x_2) + J\hat{\eta}(x_2)\tau_{22} = T_2(\hat{\eta}).$$

We therefore have the identity

$$J\hat{\eta}(x_2)\eta(x_2) - J\hat{\eta}(x_1)\eta(x_1) = JT_1(\hat{\eta})S_2(\eta) - JT_2(\hat{\eta})S_1(\eta),$$

and we note that on account of the condition J^{P_0}, the $T_1(\hat{\eta})$ and $T_2(\hat{\eta})$ will be uniquely defined, if S_1 and S_2 are given as functions of $\eta(x_1)$ and $\eta(x_2)$.

We call the expression $N_2(\hat{\eta}) = D\hat{\eta} + J\hat{\eta}\alpha$ together with the $T_1(\hat{\eta})$ so defined an *adjoint system* of system (1).*

In order to obtain the relations between the constants and functions in S_1, S_2 and T_1, T_2, we can proceed as outlined in the definition. The same relations result however if we equate the coefficients in the defining identity. This yields

$$c_1' c_2 - c_2' c_1 = -1, \qquad d_1' c_2 - d_2' c_1 = 0,$$

$$c_1' d_2 - c_2' d_1 = 0, \qquad d_1' d_2 - d_2' d_1 = 1;$$

from which we conclude that

$$c_1' = d_1, \qquad d_1' = c_1, \qquad c_2' = d_2, \qquad d_2' = c_2.$$

We find further

$$d_1 \sigma_{21} - d_2 \sigma_{11} + c_2 \tau_{11} - c_1 \tau_{21} + J(\tau_{11}\sigma_{21} - \tau_{21}\sigma_{11}) = 0,$$

$$d_1 \sigma_{22} - d_2 \sigma_{12} + d_2 \tau_{11} - d_1 \tau_{21} + J(\tau_{11}\sigma_{22} - \tau_{21}\sigma_{12}) = 0,$$

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* This definition is a generalization of the definition due to Birkhoff, these Transactions, vol. 9 (1908), p. 173. Bounitzky (loc. cit., p. 73) gives a definition of adjoint which may be generalized as follows: $S_1(\eta)$ and $T_1(\hat{\eta})$ are adjoint if for every $\eta(x_1)$, $\eta(x_2)$, $\hat{\eta}(x_1)$, $\hat{\eta}(x_2)$ for which $S_1(\eta) = 0$ and $T_1(\hat{\eta}) = 0$, we have

$$J\hat{\eta}(x_2)\eta(x_2) - J\hat{\eta}(x_1)\eta(x_1) = 0.$$

If S_1 and T_1 are adjoint according to our definition they will also be according to the Bounitzky definition, and conversely on account of J^{P_0}. The Bounitzky definition does not however seem to lend itself so readily to the derivation of the results of this section and the next.

$$c_1\sigma_{21} - c_2\sigma_{11} + c_2\tau_{12} - c_1\tau_{22} + J(\tau_{12}\sigma_{21} - \tau_{22}\sigma_{11}) = 0,$$

$$c_1\sigma_{22} - c_2\sigma_{12} + d_2\tau_{12} - d_1\tau_{22} + J(\tau_{12}\sigma_{22} - \tau_{22}\sigma_{12}) = 0.$$

These equalities express the fact that the system of kernels

$$\begin{array}{ll} c_1\tau_{21} - c_2\tau_{11}, & d_1\tau_{21} - d_2\tau_{11}, \\ c_2\tau_{12} - c_1\tau_{22}, & d_2\tau_{12} - d_1\tau_{22}, \end{array}$$

are the reciprocals of the system of kernels

$$\begin{array}{ll} d_2\sigma_{11} - d_1\sigma_{21}, & d_2\sigma_{12} - d_1\sigma_{22}, \\ c_1\sigma_{21} - c_2\sigma_{11}, & c_1\sigma_{22} - c_2\sigma_{12}. \end{array}$$

If we add the last four equations, we get the following symmetrical relation

$$\sigma_{22} + \sigma_{21} + \tau_{11} + \tau_{12} + J(\tau_{11} + \tau_{12})(\sigma_{22} + \sigma_{21})$$
$$= \sigma_{11} + \sigma_{12} + \tau_{21} + \tau_{22} + J(\tau_{21} + \tau_{22})(\sigma_{11} + \sigma_{12}).$$

With the aid of these relations we verify without much difficulty that

$$T_{01}(\hat{\eta}) + S_{02}(\eta) + JT_{01}(\hat{\eta})S_{02}(\eta) - T_{02}(\hat{\eta}) - S_{01}(\eta)$$
$$- JT_{02}(\hat{\eta})S_{01}(\eta) = \hat{\eta}(x_2) + \eta(x_2) + J\hat{\eta}(x_2)\eta(x_2)$$
$$- \hat{\eta}(x_1) - \eta(x_1) - J\hat{\eta}(x_1)\eta(x_1),$$

where

$$\begin{array}{ll} T_{01}(\hat{\eta}) = T_1(\hat{\eta}) + \tau_{11} + \tau_{12}, & T_{02}(\hat{\eta}) = T_2(\hat{\eta}) + \tau_{21} + \tau_{22}, \\ S_{01}(\eta) = S_1(\eta) + \sigma_{11} + \sigma_{12}, & S_{02}(\eta) = S_2(\eta) + \sigma_{21} + \sigma_{22}. \end{array}$$

We therefore have

THEOREM I. *If $S_1(\eta)$ and $T_1(\hat{\eta})$ are adjoint, i. e., if we have identically*

$$J\hat{\eta}(x_2)\eta(x_2) - J\hat{\eta}(x_1)\eta(x_1) = JT_1S_2 - JT_2S_1,$$

then we also have identically

$$\hat{\eta}(x_2) + \eta(x_2) + J\hat{\eta}(x_2)\eta(x_2) - \hat{\eta}(x_1) - \eta(x_1) - J\hat{\eta}(x_1)\eta(x_1)$$
$$= T_{01} + S_{02} + JT_{01}S_{02} - T_{02} - S_{01} - JT_{02}S_{01}.$$

The same relations between the σ and τ in the light of their reciprocal character give

THEOREM II. *If T_1 and T_2 are adjoint to S_1 and S_2 then from*

$$T_1(\hat{\eta}) = 0 \qquad \textit{and} \qquad T_2(\hat{\eta}) = 0$$

it follows uniquely that

$$\hat{\eta}(x_1) = 0 \qquad \textit{and} \qquad \hat{\eta}(x_2) = 0,$$

i. e., $T_1(\hat{\eta})$ is also linearly independent.

The element of arbitrariness which enters into the definition of the adjoint T_1 of S_1 is taken care of in the following

THEOREM III. *If S_2 be replaced by any other S_2' which has the same character as S_2, and if we denote by T_1 and T_1' the corresponding adjoint expressions, then there exist functions κ_1 and κ_1' of the class $\mathfrak{K}$ such that*

$$T_1 = T_1' + JT_1'\kappa_1$$

and

$$T_1' = T_1 + JT_1\kappa_1',$$

i. e., T_1 and T_1' are essentially equivalent.

For if we solve the relations

$$S_1 = c_1\eta(x_1) + J\sigma_{11}\eta(x_1) + d_1\eta(x_2) + J\sigma_{12}\eta(x_2),$$

$$S_2 = c_2\eta(x_1) + J\sigma_{21}\eta(x_1) + d_2\eta(x_2) + J\sigma_{22}\eta(x_2)$$

for $\eta(x_1)$ and $\eta(x_2)$, we find

$$\eta(x_1) = d_2 S_1 - d_1 S_2 + J\tau_{21} S_1 - J\tau_{11} S_2,$$

$$\eta(x_2) = c_1 S_2 - c_2 S_1 - J\tau_{22} S_1 + J\tau_{12} S_2.$$

Substituting these values in

$$S_2' = c_2\eta(x_1) + J\sigma_{21}'\eta(x_1) + d_2\eta(x_2) + J\sigma_{22}'\eta(x_2),$$

we find

$$S_2' = S_2 + J(\kappa_2 S_1 + \kappa_1 S_2),$$

where κ_1 and κ_2 are functions of the class $\mathfrak{K}$, depending on σ_{21}', σ_{22}', τ_{11}, τ_{12}, τ_{21}, and τ_{22}. We therefore have

$$\begin{aligned} J\hat{\eta}(x_2)\eta(x_2) - J\hat{\eta}(x_1)\eta(x_1) &= JT_1' S_2' - JT_2' S_1 \\ &= JT_1'(S_2 + J\kappa_2 S_1 + J\kappa_1 S_2) - JT_2' S_1 \\ &= J(T_1' + JT_1'\kappa_1)S_2 - J(T_2' - JT_1'\kappa_2)S_1; \end{aligned}$$

from which on account of the uniqueness of T_1 it follows that

$$T_1 = T_1' + JT_1'\kappa_1.$$

In a similar way we obtain the other relation of the theorem.

3. **On the solutions of adjoint systems.** Before taking up the relations which exist between solutions of adjoint systems we note the following lemmas.

LEMMA I. *If η is a solution of equation (A) and $\hat{\eta}$ a solution of the adjoint equation (A'), then we have for every set of adjoint expressions S_1, S_2, T_1, and T_2*

$$T_{01}(\hat{\eta}) + S_{02}(\eta) + JT_{01}(\hat{\eta})S_{02}(\eta) = T_{02}(\hat{\eta}) + S_{01}(\eta) + JT_{02}(\hat{\eta})S_{01}(\eta).$$

For by Theorem IV′ of § 11 of I,* we have for every pair of solutions η and $\hat{\eta}$ of the adjoint equations (A) and (A')

$$\hat{\eta}(x_2) + \eta(x_2) + J\hat{\eta}(x_2)\eta(x_2) = \hat{\eta}(x_1) + \eta(x_1) + J\hat{\eta}(x_1)\eta(x_1).$$

Our lemma is then an immediate consequence of Theorem I of the preceding section.

LEMMA II. *If η is any solution of the system* (2) [(2′)]

$$M_2(\eta) = 0, \qquad S_1(\eta) = 0, \qquad [N_2(\hat{\eta}) = 0, \qquad T_1(\hat{\eta}) = 0],$$

then $S_2(\eta)$ $[T_2(\hat{\eta})]$ is a solution of the equation

$$S_2(\eta) + JT_{01}(\hat{\eta})S_2(\eta) = 0, \qquad [T_2(\hat{\eta}) + JT_2(\hat{\eta})S_{01}(\eta) = 0],$$

where $\hat{\eta}$ $[\eta]$ is any solution of equation (A') $[(A)]$.

Let η be a solution of system (2). Then if η_0 is any solution of equation (A), $\eta_0 + \eta$ is also a solution of equation (A). Further by proposition (4) of § 1, we have

$$S_{01}(\eta_0 + \eta) = S_{01}(\eta_0) + S_1(\eta) = S_{01}(\eta_0),$$

$$S_{02}(\eta_0 + \eta) = S_{02}(\eta_0) + S_2(\eta).$$

Applying Lemma I to the solutions η_0 and $\eta_0 + \eta$, we have for every solution $\hat{\eta}$ of the adjoint equation (A')

$$\begin{aligned} T_{01}(\hat{\eta}) + S_{02}(\eta_0) + S_2(\eta) + JT_{01}(\hat{\eta})S_{02}(\eta_0) + JT_{01}(\hat{\eta})S_2(\eta) \\ = T_{02}(\hat{\eta}) + S_{01}(\eta_0) + JT_{02}(\hat{\eta})S_{01}(\eta_0) \\ = T_{01}(\hat{\eta}) + S_{02}(\eta_0) + JT_{01}(\hat{\eta})S_{02}(\eta_0). \end{aligned}$$

Hence

$$S_2(\eta) + JT_{01}(\hat{\eta})S_2(\eta) = 0.$$

The proof for the lemma in the brackets runs parallel.

We are now in position to derive the following theorems.

THEOREM I. *If the system* (1)

$$M_1(\eta) = 0, \qquad S_{01}(\eta) = 0$$

has a unique solution, then the adjoint system (1′)

$$N_1(\hat{\eta}) = 0, \qquad T_{01}(\hat{\eta}) = 0$$

also has a unique solution and conversely.

Suppose η is the unique solution of system (1), and let $\hat{\eta}_0$ be any solution of equation (A'), whose Fredholm determinant is not zero. Then by Lemma I we shall have for η and $\hat{\eta}_0$

$$S_{02}(\eta) + T_{01}(\hat{\eta}_0) + JT_{01}(\hat{\eta}_0)S_{02}(\eta) = T_{02}(\hat{\eta}_0).$$

* Loc. cit., p. 87.

If we consider this as an integral equation in $S_{02}(\eta)$, it has a unique solution, viz., the value of $S_{02}(\eta)$. Hence the Fredholm determinant of $T_{01}(\hat{\eta}_0)$ is not zero, which by Theorem I of § 1 is a necessary and sufficient condition for the existence of a unique solution of the system (1′).

THEOREM II. *If the system* (2)

$$M_2(\eta) = 0, \qquad S_1(\eta) = 0$$

has n-fold compatibility, then the system (2′)

$$N_2(\hat{\eta}) = 0, \qquad T_1(\hat{\eta}) = 0$$

has also n-fold compatibility, and conversely.

Let $\mu_1', \cdots, \mu_n'$ be a complete system of linearly independent solutions of the homogeneous equation

$$\mu' + JS_{01}(\eta_0)\mu' = 0,$$

η_0 being a solution of equation (A) whose Fredholm determinant is not zero. Then by Theorem II of § 1

$$\eta = \sum_{m=1}^{n} (\mu_m' + J\eta_0 \mu_m')\mu_m''$$

μ_m'' being arbitrary, is a general solution of system (2). By Lemma II

$$S_2(\eta) = \sum_{m=1}^{n} (\mu_m' + JS_{02}(\eta_0)\mu_m')\mu_m''$$

will satisfy the equation

$$\kappa + JT_{01}(\hat{\eta}_0)\kappa = 0,$$

i. e., we shall have

$$\sum_{m=1}^{n} \Big(\mu_m' + JS_{02}(\eta)\mu_m' + JT_{01}(\hat{\eta}_0)(\mu_m' + JS_{02}(\eta)\mu_m')\Big)\mu_m'' = 0.$$

Since the μ_m'' are arbitrary, we conclude that the equation

$$\mu' + JT_{01}(\hat{\eta}_0)\mu' = 0,$$

$\hat{\eta}_0$ being a solution of equation (A') whose Fredholm determinant is not zero, has a set of n solutions

$$\mu_m' + JS_{02}(\eta_0)\mu_m'.$$

These solutions are linearly independent. For suppose they were not. Then there would exist n constants $c_1, \cdots, c_n$, such that

$$\sum_{m=1}^{n} c_m(\mu_m' + JS_{02}(\eta_0)\mu_m') = \sum_{m=1}^{n} c_m \mu_m' + JS_{02}(\eta_0)\sum_{m=1}^{n} c_m \mu_m' = 0.$$

Let

$$\sum_{m=1}^{n} c_m \mu_m' = \mu'.$$

Then for every μ'' the function

$$\eta = \mu' \mu'' + J\eta_0 \mu' \mu''$$

will satisfy both of the conditions

$$S_1(\eta) = 0, \qquad S_2(\eta) = 0.$$

But from our assumption relative to S_1 and S_2 it follows that

$$\eta(x_1) = 0, \qquad \eta(x_2) = 0,$$

from which we have $\mu' \equiv 0$, which is contrary to our assumption that the μ'_m are linearly independent. It follows therefore that the expressions

$$\mu'_m + JS_{02}(\eta_0)\mu'_m$$

are linearly independent.

Now if the linear equation

$$\mu' + JT_{01}(\hat{\eta}_0)\mu' = 0$$

has at least n linearly independent solutions, then the adjoint linear integral equation

$$\mu'' + J\mu''\, T_{01}(\hat{\eta}_0) = 0$$

will also have at least n linearly independent solutions, so that by Theorem II of § 1, we know that the system (2′) has at least n-fold compatibility.

By following through a similar line of reasoning, we show that if the system (2′) has m-fold compatibility, the system (2) has at least m-fold compatibility. But from this we conclude that $m = n$, which is the desired result.

Since 0-fold compatibility of system (2) yields a unique solution for the system (1), Theorem I may be regarded as a corollary to Theorem II.

We note further that in the course of the proof we have extended Lemma II to

LEMMA II*a*. *Every solution* κ *of the equation*

$$\kappa + JT_{01}(\hat{\eta}_0)\kappa = 0, \qquad [\kappa + J\kappa S_{01}(\eta_0) = 0]$$

when $\hat{\eta}_0$ $[\eta_0]$ *is a solution of equation* (*A*′) [(*A*)] *whose Fredholm determinant is not zero, is expressible in the form* $S_2(\eta)$ $[T_2(\hat{\eta})]$, *where* η $[\hat{\eta}]$ *is a solution of the system* (2) [(2′)].

THEOREM III. *A necessary and sufficient condition for the existence of a solution of the non-homogeneous system* (3) [(3′)]

$$M_2(\eta) = \alpha_0, \qquad S_1(\eta) = \sigma_0, \qquad [N_2(\hat{\eta}) = \alpha_0, \qquad T_1(\hat{\eta}) = \tau_0]$$

is that

$$\int_{x_1}^{x_2} J\hat{\eta}\alpha_0 = -JT_2(\hat{\eta})\sigma_0, \qquad \left[\int_{x_1}^{x_2} J\alpha_0\,\eta = -J\tau_0\, S_2(\eta)\right]$$

for every solution $\hat{\eta}$ $[\eta]$ *of the adjoint homogeneous system* (2′) [(2)].

Suppose η is a solution of the system (3). Then if we let $\hat{\eta}$ be any solution of the adjoint system (2′) and apply Green's Theorem G_2*

$$\int_{x_1}^{x_2} J(\hat{\eta}M_2(\eta) + N_2(\hat{\eta})\eta) = J\hat{\eta}(x_2)\eta(x_2) - J\hat{\eta}(x_1)\eta(x_1)$$
$$= JT_1(\hat{\eta})S_2(\eta) - JT_2(\hat{\eta})S_1(\eta),$$

we get at once the condition of the theorem, i. e., the condition is necessary.

On the other hand, the condition is sufficient. For if η_1 be a particular solution of equation (C), and $\hat{\eta}$ be a solution of the system (2′), we get by the Green's Theorem G_2

$$\int_{x_1}^{x_2} J\hat{\eta}\alpha_0 = -JT_2(\hat{\eta})S_1(\eta_1).$$

Applying the conditions of our theorem, we have for any particular solution η_1 of equation (C) and every solution $\hat{\eta}$ of system (2′)

$$JT_2(\hat{\eta})(S_1(\eta_1) - \sigma_0) = 0.$$

By Lemma IIa however, every solution of the equation

$$\kappa + J\kappa S_{01}(\eta_0) = 0$$

is expressible in the form $T_2(\hat{\eta})$, i. e., we have for every such κ

$$J\kappa(S_1(\eta_1) - \sigma_0) = 0.$$

By Theorem III of § 1, this is sufficient for the existence of a solution of system (3).

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* Cf. I, loc. cit., p. 86.