PROBLEMS IN THE THEORY OF ORDINARY LINEAR DIFFERENTIAL EQUATIONS WITH AUXILIARY CONDITIONS AT MORE THAN TWO POINTS*

BY

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In this paper is considered a differential system consisting of an ordinary linear differential equation and auxiliary conditions involving linearly the values of the solution and its derivatives at interior points, as well as at the end points, of the interval over which the equation is considered. The notions of Green's function and adjoint systems, already introduced by Birkhoff† and further developed by Bôcher‡ for auxiliary conditions involving the end points only, are extended to the more general system. The problem of the expansion of arbitrary functions in terms of the characteristic solutions of the system is stated, but the convergence proof is given in another paper.§

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1. The auxiliary condition problem

Consider the differential expression

$$L(u) \equiv \frac{d^n u}{dx^n} + p_1(x)\frac{d^{n-1} u}{dx^{n-1}} + \cdots + p_{n-1}(x)\frac{du}{dx} + p_n(x)u$$

and the equation $L(u) = p(x)$, in which $p(x)$, $p_1(x)$, $\cdots$, $p_n(x)$, are functions of the real variable x, continuous together with their derivatives of all orders in the closed interval (a, b). Let $a_1 = a$, a_2, a_3, $\cdots$, $a_k = b$, be k points of this interval arranged in order of ascending algebraical magnitude. If $\phi(x)$ is any function of x which has at these points derivatives of the first

* Presented to the Society, April 29, 1916, as part of a more extensive paper, with a slightly different title; the remainder of the paper has already been published in these Transactions, vol. 18 (1917), pp. 415–442.

† Birkhoff, these Transactions, vol. 9 (1908), pp. 373–395.

‡ Bôcher, these Transactions, vol. 14 (1913), pp. 403–420.

§ See first footnote above.

$n - 1$ orders, then let

$$W_{ij}(\phi) = \alpha_{ij}\,\phi(a_j) + \alpha'_{ij}\,\phi'(a_j) + \alpha''_{ij}\,\phi''(a_j) + \cdots + \alpha^{(n-1)}_{ij}\,\phi^{(n-1)}(a_j) \qquad (j = 1, 2, \cdots, k),$$

and let

$$W_i(\phi) = W_{i1}(\phi) + W_{i2}(\phi) + \cdots + W_{ik}(\phi),$$

in which $\alpha_{ij}, \alpha'_{ij}, \alpha''_{ij}, \cdots, \alpha^{(n-1)}_{ij}$, are constants, real or complex.

If π_i represents a real or complex number, then the system

$$L(u) = p, \qquad W_i(u) = \pi_i \qquad (i = 1, 2, 3, \cdots, n),$$

consisting of the differential equation with n *auxiliary conditions,* is called the complete system; the system

$$(1) \qquad L(u) = p, \qquad W_i(u) = 0 \qquad (i = 1, 2, 3, \cdots, n),$$

is called the semi-homogeneous system; and finally the system

$$(2) \qquad L(u) = 0, \qquad W_i(u) = 0 \qquad (i = 1, 2, 3, \cdots, n),$$

is called the reduced system.

The reduced system is said to be *compatible* if it has solutions other than zero, otherwise it is *incompatible.* It has *r-fold compatibility* if it has r and only r linearly independent solutions.* By direct substitution of the general solution of $L(u) = 0$ in the auxiliary conditions $W_i(u) = 0$ it is found that

If $y_1, y_2, \cdots, y_n$, is a fundamental system of solutions of $L(u) = 0$, a necessary and sufficient condition that the reduced system have r-fold compatibility is that the following determinant be of rank $n - r$:

$$(3) \qquad D \equiv |W_i(y_j)| \qquad (i, j = 1, 2, \cdots, n).$$

If we denote by u_0 a particular solution of $L(u) = p$, then it is found in the same way that

If the reduced system has r-fold compatibility, a necessary and sufficient condition that the complete system have a solution is that the matrix obtained by bordering the determinant D on the right by the quantities $W_i(u_0) - \pi_i$, $i = 1, 2, \cdots, n$, be of rank $n - r$.

Whence, in particular, for $r = 0$,

If the reduced system is incompatible, the complete system always has a solution.

That this solution is unique follows at once from the fact that the difference of any two solutions of the complete system is a solution of the reduced system.

Assuming that the auxiliary conditions are linearly independent, we shall now define a function, $G(x, s)$, called the *Green's function* for the reduced system, by requiring that for every value of s in the interval (a, b), it shall have as a function of x the following properties:

* Bôcher, loc. cit., p. 405.

I. It is continuous in the interval (a, b), together with its first $n-2$ derivatives;

II. It satisfies $L(u) = 0$ at all points of the interval except the point s, where its $(n-1)$th derivative has a finite jump of magnitude 1;

III. It satisfies the auxiliary conditions $W_i(G) = 0, i = 1, 2, \cdots, n$.

The proof that these properties completely define the function is as follows: Since $G(x, s)$ satisfies $L(u) = 0$, it can be written in terms of a fundamental system of solutions,

$$\begin{aligned} G(x, s) &= c_1(s)y_1(x) + c_2(s)y_2(x) + \cdots + c_n(s)y_n(x), \qquad x \leqq s; \\ &= d_1(s)y_1(x) + d_2(s)y_2(x) + \cdots + d_n(s)y_n(x), \qquad x \geqq s. \end{aligned}$$

Now $G(x, s)$ and its first $n-2$ derivatives are continuous at the point s, while the $(n-1)$th derivative has a finite jump of magnitude 1 there, whence

$$\begin{aligned} c_1(s)y_1(s) + \cdots + c_n(s)y_n(s) &= d_1(s)y_1(s) + \cdots + d_n(s)y_n(s), \\ c_1(s)y_1'(s) + \cdots + c_n(s)y_n'(s) &= d_1(s)y_1'(s) + \cdots + d_n(s)y_n'(s), \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot &= \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot, \\ c_1(s)y_1^{(n-2)}(s) + \cdots + c_n(s)y_n^{(n-2)}(s) &= d_1(s)y_1^{(n-2)}(s) + \cdots \\ &\qquad + d_n(s)y_n^{(n-2)}(s), \\ c_1(s)y_1^{(n-1)}(s) + \cdots + c_n(s)y_n^{(n-1)}(s) &= d_1(s)y_1^{(n-1)}(s) + \cdots \\ &\qquad + d_n(s)y_n^{(n-1)}(s) - 1. \end{aligned}$$

If we set $d_i(s) - c_i(s) = z_i(s)$, we can write these equations in the form

$$\sum_{i=1}^{n} z_i(s)y_i^{(j)}(s) = 0 \qquad (j = 0, 1, 2, \cdots, n-2),$$

$$\sum_{i=1}^{n} z_i(s)y_i^{(n-1)}(s) = 1.$$

The determinant of these equations is the wronskian of the y's and so is not zero, since they are linearly independent. Hence the z's are uniquely determined and are continuous functions of s in the interval (a, b). Knowing the z's, we determine the c's and the d's by means of III. If we assume for purposes of proof that s is in the interval (a_l, a_m), we have

$$W_i(G) \equiv \sum_{j=1}^{n}\left(c_j \sum_{q=1}^{l} W_{iq}(y_j) + d_j \sum_{q=m}^{n} W_{iq}(y_j)\right) = 0$$
$$(i = 0, 1, 2, \cdots, n-2),$$

and on substitution of $d_j - z_j$ for c_j this becomes

$$\sum_{j=1}^{n} d_j W_i(y_j) = \sum_{j=1}^{n} z_j \sum_{q=1}^{l} W_{iq}(y_j) \qquad (i = 1, 2, 3, \cdots, n).$$

These are n equations in the n unknowns, the d's, and their determinant is D, so that if D does not vanish the unknowns are uniquely determined, and then by the relation $d_j - z_j = c_j$ the c's are also. Hence if the reduced system is incompatible it possesses one and only one Green's function.

By direct substitution of

$$U(x) = \int_a^b p(s)\,G(x,s)\,ds$$

in the system (1) it is found that the function $U(x)$ is the solution of the semi-homogeneous system, by reason of the three defining conditions for $G(x,s)$.*

If p_i has derivatives of the first $n-i$ orders then the first n derivatives of the functions z_i (called adjoint functions) exist, and satisfy the differential equation

$$M(v) \equiv (-1)^n \frac{d^n v}{ds^n} + (-1)^{n-1} \frac{d^{n-1}(p_1 v)}{ds^{n-1}} + \cdots - \frac{d(p_{n-1} v)}{ds} + p_n v = 0,$$

called the *adjoint equation*.† From this, and the explicit expressions for $G(x,s)$ in terms of the y's, we can deduce easily the following properties of $G(x,s)$ as a function of s:

I. It is continuous in the interval (a,b), together with its first $n-2$ derivatives, except at the points a_i, where it has finite jumps whose magnitudes are in general functions of x;

II. It satisfies $M(v) = 0$ at all points of the interval (a,b), except the above, and the point x, where its $(n-1)$th derivative has a finite jump of magnitude $(-1)^{n-1}$.

Since $G(x,s)$ as a function of s has discontinuities it is apparent that an *adjoint system* in the accepted sense of the term‡ does not exist. In considering the expansion problem it becomes necessary to have functions analogous to the solutions of the adjoint system in the two point case. These may be obtained as the solutions of the adjoint to the integral equation which is equivalent to the differential system, but it seems worth while to indicate briefly how they may be found without the aid of the theory of integral equations.

The fact that the Green's function as a function of its second argument has discontinuities at the points a_i suggests that it might be profitable to investigate the case where it has similar discontinuities with regard to its first argument. Such a Green's function would be obtained if the auxiliary

* It is to be noticed that down to this point the operators W_i might have been any linear operators, so that the analysis has a considerably wider range of applicability than has been indicated.

† Schlesinger, *Handbuch der Theorie der linearen Differentialgleichungen*, vol. I, p. 63.

‡ Birkhoff, loc. cit., p. 375, or Bôcher, loc. cit., p. 405.

conditions were such as to require a different solution of the equation between each two points of the set a_i, $i = 1, 2, \cdots, k$. We can define a set of conditions which will do this as follows: Let w_{ij} be of the same form as W_{ij}, but with different values for the α's, and let W_{ij} be considered as applying to the solution to the right of the point a_j, while w_{ij} applies to the solution to the left of that point. Then the auxiliary conditions are

$$\overline{W}_i(u) \equiv W_{i1}(u) + w_{i2}(u) + W_{i2}(u) + \cdots$$
$$+ w_{i,\,k-1}(u) + W_{i,\,k-1}(u) + w_{ik}(u)$$
$$(i = 1, 2, \cdots, n(k-1)).$$

When the general solution of $L(u) = 0$ is substituted into these auxiliary conditions the constants are determined differently between each two of the points a_i. If that part of the condition W_i which applies to the interval (a_l, a_m), i. e., $W_{il}(u) + w_{im}(u)$, be denoted by $\overline{W}_{il}$, then the determinant whose vanishing is the necessary and sufficient condition that the reduced system have solutions, other than the one which is identically zero throughout the interval $(a_1 b)$, becomes

$$(4)\quad D \equiv \begin{vmatrix} \overline{W}_{11}(y_1) \cdots \overline{W}_{1,\,k-1}(y_1) & \overline{W}_{11}(y_2) \cdots \overline{W}_{1,\,k-1}(y_2) & \cdots & \overline{W}_{11}(y_n) \cdots \overline{W}_{1,\,k-1}(y_n) \\ \overline{W}_{21}(y_1) \cdots \overline{W}_{2,\,k-1}(y_1) & \overline{W}_{21}(y_2) \cdots \overline{W}_{2,\,k-1}(y_2) & \cdots & \overline{W}_{21}(y_n) \cdots \overline{W}_{2,\,k-1}(y_n) \\ \cdots\cdots & \cdots\cdots & \cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots & \cdots\cdots \\ \overline{W}_{n(k-1),\,1}(y_1) \cdots \overline{W}_{n(k-1),\,k-1}(y_1) & \overline{W}_{n(k-1),\,1}(y_2) \cdots \overline{W}_{n(k-1),\,k-1}(y_2) & \cdots & \overline{W}_{n(k-1),\,1}(y_n) \cdots \overline{W}_{n(k-1),\,k-1}(y_n) \end{vmatrix}.$$

As in the case already considered, one easily finds that the necessary and sufficient condition that the complete system have one and only one solution is that the reduced system be incompatible.

If it is assumed that the auxiliary conditions are linearly independent, the Green's function for the present system is defined for all values of x and s in (a, b) by requiring that for every value of s it shall satisfy as a function of x the following conditions:

I. It is continuous in the interval (a, b), together with its first $n-2$ derivatives, except at the points a_i;

II. Apart from these points, it satisfies $L(u) = 0$ throughout the interval except at the point s, where its $(n-1)$th derivative has a finite jump of magnitude 1;

III. It satisfies the auxiliary conditions $\overline{W}_i(u) = 0$ $(i = 1, 2, \cdots, n(k-1))$.

The proof that these three conditions completely define $G(x, s)$ is parallel to that given for the simpler case.

It is now possible to define adjoint auxiliary conditions, which, taken with the adjoint equation, form the adjoint system. The differential expressions $L(u)$ and $M(v)$ are connected by Lagrange's identity,

$$vL(u) - uM(v) = \frac{d}{dx} P(u, v),$$

where P is a non-singular bilinear form in the $2n$ variables $u, u', u'', \cdots, u^{(n-1)}$; $v, v', v'', \cdots, v^{(n-1)}$, for every x in the interval. If we integrate this over the interval (a, b), we can for the purposes of integration divide it up into the sum of the integrals taken between each two successive points of discontinuity of G, i. e., the points a_i, and thus admit the possibility of using solutions of the two equations with finite jumps at these points. This gives us

$$\int_a^b [vL(u) - uM(v)]\, dx = P, \tag{5}$$

where P is now a bilinear form in the two sets of $2n(k-1)$ variables each

$$u^{(j)}(a_1^+),\ u^{(j)}(a_2^-),\ u^{(j)}(a_2^+),\ u^{(j)}(a_3^-),\ \cdots,\ u^{(j)}(a_{k-1}^+),\ u^{(j)}(a_k^-),$$

$$v^{(j)}(a_1^+),\ v^{(j)}(a_2^-),\ v^{(j)}(a_2^+),\ v^{(j)}(a_3^-),\ \cdots,\ v^{(j)}(a_{k-1}^+),\ v^{(j)}(a_k^-)$$
$$(j = 0, 1, 2, \cdots, n-1).$$

If $\overline{W}_i(u)$, $i = 1, 2, \cdots, 2n(k-1)$, be any $2n(k-1)$ linearly independent linear forms in the U's, then the U's can be expressed linearly in terms of them, and P becomes a linear form in the $\overline{W}$'s with coefficients which are linear in the v's and which we may denote by $\overline{V}_i$. So we can write

$$P \equiv \overline{W}_1 \overline{V}_{2n(k-1)} + \overline{W}_2 \overline{V}_{2n(k-1)-1} + \cdots + \overline{W}_{2n(k-1)} \overline{V}_1 .$$

In particular we may choose as the first $n(k-1)$ of the linear forms in the u's the left-hand members of our $n(k-1)$ auxiliary conditions. Then the $\overline{V}$'s with indices 1 to $n(k-1)$ when set equal to zero give the adjoint auxiliary conditions. It is easily seen that different choices of the last $n(k-1)$ of the $\overline{W}$'s would lead merely to linear combinations of these adjoint conditions, so that the system thus defined is unique. From the symmetry in the work it follows that the adjoint relation is a reciprocal one.

By definition $G(x, s)$ as a function of x satisfied the reduced auxiliary conditions. We shall now prove that as a function of s it satisfies the adjoint auxiliary conditions. By substituting, in the integral and in P, $G(x, s)$ for u and $G(s, x)$ for v, we obtain

$$\sum_{j=1}^{n(k-1)} W_{2n(k-1)-j}(G) \bar{V}_j(G) = 0 .$$

The $\bar{W}$'s here appearing are the last $n(k-1)$. We shall make $n(k-1)$ different choices of these. We obtain in this way $n(k-1)$ different sets of $\bar{V}$'s, but as we have already remarked, any set can be expressed as linear combinations of the first set. If we do this we have $n(k-1)$ linear homogeneous equations in the $n(k-1)$ variables, the first set of $\bar{V}$'s. The determinant of these equations depends on the choice of the $\bar{W}$'s, but it is easily shown that these can be chosen in such a way as to make the determinant different from zero, and it follows that the $\bar{V}$'s are all zero. That is, the Green's function as a function of its second argument satisfies the adjoint auxiliary conditions. Combining this with what we already know about $G(x, s)$ as a function of its second argument, we see that $(-1)^n G(x, s)$ is, as a function of s, the Green's function of the adjoint system.

If $y_{ij}(x)$ is defined as equal to $y_i(x)$ in the interval (a_j, a_{j+1}) and as zero elsewhere in the interval (a, b), then the n functions y and the $k-1$ intervals give in all $n(k-1)$ linearly independent solutions of $L(u)$, which it is convenient to call a *fundamental system of discontinuous solutions* with regard to the points a_i. Such a system has properties analogous to the properties of a fundamental system of solutions in the ordinary case. Any solution of $L(u) = 0$ having finite jumps at any of the points a_i, but otherwise continuous in (a, b), can be represented as a linear combination of these. We shall call any function which may have finite jumps at any of the points a_i, but satisfies $L(u) = 0$ at all other points of (a, b), a discontinuous solution (with regard to the points a_i). We have then the

THEOREM. *If the reduced system has h linearly independent discontinuous solutions the adjoint system has also.*

Because of the reciprocal relation between the two systems it is sufficient to show that it has at least h. To prove this let $y(x)$ represent any discontinuous solution of $L(u) = 0$, and form a fundamental system of discontinuous solutions, $z_{ij}(x)$ $(i = 1, 2, \cdots, n; j = 1, 2, \cdots, k-1)$. Substitution of these in the integral and in P leads to the following equations:

$$\sum_{j=1}^{n(k-1)} W_{2n(k-1)-j}(y) V_{j1}(z_i) = 0 ,$$

$$\sum_{j=1}^{n(k-1)} W_{2n(k-1)-j}(y)\,V_{j2}(z_i) = 0,$$

$$\cdots\cdots\cdots\cdots$$

$$\sum_{j=1}^{n(k-1)} W_{2n(k-1)-j}(y)\,V_{j,\,k-1}(z_i) = 0 \qquad (i = 1, 2\ \cdots, n).$$

These may be regarded as $n(k-1)$ homogeneous linear equations to determine the constants $W_{2n(k-1)-j}(y)$, $j = 1, 2, 3, \cdots, n(k-1)$. Their determinant is just the determinant whose rank determines the number of linearly independent discontinuous solutions of the adjoint system, as will be seen by comparing it with (4), which is the corresponding determinant for the reduced system. We wish then to show that the rank of this determinant is at most $r - h$, which we do by showing that the system of equations has at least h independent solutions. To this end let us substitute the h linearly independent solutions of the reduced system in place of the solution $y(x)$ in the W's. This gives h solutions of the system of equations, which can easily be shown to be linearly independent. So the theorem is proved.

Of course these solutions which we have been dealing with may in particular cases be continuous over the whole interval (a, b). We can regard our system (2) as a special case in which the first n auxiliary conditions are the conditions there given, and the rest of the set are the $n(k-2)$ conditions that the solution of the system together with its first $n-1$ derivatives be continuous across each of the $k-2$ points a_i interior to the interval (a, b). We have shown, then, how a system adjoint to the system (2) may be defined, and from now on the discussion will be limited to such a system and its adjoint.

2. Characteristic numbers and solutions

From the system (2) is formed the system

$$(6) \qquad L(u) + \lambda u = 0, \qquad W_i(u) = 0 \quad (i = 1, 2, 3, \cdots, n),$$

in which λ is a complex parameter with the whole finite plane for its domain. The adjoint system can be written

$$(7) \qquad M(v) + \lambda v = 0, \qquad V_i(v) = 0 \quad (i = 1, 2, 3, \cdots, n(k-1)).$$

The solutions of $L(u) + \lambda u = 0$ are all functions of λ, and so the determinant (3) is also. The values of λ for which this determinant vanishes are the values for which the system above will have solutions other than the identically zero one. Such values are called *characteristic numbers*, and the corresponding solutions are called *characteristic solutions*. The characteristic numbers of the system and its adjoint are, by the theorem just proven, the same; and furthermore the number of linearly independent discontinuous solutions corresponding to any one characteristic number is the same for both systems.

If u and v are solutions of (6) and (7) respectively, belonging to different characteristic numbers, then

$$\int_a^b u(x)v(x)\,dx = 0,$$

as will readily be seen by substituting them in the integral (5).

If we assume that each characteristic number has but one corresponding characteristic solution, and try to expand an arbitrary function in terms of these solutions, and if we assume further that the series is such that it can be multiplied through by a continuous function and integrated term by term, then we find that

$$f(x) \equiv \sum_{i=1}^{\infty} c_i u_i(x), \tag{8}$$

where

$$c_i \equiv \frac{\int_a^b f(x)v_i(x)\,dx}{\int_a^b u_i(x)v_i(x)\,dx}.$$

Now the Green's function of the system (6) is, of course, a function of λ, and is defined for all values of λ save the characteristic numbers. The fundamental system $y_1, y_2, y_3, \cdots, y_n$, may be chosen analytic in λ, whence $G(x, s; \lambda)$ is also, except possibly at the characteristic numbers. If $\lambda = \lambda_i$ is a characteristic number to which corresponds but one independent characteristic solution, and for which $G(x, s; \lambda)$ has a pole of the first order, then the residue at the pole is

$$\frac{u_i(x)v_i(s)}{\int_a^b u_i(x)v_i(x)\,dx},$$

and the integral in the denominator is not zero. The proof given by Birkhoff* applies here.

Under these conditions the terms of the expansion (8) are

$$\int_a^b R_i(x, s)f(s)\,ds, \tag{9}$$

in which $R_i(x, s)$ is the residue corresponding to the ith characteristic number, and the sum of the first m terms is

$$\frac{1}{2\pi i}\int_\Gamma\int_a^b G(x, s; \lambda)f(s)\,ds\,d\lambda, \tag{10}$$

where Γ is a contour in the λ-plane enclosing the first m characteristic numbers.

* Birkhoff, loc. cit., p. 378.

In a previous paper* the author has proved the convergence of the integral (10) when suitable restrictions are placed on $f(x)$ and the auxiliary conditions. Of course in the more general cases the expansion (9) cannot be written in the form (8), that is, the residue cannot be expressed linearly in terms of the characteristic and adjoint functions alone. The new functions which must be introduced have been defined by means of integral equations by Goursat,† but apparently have never been defined for the two point case in terms of the differential system. Such a definition has been formulated by the author, and the accompanying theory forms the subject of another paper.

* Wilder, loc. cit.

† Goursat, Annales de la Faculté des Sciences de Toulouse, series 2, vol. 10 (1908), pp. 5–98.