A FUNDAMENTAL SYSTEM OF FORMAL COVARIANTS MODULO 2
OF THE BINARY CUBIC*

BY

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If a binary form of order \( m \),

\[ f_m = (a_0, a_1, \ldots, a_m x_1, x_2)^m, \]

whose coefficients are arbitrary variables, be transformed by the group \( A \) of all linear substitutions on \( x_1, x_2 \), whose coefficients are least positive residues modulo \( p \), a prime number, there is brought into existence an infinitude of rational integral functions of \( a_0, \ldots, a_m, x_1, x_2 \), which are invariants under the group. Whether this infinite system possesses the property of finiteness, in general, is an unsolved problem, but in this paper I show that, when the modulus is 2, the system of covariants of a cubic \( f_3 \) is finite and that the fundamental set consists of twenty quantics. This system of covariants, five of which are pure invariants, is derived in explicit form.

The methods of generation and proof of the completeness of the fundamental set are developed from the point of view emphasized in a paper on the formal modular invariant theory, by the present writer, in volume 17 of these Transactions.† These methods presuppose a knowledge of a fundamental system of formal seminvariants of the given ground form; but this seminvariant system has been given previously‡ for \( f_3 \) and the modulus 2, by Dickson.

1. RESUMÉ OF METHODS

We recapitulate in (a), (b), (c), (d) certain processes, previously given, which are apropos in the developments relating to \( f_3 \). In (e), (f) novel principles are developed.

* Presented to the Society, April 28, 1917.
† The following papers by the present writer will be referred to by number:
III. These Transactions, vol. 17 (1916), p. 545.
‡ Dickson, Madison Colloquium Lectures (1913), p. 53.
(a) **Transvection.** In I it was shown that transvection between a form, as \( f_3 \), and the forms of the complete system of universal covariants of the group \( A \pmod{p} \) yields numerous formal concomitants \( \pmod{p} \). These universal covariants are*

\[
L = x^p_1 x_2 - x_1 x^p_2, \quad Q = (x^p_1 x_2 - x_1 x^p_2) \div L.
\]

(b) **Modular polars.** In II the following invariantive operators \( \pmod{p} \) were introduced in connection with a binary \( m \)-ic \( f_m \):

\[
E = x^2_1 \frac{\partial}{\partial x_1} + x^2_2 \frac{\partial}{\partial x_2}, \quad w = a^n_0 \frac{\partial}{\partial a_0} + \cdots + a^n_m \frac{\partial}{\partial a_m}.
\]

(c) **Concomitants of the first degree.** Every form of order \( > 3 \) is shown in III to be reducible modulo 2 in terms of first degree invariants and first degree covariants of orders 1, 2, and 3. A set of concomitants \( \pmod{2} \) of \( f_m \), of degree 1, is the following:

\[
K = a_1 + \cdots + a_{m-1}, \quad K_1 = (a_0 + K)x_1 + (K + a_m)x_2,
\]

\[
K_2 = a_0 x_1^2 + K x_1 x_2 + a_m x_2^2.
\]

These three exist for all orders. If \( m \) is odd there exists a cubic covariant

\[
K_{m3} = a_0 x_1^3 + I_1 x_1^2 x_2 + I_2 x_1 x_2^2 + a_m x_2^3 \quad (I_1 + I_2 = K).
\]

(d) **Copied forms.** If \( f_m, g_n \) are two binary forms and \( \sigma \) is a system of modular concomitants of \( g_n \), then a system for any covariant \( F_n \) of \( f_m \), constructed on the model of \( \sigma \), is a system of concomitants of \( F_n \).

(e) **Hexadic scales.** There exist, in general, an infinite number of covariants \( \pmod{p} \) having one and the same seminvariant leading coefficient. Let \( F_M = C_0 x_M^1 + C_1 x_1 x_2 + \cdots \) be any covariant modulo 2, of odd order \( M \), of \( f_m \), and construct concomitants of \( F_M \) on the models of \( K, K_1, K_2, K_{m3} \) (cf. (2), (3)). These copied forms are concomitants of \( f_m \), viz.,

\[
D = C_1 + \cdots + C_{M-1}, \quad D_1 = (C_0 + D)x_1 + (D + C_M)x_2,
\]

\[
D_2 = C_0 x_1^2 + Dx_1 x_2 + C_M x_2^2,
\]

\[
F_{M3} = C_0 x_1^3 + J_1 x_1^2 x_2 + J_2 x_1 x_2^2 + C_M x_2^3 \quad (J_1 + J_2 = D).
\]

**Lemma.** Corresponding to any given cubic covariant \( \pmod{2} \) of \( f_m \), as,

\[
F_3 = C_0 x_1^3 + C_1 x_1^2 x_2 + C_2 x_1 x_2^2 + C_3 x_2^3
\]

there exists a definite cubic covariant \( \Gamma \) whose leading coefficient is the invariant

\* Dickson, these *Transactions*, vol. 12 (1911), p. 75, and Madison Colloquium Lectures (1913), p. 33.
To prove the covariancy of $\Gamma$ we assume that $F_3$ is a covariant, necessary and sufficient conditions for which are (1) homogeneity, (2) invariancy under the permutational substitution $s = (a_0 \ a_m) (a_1 \ a_{m-1}) \cdots (x_1 \ x_2)$, and (3) invariancy under the operation of transforming $f_m$ by $T : x_i = x'_i + x_2$, $x_2 = x'_2$ (mod 2). Let the increments of $C_i$ ($i = 0, \cdots, 3$), when $f_m$ is transformed into $f'_m$ by $T$, be $\delta C_i$ ($i = 0, \cdots, 3$) respectively. Then if $F'_3$ is the $F_3$ function constructed from $f'_m$,

$$F'_3 = C_0 x_1^3 + (C_0 + C_1 + \delta C_1) x_1^2 x_2 + (C_0 + C_2 + \delta C_2) x_1 x_2^2 + (C_0 + C_1 + C_2 + C_3 + \delta C_1 + \delta C_2 + \delta C_3) x_2^3 \equiv F_3 \ (\text{mod} \ 2),$$

whence follows

$$\delta C_0 = 0, \quad \delta C_1 = C_0, \quad \delta C_2 = C_0, \quad \delta C_3 = C_0 + C_1 + C_2 \ (\text{mod} \ 2).$$

Constructing $\Gamma'$ and applying (6) we obtain immediately $\Gamma' \equiv \Gamma \ (\text{mod} \ 2)$, which proves the lemma.

Application of this lemma to $F_{m0}$ of (4) gives the covariant

$$\Gamma_m = D x_1^3 + (C_0 + C'_M + J_1) x_1^2 x_2 + (C_0 + C'_M + J_2) x_1 x_2^2 + D x_2^3.$$  

Observe that $J_1, J_2$ are interchanged by the substitution $s$. Hence, since $J_1 + J_2 = D$, the invariant leading coefficient of $\Gamma_m$ can contain no term $a'_i a'_j a'_k \cdots$ which is left unaltered by $s$. This is true of all invariants which lead cubic covariants, for, if the leading coefficient $C_0$ of $F_3$ is an invariant, we have $C_0 = C_3$, and $\delta C_3 = 0$. Then (6) gives

$$C_0 \equiv C_1 + C_2 \ (\text{mod} \ 2),$$

and, as the covariancy of $F_3$ requires that $C_1$ and $C_2$ be interchanged by $s$, any term of $C_1$ which is left unaltered by $s$ occurs also in $C_2$ and therefore has a zero coefficient, modulo 2, in the sum $C_0$.

We shall designate the six interrelated concomitants $F_M, F_{m0}, \Gamma_M, D, D_1, D_2$ as the hexadic scale* for the covariant $F_M$ of odd order $M$. Every covariant of odd order furnishes such a scale of concomitants.

(f) Tetradic scale. If $F_M$ is a covariant mod 2 of even order $M$ no cubic covariant corresponding to $F_{m0}$ exists, but we have the interrelated forms $F_M, D, D_1, D_2$, which, accordingly, will be called a tetradic scale for $F_M$.

* The term scale was used by Sylvester in the sense of a fundamental system, but this designation has become practically obsolete.
Note that the reducibility of $F_M$ does not imply the reducibility of the forms in the scale for $F_M$, but the covariant $\Gamma_M$ in any hexadic scale is reducible in terms of other concomitants in the scale, as follows:

$$\Gamma_M = QD_1 + F_{M_3} \pmod{2}.$$  

2. The general seminvariant of $f_3$

If

$$f_3 = a_0 x_1^5 + a_1 x_1^4 x_2 + a_2 x_1 x_2^4 + a_3 x_2^5,$$

the fundamental system of formal seminvariants modulo 2, given by Dickson, is composed of

$$a_0, \quad K = a_1 + a_2, \quad \delta_{00} = (a_0 + K + a_3) a_3,$$

$$\Delta = a_0 a_3 + a_1 a_2, \quad \beta = a_0 a_1 + a_1^2.$$

With these may be associated the following invariants:* 

$$K, \quad \Delta, \quad I = a_0^2 + a_0 K + \delta_{00}, \quad k = a_0 \delta_{00},$$

$$g = \beta^2 + \beta (\Delta + K^2) + (\Delta + \delta_{00}) (\beta + a_0 K + K^2).$$

We have noticed previously, in III, the following syzygies connecting seminvariants. They result immediately from (9) and (10).

$$a_0^5 + a_0^2 K + a_0 I + k \equiv 0,$$

$$\beta^2 + \beta (a_0^2 + a_0 K + I + K^2) + (a_0^2 + a_0 K + I + \Delta) (a_0 K + K^2) + g \equiv 0 \pmod{2}.$$  

It will be convenient, in this paper,† to abbreviate the invariant $g + I\Delta + \Delta^2$ as $g_1$. Thus,

$$g \equiv g_1 + I\Delta + \Delta^2 \pmod{2}.$$

Any seminvariant $\phi$ of $f_3$, being a polynomial in the seminvariants (9), is a polynomial in $a_0, K, I, \beta, \Delta$,

$$\phi = \phi(a_0, \beta, K, \Delta, I).$$

Hence when the congruences (11) are used as reducing moduli with respect to powers of $a_0$ and of $\beta$, $\phi$ can be reduced to the form

$$\phi = J_0 + J_1 a_0 + J_2 a_0^2 + (\Gamma_0 + \Gamma_1 a_0 + \Gamma_2 a_0^2) \beta,$$

where $J_i, \Gamma_i$ $(i = 0, 1, 2)$ are invariants expressed as polynomials in $K, \Delta$, 

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* Cf. Dickson, loc. cit.; and III, pp. 554, 555.

† The advantage of this change is that $g$ contains terms which are symmetrical under $s$ (§ 1) and hence $g$ cannot be the leading coefficient of any covariant of odd order, whereas we shall determine a cubic covariant led by $g_1$. 

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3. Construction of covariants of $f_3$

The hexadic scale for the quantic $f_3$ itself, composes a system of first degree concomitants. The forms in this scale are $f_3, G, K, K_1, K_2$, where (cf. III, p. 555)

$$G = Kx_1^3 + (a_0 + a_1 + a_3)x_1^2 x_2 + (a_0 + a_2 + a_3)x_1 x_2^2 + K x_2^3$$

(14)  

$$= QK_1 + f_3 \quad (\text{mod } 2).$$

The hessian of $f_3$ is both an algebraical and a formal modular covariant, viz.,

$$H = sx_1^2 + \Delta x_1 x_2 + (a_1 a_2 + a_2^2) x_2^2 \quad (s = a_0 a_2 + a_1^2).$$

The tetradic scale for $QH$ consists of $QH, H, G_1, \text{ and } \Delta$, where

$$G_1 \equiv (s + \Delta) x_1 + (\Delta + a_1 a_2 + a_2^2) x_2 \quad \text{[ } = (H, L)^2].$$

A quadratic covariant led by $f_3$ is

$$t \equiv K K_2 + H \equiv \beta x_1^2 + (\Delta + K^2) x_1 x_2 + (a_2 a_3 + a_2^2) x_2^2,$$

and in the tetradic scale for $Qt$ occurs

$$t_1 \equiv (\beta + \Delta + K^2) x_1 + (\Delta + K^2 + a_2 a_3 + a_2^2) x_2$$

(16)  

$$= G_1 + KK_1 \quad (\text{mod } 2).$$

Apply to the forms just derived the modular operator

$$w = a_0^2 \frac{\partial}{\partial a_0} + a_1^2 \frac{\partial}{\partial a_1} + a_2^2 \frac{\partial}{\partial a_2} + a_3^2 \frac{\partial}{\partial a_3}.$$  

Thus we get

$$C_1 = wK_1 = (a_0^2 + K^2) x_1 + (K^2 + a_2^2) x_2,$$

(17)  

$$C_2 = wK_2 = a_0^2 x_1^3 + K^2 x_1 x_2 + a_2^2 x_2^3 = K_1^2 + K^2 Q \quad (\text{mod } 2),$$

$$P = wq_3 = a_0^2 x_1^3 + a_1^2 x_1 x_2 + a_2^2 x_1 x_2^2 + a_3^2 x_2^3.$$  

In order to perform the reductions necessary to isolate the complete system sought we shall need, among other forms, quadratic covariants led by the
seminvariants $a_0 \beta$ and $a_0^3 \beta$ respectively, and cubic covariants led by $B$, $a_0 B$, and $a_0^3 B$. These are derived by forming certain scales of concomitants, as follows:

Construct the hexadic scale for the quintic covariant $f_3 t$, and we have, neglecting reducible forms,

$$F_{s3} = a_0 \beta x_1^5 + (a_0^3 a_3 + a_1 a_2 + a_0 a_1 a_3 + a_0 a_2 a_3) x_1^2 x_2$$

$$+ (a_0 a_1^2 + a_1 a_2 + a_0 a_2 a_3 + a_0 a_1 a_3) x_1 x_2^2 + (a_2 a_1^2 + a_1^2 a_3) x_2^3,$$

(18)

$$D_2 = a_0 \beta x_1^5 + (k + K\Delta) x_1 x_2 + (a_2 a_1^2 + a_1^2 a_3) x_2^2,$$

$$D_1 = (a_0 \beta + k + K\Delta) x_1 + (k + K\Delta + a_2 a_3^2 + a_0 a_3) x_2.$$

Next we form the hexadic scale for the quintic $tP$, giving

$$F'_{s3} = a_0^3 \beta x_1^5 + (a_0^3 a_3 + a_1 a_2 + a_0 a_1 a_3 + a_0 a_2 a_3 + a_0 a_1^3 + a_0 a_1 a_3 + a_0 a_2 a_3 + a_0^3 a_3$$

$$+ a_0 a_1 a_2 + a_2 + a_1 a_2 + a_1 a_3 + a_1^2 a_3 + a_0 a_1^2 + a_0 a_2 a_3 + a_0 a_1 a_3 + a_0 a_2 a_3$$

$$+ a_0 a_1 a_2 + a_1 a_2 + a_1 a_3 + a_1^2 a_3 + a_0 a_1^2 + a_0 a_2 a_3 + a_0 a_1 a_3 + a_0 a_2 a_3$$

(19)

$$D'_2 = a_0^3 \beta x_1^5 + (I \Delta + K^2 + g) x_1 x_2 + (a_2 a_1^2 + a_1^2 a_3) x_2^2,$$

$$D'_1 = (a_0^3 \beta + I \Delta + K^2 + g) x_1 + (I \Delta + K^2 + g + a_2 a_3^2 + a_0 a_3) x_2.$$

Again, the scale for the quintic $Ql$, where $l$ is the cubic

$$l = QG_1 + Kf_3 = Bx_1^5 + (a_0 a_2 + a_1 a_2 + a_1 a_3 + a_2^2) x_1^7 x_2$$

$$+ (a_0 a_2 + a_1 a_2 + a_1 a_3 + a_1^2) x_1 x_2^2 + (a_2 a_1^2 + a_1^2 a_3 + a_1 a_2 a_3 + a_1 a_3) x_2^3,$$

(20)

furnishes the covariants

$$l_2 = Bx_1^5 + K^2 x_1 x_2 + (\Delta + a_2 a_3 + a_2^3) x_2^3,$$

$$t_1 = (B + K^2) x_1 + (K^2 + \Delta + a_2 a_3 + a_2^3) x_2,$$

(21)

and it is now evident that cubic covariants, say $F$ and $F'$, led by the respective seminvariants $a_0 B$, $a_0^3 B$ may also be constructed. For $F$ is the cubic covariant (the one not led by an invariant) in the hexadic scale for the quintic $f_3 l_2$ and $F'$ is the corresponding cubic in the scale for $Pl_2$. We also write $F$ and $F'$ explicitly, but they will be proved to be reducible (cf. (26)):

$$F = a_0 Bx_1^5 + (a_0^2 a_3 + a_0 a_1 a_3) x_1^2 x_2 + (a_0 a_1^2 + a_0 a_2 a_3) x_1 x_2^2$$

$$+ (a_0 a_1^2 + a_1 a_2 + a_0 a_2 a_3 + a_1 a_3) x_2^3,$$

(22)

$$F' = a_0^3 Bx_1^5 + (a_0^3 a_3 + a_0^2 a_1 a_3 + a_0^2 a_1 a_2 + a_0^2 a_2 a_2 + a_0^2 a_1 a_3 + a_0^2 a_2 a_3$$

$$+ a_0 a_1 a_2 + a_0 a_1^2 + a_1 a_2 + a_1 a_3 + a_1 a_3 + a_0 a_2 a_3 + a_0 a_2^2 a_3 + a_0 a_2 a_3 + a_1 a_2 a_3 + a_1 a_2 a_3$$

$$+ a_0 a_1 a_2 + a_1 a_3) x_1 x_2^2 + (a_0 a_3 + a_1 a_2 a_3 + a_2 a_3 + a_2 a_3 + a_3 a_3) x_2^3.$$
4. Covariants of $f_3$, whose leading coefficients are invariants

Reduction methods to be employed in the next section require an explicit knowledge of all covariants which have as leading coefficients pure invariants which are polynomials in the invariants $K, \Delta, k, I, g_1$, homogeneous as to $a_0, \cdots, a_3$. We can write the general polynomial in question in the form

$$\Phi = f(I, \Delta) + K\psi_1 + k\psi_2 + g_1\psi_3,$$

where $f(I, \Delta)$ is a polynomial in $I$ and $\Delta$ only, and $\psi_i$ ($i = 1, 2, 3$) are polynomials involving, in general, all five invariants.

**Lemma.** No covariant of odd order exists having $f(I, \Delta)$ as a leading coefficient.

In proof of this we show that every polynomial in $I$ and $\Delta$ alone, which is homogeneous in $a_0, \cdots, a_3$, necessarily has a term which is left unaltered by the substitution $s = (a_0 a_3)(a_1 a_2)$. In fact the only symmetrical term in $I^r$ is $a_o a_3^r$,* while all terms in $\Delta^s$ are symmetrical under $s$. If $I^r \Delta^s$ is the term containing the highest power of $\Delta$ in $f(I, \Delta)$, then the term

$$\tau = a_o^r a_3^s a_2^r a_1^s$$

certainly occurs in $f$ with a numerical coefficient which is $\equiv 0 \pmod{2}$. But, as shown in § 1(e), no covariant of odd order can be led by an invariant containing a term $\tau$ unaltered by $s$. This proves the lemma.

**Lemma.** There exists both a quadratic and a cubic covariant (but no linear covariant) led by each one of the three invariants $K, k, g_1$.

The quadratic covariants are the products of the respective invariants $K, k, g_1$ by $Q$.

The cubic led by $K$ is the covariant $G$ in (14).

A cubic covariant led by $k$ is found by constructing the following polynomial in concomitants derived in § 3:

$$T = QD_1 + F_{63} + \Delta G;$$

(23)  $$T = kx_1^3 + (a_0 a_1 + a_0 a_1^2 + a_2 a_2^2 + a_3 a_3) x_1 x_2 + (a_2 a_2^2 + a_3^2 a_3$$

$$+ a_0 a_1 + a_0 a_1^2 + a_2 a_2 + a_0 a_1 + a_2 a_2 + a_3 a_3 + a_1 a_1 + a_2 a_2 + a_0 a_1 + a_2 a_2 + a_3 a_3$$

$$+ a_0 a_3) x_1 x_2^2 + kx_1^3.$$  

Finally, a cubic covariant whose leader is the invariant $g_1 = I \Delta + \Delta^2 + g$ is the covariant led by an invariant, belonging to the hexadic scale for $tP$.

* The expression $z_2$ is the only term in the expansion of $(z_1 + z_2 + \cdots + z_7)^n$ which has an odd coefficient and is left unaltered by $r = (z_1 z_7)(z_2 z_6)(z_3 z_4)(z_4)$.  

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By (8) this covariant, say \( E \), is reducible as follows (cf. (19)):

\[
E = QD' + F'3 \pmod{2}.
\]

(24)

Combining the preceding results in this section we conclude as in the following

**Lemma.** The most general form for a pure invariant leading coefficient of a covariant of \( f_3 \), of odd order, is

\[
S = K\psi_1 + k\psi_2 + g_1 \psi_3,
\]

and the following quantic is a cubic covariant led by \( S \):

\[
\Psi = \psi_1 G + \psi_2 T + \psi_3 E.
\]

(25)

5. The fundamental system of covariants of \( f_3 \)

Every covariant \( \pmod{2} \) of even order, of \( f_3 \), is of the form \( R = \phi x_1^{2h} + \cdots \), where the leading coefficient is the \( \phi \) function of (12). We shall construct, as a rational integral function of the concomitants which were derived in the two preceding sections, another covariant \( C \) whose seminvariant leading coefficient is \( \phi \). Then \( R - C \), although not vanishing in general, will always be congruent to the product of a covariant \( C' \) of odd order \( 2h - 3 \) by \( L \). This consideration, the method of which is due to Dickson, evidently furnishes a general reduction method,* since the same process can be applied to \( C' \) and to the covariants analogous to \( C' \), in succession.

For example, if we subtract \( F_{33} + A/3 \) from \( F \) (cf. (18) and (22)), and reduce the factor \( C'' \) in the remainder, we get the first relation below. The second relation is given by performing similar operations in connection with the covariant \( F' \) of (22),

\[
F = F_{33} + A/3 + LK\Delta,
\]

(26)

\[
F' = F'_{33} + \Delta P + LK^2 \Delta \pmod{2}.
\]

In the general case, when \( R = \phi x_1^{2h} + \cdots \) \( (h \equiv 1) \), we find

\[
R = J_0 Q^h + J_1 K_2 Q^{h-1} + J_2 C_2 Q^{h-1} + \Gamma_0 tQ^{h-1} + \Gamma_1 D_2 Q^{h-1}
\]

\[
+ \Gamma_2 D_2 Q^{h-1} + LC' \pmod{2},
\]

(27)

where \( C' = 0 \) if \( h = 1 \), and \( C' \) is a covariant of odd order \( 2h - 3 \) if \( h > 1 \).

Next, when \( R \) is of odd order \( 2h + 1 > 1 \), namely \( R = \phi x_1^{2h+1} + \cdots \), we deduce by use of the form (13) of \( \phi \), involving \( B \), and the last lemma in Section 4,

\[
R = (\Psi + R_1 f_3 + R_2 P + S_0 l + S_1 F + S_2 F')Q^{h-1} + LC' \pmod{2},
\]

(28)

where, if \( h = 1 \), \( C' \) is an invariant, and, if \( h > 1 \), \( C' \) is a covariant of even order \( 2h - 2 \).

After we have applied these processes successively to the covariants \( C'' \), we shall have reduced all covariants of order \( \neq 1 \), excepting the irreducible concomitants of orders 0, 1, 2, 3 in terms of which the covariants in formulas (27), (28) are explicitly constructed, as in formulas (14) to (26).

Consider next the covariants which are linear in \( x_1, x_2 \), all being of the form
\[
\lambda = \phi x_1 + \phi_1 x_2.
\]

From (12) we have, identically,
\[
\phi = N + J_1 (a_0 + K) + J_2 (a_0^2 + K^2) + \Gamma_0 (\beta + \Delta + K^2)
+ \Gamma_1 (a_0 \beta + k + K\Delta) + \Gamma_2 (a_0^2 \beta + g_1) \pmod{2},
\]
in which \( N \) is an invariant (cf. (11));
\[
N = J_0 + J_1 K + J_2 K^2 + \Gamma_0 \Delta + \Gamma_0 K^2 + \Gamma_1 k + \Gamma_1 K\Delta + \Gamma_2 g_1 \pmod{2}.
\]

We can construct a linear covariant led by each parenthesis in (29) (cf. (30)).

No linear covariant is led by an invariant. Hence, assuming \( \lambda \) to be a covariant, \( N = 0 \pmod{2} \). Making use of the linear covariants in the various scales of concomitants in Section 3, we have,
\[
\lambda = J_1 K_1 + J_2 C_1 + \Gamma_0 t_1 + \Gamma_1 D_1 + \Gamma_2 D'_1 \pmod{2}.
\]

Hence all linear covariants are reducible in terms of the invariants \( K, \Delta, k, I, g_1 \) and the five covariants \( K_1, C_1, t_1, D_1, D'_1 \).

Giving attention, now, to the covariants entering the formulas (27), (28), (30), expressed as rational integral functions of other covariants by the explicit formulas given in Sections 3 and 4, we summarize our conclusions, in the following

**Theorem.** A fundamental system of formal covariants modulo 2 of the binary cubic quantic \( f_3 \) is composed of twenty forms, as follows: Five invariants \( K, \Delta, k, I, g_1 \); five linear covariants \( K_1, C_1, t_1, D_1, D'_1 \); four quadratic covariants \( K_2, t, D_2, D'_2 \); and four cubic covariants \( f_3, P, F_{33}, F'_{33} \), together with the two universal covariants \( L, Q \).

6. **Syzygies**

The syzygies connecting the members of this fundamental system are legionary. In the paper quoted as III above I gave a theorem which establishes the existence and furnishes a method of construction of an infinitude of syzygies, although this theorem does not directly furnish all such relations.

Each identity (11) connecting the fundamental seminvariants of \( f_3 \) furnishes a
syzygy, for if we substitute in one of these, for each seminvariant, a covariant which the latter leads, paying attention to considerations of homogeneity, we get a reducible covariant which is congruent to $L$ times a covariant $C'$, and reduction of $C'$ leads to a syzygy. The $\Sigma_1$ below was constructed from the first relation in (11); $\Sigma_2$ by the before-mentioned theorem:

$$\Sigma_1 = K_1^2 K_2 + K_2^2 K_3 Q + KK_1^2 Q + K_3^2 Q^2 + IK_3 Q$$

$$+ kQ^2 + K_2 K_1 L + IK_1 L \equiv 0 \pmod{2},$$

(31)

$$\Sigma_2 = t f_3 + K_2 t_1 Q + KK_1 K_2 Q + KK_2 f_3 + \Delta f_3 Q + D_2 L$$

$$+ KK_1^2 L + K_3 LQ + \Delta KLQ \equiv 0 \pmod{2}.$$ 

Since a syzygy is a polynomial in the fundamental concomitants we can prove that all are expressible in terms of a finite set of irreducible ones, $\Sigma_1, \ldots, \Sigma_r$, by applying Hilbert's theorem, replacing the equations customarily understood in this theorem by identical congruences modulo 2. That is, any syzygy $\Sigma$ may be put in the form

$$\Sigma \equiv \sigma_1 \Sigma_1 + \sigma_2 \Sigma_2 + \cdots + \sigma_r \Sigma_r \pmod{2},$$

where $\sigma_1, \ldots, \sigma_r$, being polynomials in the concomitants, are themselves concomitants of $f_3$.

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