SPACE INVOLUTIONS DEFINED BY A WEB OF QUADRICS*

BY

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1. Properties of the involution

1. **Statement of problem.** The following paper is concerned with the study of the transformation between the two spaces (x'), (x) which is defined by the equations

$$\rho x_i' = \phi_i(x) \qquad (i = 1, 2, 3, 4), \tag{1}$$

where $\phi_i(x) = 0$ is the equation of a quadric surface. It has been treated synthetically by Reye† but only incidentally in connection with line congruences. In the existing memoirs the involution of the whole of space is not considered except in the special case in which the system of quadrics have six common points; this special case has been extensively studied.‡

2. **Images of planes, lines, and points.** A plane s_1 in (x') goes into a quadric surface s_2 in (x), belonging to a definite web. A line c_1' in (x') has for image a space quartic c_4 of genus 1 in (x); the line is the basis of a pencil of planes, and the image quartic is the basis of the pencil of image quadric surfaces. A point P' in (x') has 8 image points in (x); the point P' is the vertex of a bundle of planes, and the image points are the eight basis points of the bundle of image quadric surfaces.

A plane s_1 in (x) goes into a Steiner surface s_4' in (x'), since by means of the equation of the plane, the coördinates of a point on the image surface are expressible as quadratic functions of three homogeneous parameters. Since the coefficients a_i in the equation $\sum a_i x_i = 0$ of a plane do not enter the equation of the image surface linearly, the system of Steiner surfaces in (x'), which appear as images of the whole system of planes of (x), is not linear.

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† T. Reye, *Ueber die reciproke Verwandtschaft von F Systemen und ϕ^2-Geweben und die quadratischen F^2-Systeme achter Stufe*, Journal für die reine und angewandte Mathematik, vol. 82 (1876), pp. 173–206, and *Ueber Strahlensysteme zweiter Classe und die Kummersche Fläche vierter Ordnung mit sechzehn Knotenpunkten*, ibid., vol. 86 (1879), pp. 84–107.

‡ For the literature of this case, see Snyder, *An application of the (1, 2) quaternary correspondence to the Kummer and Weddle surfaces*, these Transactions, vol. 12 (1911), pp. 354–366.

A line c_1 in (x) goes into a conic c_2' in (x'), since by means of the equations of c_1 the coördinates of a point on the image locus can be expressed as quadratic functions of two homogeneous parameters.

A point P in (x) has a single point P' in (x') for image; this appears directly from equation (1).

3. **Surfaces of coincidences and of branch points.** A point in (x') which has two coincident images in (x) is called a *branch-point.* Let P' be a branch-point and P the corresponding coincidence. To the bundle of planes through P' corresponds a bundle of quadrics having a common tangent line at P. Let $P \equiv (0, 0, 0, 1)$, and let $x_1 = 0$, $x_2 = 0$ be the equations of the common tangent line. We may take the equations of the three linearly independent quadrics through P in the forms $x_1 x_4 + f = 0$, $x_2 x_4 + \psi = 0$, $(a x_1 + b x_2) x_4 + \theta = 0$, wherein f, ψ, θ are quadratic forms in x_1, x_2, x_3. In this bundle is a cone having its vertex at P, hence we have the theorem:

THEOREM I: *The surface of coincidences is the locus of the vertices of the cones contained in the web of quadrics.*

The equation $\sum \lambda_i \phi_i = 0$ represents a cone when each of its first partial derivatives as to x_i vanishes. By eliminating λ_i from these equations we obtain the equation of the surface of coincidences; it is the jacobian of the web of quadrics. It will be denoted by K_4. The c_4 image of a straight line of (x') meets K_4 in 16 points, hence:

THEOREM II: *The locus of branch points is a surface of order* 16.

It will be denoted by L_{16}'. The complete image of L_{16}' is of order 32; it consists of K_4 counted twice, and of a residual surface of order 24, which we denote by R_{24}.

A quadric of the web meets K_4 in a c_8 of genus 9, hence a plane section of L_{16}' is of genus 9. A cone of the web meets K_4 in a c_8 of genus 8, having a double point at the vertex of the cone. The image plane meets L_{16}' in a curve of genus 8, having a double point at the image point of the vertex of the cone. It is a tangent plane to L_{16}', hence we may say:

THEOREM III: *The images of the quadric cones of the web are the tangent planes to* L_{16}'.

We shall now prove the theorem:

THEOREM IV: *The surface K_4 contains ten straight lines.* We have seen that in the web of quadrics are ∞^2 cones, the vertices of which lie on K_4. If a quadric of the web is composite, every point of its line of vertices lies on K_4; if (x) is any point on the line common to the two component planes, then for (x) every first minor of the discriminant of the quadric must vanish. In the system $\sum \lambda_i \phi_i = 0$ are ten composite quadrics.*

* Salmon, *Algebra.* Lesson 19 in the 4th edition, 1885.

4. **Particular lines.** Given a point P' in (x'). Its images in (x) are $P_1, P_2, \cdots, P_8$. The bundle of planes through P' goes into the bundle of quadrics through the eight associated points P_i. A straight line through P' is determined by one other point; a c_4 through all P_i is determined by one other point. If an additional point is chosen on the straight line $P_i P_k$, the associated c_4 has three collinear points, hence consists of the line $P_i P_k$ and a space cubic which meets the line in two points. The image of this composite quartic is a straight line in (x') which touches L'_{16} at the images of the points of intersection of $P_i P_k$ and its residual cubic. Thus, any line in (x) which joins two associated points has a line for image in (x'). Between these two lines exists a $(1, 2)$ correspondence, the double points of which are the points of intersection of $P_i P_k$ and K_4, not on the residual cubic. The lines $P_i P_k$ are all double tangents to R_{24}. Through any point P of (x) seven such bitangents can be drawn. The image lines are all bitangents to L'_{16}. Through any point P' of (x') twenty-eight such bitangents can be drawn, since there are 28 lines $P_i P_k$ associated with any point P'.

5. **Images of the ten lines on K_4.** It has been seen that the web of quadrics $\sum \lambda_i \phi_i = 0$ contains ten pairs of planes, and that the line of intersection of the planes of each pair lies on K_4. The image of each line γ_i is a conic γ'_i on L'_{16}. Since the pair of planes constitutes a quadric of the web, and the image of a quadric of the web is a plane, it follows that the plane of the conic is the image plane of the composite quadric. Since every point of the line is on K_4, the plane of the image conic touches L' at every point of the conic.

The curve of intersection of a quadric of the web with K_4 has for image a plane section of L'_{16}. In the case of a composite quadric, the curve of intersection with K_4 consists of the line of vertices taken twice and of two elliptic cubic curves. Hence the plane of each singular conic γ'_i meets L'_{16} in two elliptic curves, each of order six.

6. **Successive images of a plane.** A plane s_1 of (x) goes into a Steiner surface s'_4, having three concurrent double lines p'_1, p'_2, p'_3.

The complete image of s'_4 consists of s_1 and of a residual surface s'_7 of order 7. The plane s_1 meets K_4 in a plane quartic curve through which s'_7 passes; the residual intersection of s_1 and s'_7 consists of three lines p_1, p_2, p_3, images of the double lines of s'_4. The curve (s_1, K_4) goes over into a curve of order 8 which is a curve of contact of s'_4 and L'_{16}. The residual curve of intersection of s'_4 and L'_{16} is a c'_{48}; it is the image of the curve (s_1, R_{24}).

The congruence of lines $P_i P_k$ in (x) is of order 7 and class 3; it has R_{24} for focal surface.

A general line c_1 meets K_4 in 4 points; its image in (x') is a conic c'_2 which touches L'_{16} in the images of the points of intersection. The residual 24 points of intersection of c'_2 and L'_{16} are images of the points (c_1, R_{24}).

A line $P_i P_k$ meets K_4 in four points, two of which are on the residual c_3. The image line is bitangent to L'_{16} and meets it in 12 other points. The points of tangency are images of the points (c_1, c_3), and they have points of tangency of c_1 and R_{24} for residual images. Two of the residual points of c'_1, L'_{16} are coincidences on K_4; the remaining points have each a pair of intersections of c_1 with R_{24} for images. The same 10 points have for residual images the 10 residual intersections of c_3 with K_4.

7. **Two planes and their images.** Let p_1, p_2 be two planes in (x), and s'_1, s'_2 their image Steiner surfaces in (x'). The image of the line (p_1, p_2) is the conic p' common to both Steiner surfaces, which also intersect in a residual c'_{14} having 10 points on p'. To obtain further properties, pass a plane through a double line c'_1 of s'_1. It meets s'_1 in a conic c'_2 and s'_2 in a quartic c'_4. The line c'_1 meets s'_2 in four points $P'_1, \cdots, P'_4$; through P'_1 pass p' and c'_{14}; through each of the others c'_{14} passes twice. The curves c'_2, c'_4 meet in 8 points, through 7 of which c'_{14} passes, and the other lies on p'. The partial image of c'_1 is c_1 in p_1. The residual s'_7, image of s'_2, meets p_1 in a curve of order 7, having one point on (p_1, p_2). The other six are three pairs of images of P'_2, P'_3, P'_4. The plane through c'_1 goes into a quadric of the web through c_1. The plane p_1 cuts the quadric in c_1 and another line $\bar{c}_1$.

The plane p_2 meets the quadric in a conic c_2 which has a point on c_1 and one on $\bar{c}_1$. The image of $\bar{c}_1$ is the conic c'_2, and that of c_2 is c'_4. Since (p_1, p_2), $\bar{c}_1$, and c_2 have a point in common, it follows that p', c'_2, c'_4 have a point in common. The curve c'_{14} does not pass through this point. Since c'_{14} has three double points and one simple intersection on each double line of s'_1, s'_2, its genus is 6.

8. **Double curve and cuspidal curve on L'_{16}.** It has been seen that the locus of points in (x') which have two coincident images in (x) is L'_{16}, and that the locus of the coincidences is K_4. On L'_{16} is a cuspidal curve, each point of which has at least three coincident images and a double curve, each point of which has a pair of coincident images. The points of intersection of these two curves are either points to which correspond four coincident images, or a coincidence of three image points at one point and a simultaneous coincidence of two at another. The orders of these curves are the number of cusps and double points in a general plane section of L'_{16}.

Since the order and genus of a general plane section are known, all the characteristic numbers can be determined when one more is known. The bitangents of the plane section are the lines whose images in (x) are composite quartic curves. Of the bitangents, 10 are the lines in which the plane of section meets the ten singular planes. The image of the plane of section is a quadric of the web. If a line in (x') has for image in (x) a line and a residual cubic, the image line is a generator of the quadric and joins a pair of associated points P_i, P_k.

Such a line is a component of the basis curve of a pencil of quadrics contained in the web. Let $x_1 x_3 - x_2 x_4 = 0$ be the equation of a quadric. A generator of one system has equations $x_1 = \lambda x_2$; $\lambda x_3 = x_4$. Any quadric of the web has an equation of the form $\sum a_i \phi_i = 0$. Replace x_1 by λx_2, x_4 by λx_3, equate the coefficients of x_2^2, $x_2 x_3$, x_3^2 to zero and eliminate the a_i. The result is a sextic in λ, hence 6 generators of the λ-system join associated points. The same quadric contains 6 generators of the other stysem which join associated points.

Hence the total number of bitangents is 22. From Plücker's equations we now obtain $\delta = 60$, $\kappa = 36$, from which we may state the theorem:

THEOREM. *The surface L'_{16} has a double curve of order* 60 *and a cuspidal curve of order* 36.

9. **Contact and intersection curves on K_4.** When the point P' describes c'_{36} on L'_{16}, the triple image P is on K_4, and the remaining 5 images are on R_{24}. But P is also on R_{24}, and since a definite direction is associated with every point of K_4 (the limit of the direction $P_i P_k$ as $P_i \doteq P_k$) it follows that K_4 and R_{24} touch each other along the curve described by P. This curve is in $(1, 1)$ correspondence with c'_{36}, and is of order 18. When P' is a point on c'_{60} on L'_{16}, two of its images coincide at P_1, two more at P_2, and there are four other images on R_{24}. P_1 is on K_4, hence P_2 is on R_{24}; but P_2 is on K_4, hence P_1 is on R_{24}. Hence K_4, R_{24} intersect along the curve described by P_1 and P_2 as P' describes c'_{60}. The locus of P_1 (and P_2) is of order 60; its image in (x') is c'_{60} counted twice.

10. **Special properties of γ_i, γ'_i.** The image of any straight line γ on K_4 is a singular conic γ' on L'_{16} (No. 5). The complete image of γ' consists of γ counted twice and of a residual cubic g_3 in each component plane π_1, π_2 of the composite quadric of the web belonging to γ. These two cubics both lie on R_{24}, but not on K_4. Each plane touches R_{24} along g_3. Each plane π_i meets K_4 in a cubic c_3, passing through the three points (γ, g_3), and meeting g_3 in six others. The image of each c_3 is a c'_6 in the plane π' of γ'. Together they constitute the residual intersection of π' with L'_{16}. Each c'_6 touches γ' in three points, images of the points (γ, c_3), and cuts it in six points, images of the intersections (g_3, c_3). The complete image of c'_6 is c_3 counted twice, a residual c_6 in π_1, and a c_{12} in π_2. Each plane π is tangent to the contact curve c_{18} in the three points (γ, c_3). The remaining 9 points of intersection of π with c_{18} (not on γ) are on c_3, hence c'_6 has 9 cusps. At each of the three contacts of c'_6 with γ' the plane π' meets the cuspidal curve c'_{36} in three points. At each intersection of c'_6 with γ' the plane π' touches the double curve c'_{60}; the residual intersections of c'_{60} with π' are at the 36 points common to the two sextics.

11. **Complete images of cuspidal and double curves.** The complete image

of c_{36}' is c_{18} counted three times and a residual c_{90} which is a cuspidal curve on R_{24}. The image of c_{60}' is c_{60} counted twice and a residual c_{120} which is a double curve on R_{24}. The curve c_{90} meets K_4 in 360 points. Let t of these points T be on c_{18} and c_{60}, that is, points at which four images of a point in (x') coincide; let there be m points M on c_{60} such that for each M the corresponding point in (x') has two images at M and three at an intersection of c_{18} and c_{60}. At the points T the curve c_{90} has contact with K_4, hence

$$2t + m = 360.$$

The curve c_{120} meets K_4 in 480 points. Let there be p points in (x') whose images in (x) consist of three coincidences at P_1, P_2, P_3, and of two residual points. We have the relation

$$3p + m = 480.$$

If we apply the Zeuthen formula for the number of coincidences in a multiple correspondence to the $(2, 1)$ correspondence between c_{60}, c_{60}' and the $(4, 1)$ between c_{120}, c_{60}' we obtain the relation

$$p = t.$$

Hence

$$p = t = m = 120.$$

The curves c_{18}, c_{60} touch in t points T, and intersect in the m points M. Hence they have $2t + m = 360$ common points.

12. **Web with one basis point.** Let $A = (0, 0, 0, 1)$ be on all the quadrics of the web. The equations of transformation may be expressed in the form

$$\rho x_i' = x_4 x_i + \phi_i \qquad (i = 1, 2, 3),$$

$$\rho x_4' = \phi_4,$$

wherein ϕ_i are homogeneous and quadratic in x_1, x_2, x_3. The point A is a fundamental point; its image is the plane $x_4' = 0$. Not all the images of a point in $x_4' = 0$ are at A; the residual locus is the cone $\phi_4 = 0$ having A for vertex.

Every line through A joins two associated points, hence each has a line in (x') for image. The image of the line $x_i = \lambda_i r$ $(i = 1, 2, 3)$ is the line

$$x_i' = x_4 \lambda_i + \phi_i(\lambda) r, \qquad x_4' = \phi_4(\lambda) r.$$

The points in which these lines in (x') meet the singular plane $x_4' = 0$ are projective with the lines of the bundle at A. To the directions of the generators of the cone $\phi_4 = 0$ correspond the points of a singular conic $\phi_4' = 0$ in $x_4' = 0$. The cone $\phi_4 = 0$ is the tangent cone to K_4 at A, which is a double point on the surface.

Two general quadrics of the web intersect in a quartic curve which meets K_4 in 14 points apart from A. Hence L' is of order 14.

A quadric of the web meets K_4 in a curve of order 8 and genus 8, hence a plane section of L'_{14} is of genus 8. The plane $x'_4 = 0$ touches L'_{14} along the conic $\phi'_4 = 0$, $x'_4 = 0$. The surface K_4 has an equation of the form

$$x_4^2\, \phi_4 + x_4\, \psi + f = 0.$$

Each generator of $\phi_4 = 0$ meets K_4 in one point apart from A, on the surface $x_4\, \psi + f = 0$. The curve of intersection is of order 8 and has a six-fold point at A. The image is the curve

$$x'_i = \phi_i\, \psi - f x_i, \qquad i = 1, 2, 3, \qquad x'_4 = 0,$$

in which the parameters x_1, x_2, x_3 are connected by the relation $\phi_4 = 0$. This curve is rational and of order 10; it is the residual intersection of $x'_4 = 0$ with L'_{14}. The images of the lines of $\phi_4 = 0$ are tangents to this curve c'_{10}.

Since L' is of order 14, it follows that the order of R is 20. A straight line meets L'_{14} in 14 points; its image c_4 meets R_{20} in 70 points not at A. Since c_4 has a simple point at A, it follows that R_{20} has a 10-fold point at A.

A general line l through A meets K_4 in two residual points P_1, P_2. Its image is a line l' tangent to L'_{14} at P'_1, P'_2 and meeting L'_{14} in ten other points, images of the residual intersections of l with R_{20}.

Any point P' in (x') has 8 images in (x), one of which is A; through A can be drawn seven lines of the bundle to the other associated points, hence the image congruence in (x') is of order 7. Since any quadric of the web contains two lines through A, the image congruence is of class 2. Since the points of $x'_4 = 0$ are projective with the lines through A, it follows that no two lines not in $x'_4 = 0$ can meet in a point of $x'_4 = 0$. If P' is chosen in $x'_4 = 0$, it follows that of the seven lines of the congruence through P', six must be in $x'_4 = 0$, and be tangent to c'_{10}. Hence c'_{10} is of class 6; it must therefore have 24 double points and 12 cusps.

The image of l' is l and a residual c_3 meeting K_2 (the cone $\phi_4 = 0$) in 6 points, but not passing through A. The six generators of K_2 through (c_3, K_2) have for images the 6 tangents to c'_{10} through P'. These 6 points and the direction of l at A are the 7 images of P', apart from A itself.

The lines of A, projective with the points of c'_{10}, are the generators of the tangent cone K_{10} to R_{20} at A. This cone has 24 double generators and 12 cuspidal generators.

A plane section of L'_{14} has 23 bitangents, those accounted for in the general case and the section with $x'_4 = 0$.

The cuspidal curve of L'_{14} is therefore of order 30, and the double curve of order 40. The contact curve of R_{20} and K_4 is still of order 18, and has a six-fold point at A. The curve of intersection is of order 44 and has an 8-fold point at A. The two tangent cones K_2, K_{10} touch each other along the six tangents to the contact curve, and intersect in the eight tangents to the curve of intersection.

The curve c_{10} and the conic $\phi'_4 = 0$ touch at the six image points in $x'_4 = 0$, and intersect at the eight image points. The cuspidal curve c'_{30} has second order contact with $\phi'_4 = 0$ at the six points, while the double curve c'_{40} has simple contact with $\phi'_4 = 0$ at the eight points.

The complete image of c'_{30} is the contact curve c_{18} counted three times, and a residual cuspidal curve c_{66} on R_{20}, having a 12-fold point at A; the tangents at A being the images of the cusps of c'_{10}. The complete image of c_{40} is the curve of intersection c_{44} counted twice, and a residual double curve c_{72} on R_{20}, having a 24-fold point at A, the tangents being the images of the double points of c'_{10}.

Hence we have

$$2t + m = 264 - 24 = 240,$$

$$3p + m = 288 - 48 = 240,$$

and from Zeuthen's formula

$$3p = 2t.$$

The complete image of c'_{10} is c_8 counted twice and a residual c_{24} having an 8-fold point at A. The complete image of $\phi'_4 = 0$, $x'_4 = 0$ is A and a curve of order 8 having a six-fold point at A. The surfaces R_{20}, $\phi_4 = 0$ touch along this curve and intersect in c_{24}.

13. **Curves on composite quadrics.** Given a line γ on K_4, let π_1 pass through A. The image of γ is γ' in π'. Of the eight images in (x) of a point on π', four are in π_2, and four in π_1, of which one is fixed at A. The complete image of γ' is γ counted twice, a nodal cubic g_1 in π_1, with node at A, and a cubic g_2 in π_2. Both planes touch R_{20} along these cubics. The plane π_1 meets K_4 in a cubic $c_{3,1}$ having a double point at A. Of the nine points $(g_1, c_{3,1})$, three are on γ, four are at A, and two others. The image of $c_{3,1}$ is c'_4 and of $c_{3,2}$ is c'_6 in π'; together they constitute the residual section of π' with L'_{14}. The curve c'_4 touches γ' in three points, images of $(\gamma, c_{3,1})$, and cuts it in two points, images of $(g_1, c_{3,1})$ not at A nor on γ. The complete image of c'_6 is $c_{3,2}$ counted twice, a residual c_6 in π_2, and a c_{12} in π_1, with a six-fold point at A. The complete image of c'_4 is $c_{3,1}$ counted twice, a conic in π_1 not passing through A, and a c_8 in π_2.

The plane π_1 meets the contact curve c_{18} in 9 points on γ, six points at A, and three other points on $c_{3,1}$. Hence c'_4 has three cusps.

The plane π' meets the nodal curve c'_{40} in the six points of intersection (c'_6, γ') each counted twice, the two points of intersection (c'_4, γ') each counted twice, and in the 24 intersections (c'_4, c'_6).

14. **Web with two basis points.** Let $A \equiv (0, 0, 0, 1)$, $B \equiv (0, 0, 1, 0)$ be the two basis points. The equations of transformation may be expressed in the form

$$\begin{aligned} x_1' &= x_4 x_3 + a_1 x_3 x_1 + \phi_1(x_1, x_2), \\ x_2' &= x_4 x_2 + a_2 x_3 x_1 + \phi_2(x_1, x_2), \\ x_3' &= x_4 x_1 + \qquad\quad + \phi_3(x_1, x_2), \\ x_4' &= \qquad\qquad a_4 x_3 x_2 + \phi_4(x_1, x_2). \end{aligned}$$

The image of the point A is the plane $x_4' = 0$, and of the point B is the plane $x_3' = 0$. Any point of the line AB has for image the single point $(AB)' \equiv (1, 0, 0, 0)$. The complete image in (x) of $(AB)'$ is the line AB and four points not on the line. The image of the line $x_3' = 0$, $x_4' = 0$ consists of the line AB and the cubic curve $x_4 x_1 + \phi_3 = 0$, $a_4 x_3 x_2 + \phi_4 = 0$ which meets AB in two points. The four residual points, images of $(AB)'$, are the intersections of the cubic and the planes

$$a_4 x_2^2 \phi_3 + a_2 x_1^2 \phi_4 - a_4 x_1 x_2 \phi_2 = 0 \text{ through } AB.$$

The points A, B are both double points on K_4; the cone $a_3 x_3 x_2 + \phi_4 = 0$ is tangent cone to K_4 at A, and $x_4 x_1 + \phi_3 = 0$ is tangent cone to K_4 at B. The line AB lies entirely on K_4.

The surface L' is of order 12; the planes $x_3' = 0$, $x_4' = 0$ are both singular, touching it along conics. The point $(AB)'$ lies on both conics and is a double point on L'. The residual surface R is of order 16; it contains the residual cubic, image of $x_3' = 0$, $x_4' = 0$.

The image of a line l through A is a line l' meeting $x_4' = 0$ at Q' and $x_3' = 0$ at T'. The point Q' is the image of the direction of l at A; T' is the image of the direction of the cubic curve, residual to l, at B. The line l' is bitangent to L_{12}' and meets it in 8 other points. Hence l meets R_{16} in 8 points apart from A, so that A is an 8-fold point on R_{16}. Similarly, B is 8-fold on R_{16}. The line AB does not lie on R_{16}.

The residual cubic of l meets the cone $a_4 x_3 x_2 + \phi_4 = 0$ at B and at five other points. The five generators of the cone which pass through these points have for images five tangents to the curve c_8', the residual section of L_{12}' by $x_4' = 0$, apart from the conic of contact. The curve c_8' is of class 5 and is rational, hence it has 12 double points and 9 cusps. The tangent cone K_8 to R_{16} at A has therefore 12 double generators and 9 cuspidal generators.

A general quadric of the web meets K_4 in a curve of genus 7, hence a plane section of L_{12}' is of genus 7; since it has 24 bitangents, we conclude that the double curve of L_{12}' is of order 24, and the cuspidal curve is of order 24. The curve c_8' touches the singular conic of its own plane in 5 points, and intersects it in 6. The contact curve of R_{16}, K_4 is of order 17, and has a 5-fold point at A and at B. The curve of intersection is of order 30 and has a 6-fold point at A and at B. The tangent cones to K_4 and R_{16} at A touch along the generators which are images of the points of contact of c_8' and the singular conic in $x_4' = 0$; they intersect along the generators which are images of the six points of intersection. The cuspidal curve on R_{16} is of order 45, with A and B as 9-fold points; the double curve is of order 36, having A, B each 12-fold.

We now have the relations

$$2t + m = 180 - 36 = 144,$$

$$3p + m = 144 - 48 = 96,$$

$$3p = t,$$

hence $t = m = 48$, $p = 16$.

The image of c_8' is the c_7, intersection of K_4 and the tangent cone at A, counted twice, and a c_{18} having A, B each 6-fold. The image of the conic in $x_4' = 0$ is AB, and another c_7 with A 5-fold, B simple. The curve c_{18} and this c_7 counted twice is the complete intersection of R_{16} and $a_4 x_3 x_2 + \phi_4 = 0$.

15. **Two basis points; composite quadrics.** The lines γ are now of two different types. In the first type γ_1 one plane π_1 contains both A and B. In the second type γ_2 one plane π contains A, the other B.

The image of γ_1 is γ_1' in π_1'. The images of a point in π_1' lie four in π_2, and two in π_1, besides A and B. The complete image of γ_1' is γ_1 counted twice, the line AB, a conic g_1 through A and B, and a cubic g_2 in π_2. The plane π_1 meets K_4 in γ_1, AB, and a conic c_2 through A and B. The conics g_1, c_2 intersect on γ_1. The image of c_2 is a conic c_2' having contact with γ_1' at the images of (g_1, c_2). The plane π_2 meets K_4 in a cubic c_3 whose image in π_1' is c_6', which touches γ_1' in three points and intersects it in six.

The complete image of c_2' consists of c_2 counted twice, no other locus in π_1, and a c_4 in π_2. The complete image of c_6' is c_3 counted twice, a c_6 in π_2, and a c_{12} in π_1 having a six-fold point at A and at B. The plane π_1 meets the contact curve c_{17} in 7 points on γ, 5 at A, and 5 at B. The plane π_1' meets the cuspidal curve c_{24}' in the two points of contact (γ_1', c_2') and the three points of contact (γ_1', c_6'), each counted three times, and in the nine cusps of c_6'. The nodal curve c_{24}' meets π_1' in the six intersections (γ_1', c_6') each counted twice, and in the twelve intersections (c_2', c_6'). The plane π_2 meets the contact curve c_{17} in 8 points on γ_1 and nine points on c_3.

In case of a line γ_2, the two component planes enter symmetrically. Of the variable images of a point in π_2', three are in each plane. The complete image of γ_2' consists of γ_2 counted twice, and a nodal cubic g in each plane π.

The plane π meets K_4 in a nodal cubic c_3, meeting g in (γ_2, c_3) and in two other points not at the double point. The image of c_3 is c'_4, which touches γ'_2 in three points, images of (γ_2, c_3), and intersects it in two points, images of (g, c_3). The complete image of c'_4 is c_3 counted twice, a c_7 in π_2 with a double point at B, and a c_3 in π_1 with a double point at A. The contact curve c_{17} meets π in 9 points on γ_2, 5 points at A, and in three points on c_3. Hence c'_4 has 3 cusps.

16. **Three basis points.** Let A, B, C be three points common to all the quadrics of the web. Then A, B, C are all double points on K_4, and the lines AB, BC, CA lie on the surface, besides the lines γ_i. These three lines go into double points C', A', B' on L', and the vertices go into conics of contact.

One line γ_1 lies in the plane ABC, and one component plane of the composite quadric to which it belongs is the plane ABC. The image of this composite quadric is the plane $A'B'C'$.

The bundle A, B, C have congruences of bitangents of L' for images, each of order 5 and class 2.

L' is now of order 10, and any plane section is of genus 6, since it is the image of a $(4, 4)$ curve on a quadric, but having three double points. The cuspidal curve on L'_{10} is of order 18; the double curve is of order 12.

The surface R is of order 12, and has A, B, C for six-fold points. On L'_{10} the cuspidal curve is of order 18, and the double curve of order 12. The contact curve of K_4, R_{12} is of order 15, and has $3P_4$. The curve of intersection is of order 18 and has $3P_4$. The cuspidal curve on R_{12} is of order 27 and has $3P_6$. The double curve on R_{12} is of order 12 and has $3P_4$. The tangent cones A, B, C have 4 tangents with 4 point contact with K_4. In α', the residual curve to the singular conic is of order 6, class 4, and genus 0; it has 6 cusps and 4 double points.

This curve meets the conic in 4 contacts, each counting for 3 cusps, and 4 intersections, each counting as 2 double points.

We now have

$$2t + m = 108 - 36 = 72,$$

$$m = 48 - 24 = 24,$$

$$p = 0.$$

Hence

$$t = m = 24.$$

The lines γ are of two kinds; γ_1 lies in the plane ABC, and the other component plane passes through no basis point. The image γ'_1 passes through the three double points of L'_{10}. The complete image of γ'_1 consists of γ_1 counted twice, the three lines AB, BC, CA, and a cubic g_3 in π_2.

The plane ABC of K_4 has no residual curve; the plane π_1 meets K_4 in a cubic c_3 whose image in π_1' is a c_6'. The contact curve c_{15} has 9 points on c_3, hence c_6' has 9 cusps. It touches γ_1' in three points and meets it in six.

The composite quadrics through the remaining lines γ_2 consist of a plane π_1 through two basis points, and a plane π_2 through the other, hence γ_2 meets but one line as AB. The image conic γ_2' passes through one double point. The complete image in (x) of γ_2' consists of γ_2 counted twice, the line AB, a conic g_2 in π_1 through A and B, and a cubic g_3 in π_2, having a double point at C. The plane π_1 meets K_4 in a conic c_2 through A and B, and meeting g_2 on γ_2; the plane π_2 meets K_4 in a cubic c_3 having a double point at C. The curves (g_3, c_2) meet in two points apart from γ_2 and C. The image of c_2 is c_2' having double contact with γ_2'; the image of c_3 is c_4' touching γ_2' in 3 points.

The plane π_1 meets the contact curve c_{15} in 7 points on γ_2 and 8 points at A and B; the plane π_2 meets it in 8 points on γ_2, 4 points at C, and 3 points on c_3. Hence c_4' has 3 cusps.

17. **Four basis points.** Let A, B, C, D, the vertices of a proper tetrahedron, be fixed basis points. K_4 now has four double points and six additional lines. The surface L' is of order 8, has 14 singular planes, and 6 nodes. A plane section is of genus 5. The cuspidal curve is of order 12, and the double curve is of order 4.

The bundles of lines through A, B, C, D have for images congruences of order 4, class 2, having L_8' for complete focal surface.

The surface R is of order 8, and has a 4-fold point at each basis point. The contact curve is of order 12 and has 4 three-fold points; the curve of intersection is of order 8, and has double points at the basis points.

We now have

$$p = 0, \qquad m = 0, \qquad t = 12.$$

The lines γ are again of two kinds; in one case, π_1 contains three points, π_2 one; in the other, each contains two. The image plane of the former contains a c_4' with three cusps; in the latter, we have two conics, each having double contact with γ'. R_8 has a cuspidal curve of order 12, having a triple point at each basis point. It has no double curve.

18. **Five basis points.** The surface K_4 now has five double points and ten additional lines. The surface L' is of order 6, has fifteen singular planes, and ten double points. It has a cuspidal curve of order 6, and no double curve. The surface R is of order 4 and has the basis points for nodes. It has no cuspidal or double curve. The curve of contact is of order 8 and has double points at the basis points. There is no curve of intersection. K_4, R_4, L_6' are all in $(1, 1)$ correspondence.

In this case

$$p = m = t = 0.$$

The lines γ are all of the same kind; one plane passes through three basis points, the other through two. In every singular plane the residual section is a conic having double contact with the singular conic.

The images of the 5 bundles are congruences of order 3, class 2, having L'_6 for complete focal surface.

19. **Web with a basis line.** Most of the properties of this and other special cases have already been developed.*

Let d be the common basis line. It is a three-fold line on K_4, which is now a ruled surface. Every coincidence not on d belongs to a composite quadric. The images of the points of d are generators of a quadric F'_2 in (x'). L' is a developable surface of order 6; the cuspidal curve is of order 6, and the double curve of order 4. The image of L'_6 consists of K_4 taken twice and a developable quartic having a cuspidal cubic and no double curve. The image of the cuspidal curve c'_6 consists of a curve of contact c'_5 of order 5 counted three times, meeting d in four points, and the cuspidal curve of R_4. The image of the double curve c'_4 is the curve of intersection of K_4, R_4. It is of order 6 and meets d in the points (d, c'_5). The configurations in the cases having additional basis elements can now be obtained readily. Similarly for other special webs.†

2. Transformations of K_4

20. **Systems of curves on K_4.** The surface K_4 is a particular case of those having an equation of the form $|a_1\, b_2\, c_3\, d_4| = 0$, in which the elements are general linear functions of (x).‡ The general surface contains two triply infinite systems of sextics of genus 3, C_6 and $\overline{C}_6$, each cut from the surface by the cubics through a curve of the other. The system of plane sections $|C_4|$ and either system of sextics constitute a minimum base on the surface. In the case of K_4 it is necessary to enlarge the base on account of the lines γ_i. We shall now prove the following theorem.

Theorem. *The basis number of K_4 is* 11. *The systems $|C_4|$, $|C_6|$ and nine lines γ_i constitute a minimum base.*

Consider the transformation defined by $x'_i = D_i(x)$, D_i being the cofactor of d_i in the determinant $|a_1\, b_2\, c_3\, d_4|$. It is involutorial and under it

$$C_4 \sim C_6, \qquad C_6 \sim C_4.$$

The image of any line γ_i is a space cubic (γ_i). Since $[C_4, (\gamma_i)] = 3$, hence γ_i is a trisecant of every curve of the system C_6. Any cubic F_3 of the system

* See *Encyklopädie der mathematischen Wissenschaften*, III, C, 2, no. 143 for the literature.

† In particular see *Encyklopädie*, l. c., no. 142.

‡ Snyder and Sharpe, *Certain quartic surfaces belonging to infinite discontinuous cremonian groups*, these Transactions, vol. 16 (1915), pp. 62–70.

D which meets K_4 in (γ_i) also cuts K_4 in another cubic. Since $C_4 \sim C_6$, the residual $C_4 - \gamma_i$ is transformed into a cubic of genus 1, hence a plane cubic.

Since $[\gamma_i, C_4] = 1$, it follows that $[(\gamma_i), C_6] = 1$. But $[\gamma_i, C_4 - \gamma_i] = 3$, hence $[(\gamma_i), (C_4 - \gamma_i)] = 3$, and $[(C_4 - \gamma_i)]$ meets γ_i in three points. A plane cubic lying in a plane through γ_i is therefore transformed into a plane cubic whose plane passes through γ_i. It follows further that $[\gamma_i, (\gamma_i)] = 0$, but $[\gamma_i, (\gamma_k)] = 2$.

The surface of trisecants of C_6 is of order 8 and genus 3, having C_6 for triple curve. The intersection of this ruled surface with K_4 consists of the ten lines γ_i and a curve of order 4 and genus 3, hence a plane curve. Thus,

$$8C_4 \equiv 3C_6 + \sum\gamma_i + C_4 \qquad \text{or} \qquad 7C_4 \equiv 3C_6 + \sum\gamma_i.$$

The 12 systems $|C_4|$, $|C_6|$, γ_i are therefore not independent.

But if we form the determinant of eleven systems, omitting γ_1, it is found to be different from zero. Hence these eleven are independent. It is convenient, however, to retain all 12 systems which satisfy the preceding identity.

21. **Transformation of conjugate points on K_4.** When we can express the images of the curves of the base under any transformation, the image of any other system is determined. The transformation $x'_i = D_i$ is that of conjugate points as to the web. It is defined by the equations

$$S \qquad \begin{aligned} C_4 &\sim C_6, \\ C_6 &\sim C_4, \\ \gamma_i &\sim C_6 - C_4 + \gamma_i \equiv (\gamma_i). \end{aligned}$$

This transformation will be denoted by S.

22. **Transformation by bisecants of $C_6 - C_4 + \gamma_i$.** The bisecant of the space cubic (γ_i) through P on K_4 meets K_4 in the residual point P'. The transformation $P \sim P'$ is involutorial. The bisecants from points of a plane section C_4 describe a ruled surface R_{10} of order ten on which (γ_i) is five-fold. The residual intersection with K_4 is the image of C_4, apart from the nine lines $\gamma_k\,(k \neq i)$. It has the symbol $10C_4 - C_4 - 5(C_6 - C_4 + \gamma_i) - (\sum\gamma - \gamma_i)$. Hence

$$C_4 \sim 7C_4 - 2C_6 - 4\gamma_i.$$

The bisecants of (γ_i) from points on γ_i describe a ruled surface R_4 of order 4, having (γ_i) for double curve. Since no other γ_k intersects γ_i, γ_k can not lie on the ruled surface. The complete intersection of R_4 and K_4 consists of γ_i, (γ_i) counted twice, and a residual curve of order 9, the proper image of γ_i. Hence

$$\gamma_i \sim 6C_4 - 2C_6 - 3\gamma_i.$$

The lines γ_k are all bisecants of (γ_i), hence remain invariant. The image of C_6 could be obtained in the same way, but is found more easily from the fact that the identity must remain invariant. We find

$$C_6 \sim 12C_4 - 3C_6 - 8\gamma_i.$$

Collecting these results, we have

$$T \qquad \begin{aligned} C_4 &\sim 7C_4 - 2C_6 - 4\gamma_i, \\ C_6 &\sim 12C_4 - 3C_6 - 8\gamma_i, \\ \gamma_i &\sim 6C_4 - 2C_6 - 3\gamma_i, \\ \gamma_k &\sim \gamma_k. \end{aligned}$$

This transformation will be designated by T. There are ten different transformations of type T.

23. **Transformations by transversals of γ_i, γ_k.** This transformation is also involutorial. The transversals which meet C_4 form an R_6 having γ_i, γ_k for triple lines, hence

$$C_4 \sim 5C_4 - 3\gamma_i - 3\gamma_k.$$

Each curve of $|C_4 - \gamma_i|$, $|C_4 - \gamma_k|$ is invariant, hence

$$\begin{aligned} \gamma_i &\sim 4C_4 - 2\gamma_i - 3\gamma_k, \\ \gamma_k &\sim 4C_4 - 3\gamma_i - 2\gamma_k. \end{aligned}$$

The transversals of γ_i, γ_k and any other line γ_n describe a quadric surface, so that

$$\gamma_n \sim 2C_4 - \gamma_i - \gamma_k - \gamma_n.$$

Using the same method as before for C_6 we obtain

$$C_6 \sim 6C_4 - C_6 - 3\gamma_i - 3\gamma_k.$$

There are 45 transformations of this type; they will be designated by U.

24. **Transformations by secants of γ_i, (γ_k).** This is also involutorial. Using the same methods as before, we find for this transformation

$$V \qquad \begin{aligned} C_4 &\sim 12C_4 - 3C_6 - 3\gamma_i - 7\gamma_k, \\ C_6 &\sim 17C_4 - 4C_6 - 3\gamma_i - 11\gamma_k, \\ \gamma_i &\sim 5C_4 - C_6 - 2\gamma_i - 3\gamma_k, \\ \gamma_k &\sim 11C_4 - 3C_6 - 3\gamma_i - 6\gamma_k, \\ \gamma_n &\sim 3C_4 - C_6 - \gamma_i - \gamma_k - \gamma_n. \end{aligned}$$

There are 90 of these transformations; they will be designated by V.

The product of any two of the transformations S, T, U, V is non-periodic, hence these operations generate a discontinuous group of infinite order.

25. **Transformation of K_4 into the symmetroid.** Under the transformation $x'_1 = A_1$, $x'_2 = B_1$, $x'_3 = C_1$, $x'_4 = D_1$, the equation of K_4 is transformed into $\Delta' = 0$, in which Δ' is a symmetric determinant, the elements being linear functions of x'. This is the equation of the symmetroid.*

The fundamental sextic in (x) belongs to the system C_6. The curves of the other system $\bar{C}_6$ are transformed into plane sections C'_4 of the symmetroid. The plane sections C_4 of K_4 are transformed into a system of sextics C'_6 on the symmetroid, passing through the ten double points P'_i images of the lines γ_i. On account of the symmetry, there is only one such system of sextics. They are cut out in pairs by the cubic surfaces through the ten double points P', which are the images of the planes in (x).

Hence

$$2C'_6 = 3C'_4 - \sum P'_i;$$

this is the transform of the identity in (x). The relations between the two systems are now

$$C_4 \sim C'_6, \qquad C_6 \sim 3C'_6 - C'_4, \qquad \gamma_i \sim P'_i.$$

26. **The general quartic surface through a sextic curve of genus 3.** The transformation $x'_i = D_i$ sends $F = |a_1\, b_2\, c_3\, d_4| = 0$ into a similar quartic $F' = 0$, the transformation $x'_1 = A_1$, $x'_2 = B_1$, $x'_3 = C_1$, $x'_4 = D_1$ sends $F = 0$ into a similar quartic $F'' = 0$. By a transformation of the same kind $F' = 0$ can be transformed into $F = 0$ or into $F'' = 0$, and $F'' = 0$ into $F = 0$ or into $F' = 0$. Hence we may transform $F = 0$ into $F' = 0$, then $F' = 0$ into $F'' = 0$, and finally $F'' = 0$ into $F = 0$. The series of transformations may be expressed as follows:

$$T_1 \begin{array}{l} C_4 \sim C'_6, \\ C_6 \sim C'_4, \end{array} \qquad T_2 \begin{array}{l} C'_4 \sim 3C''_4 - C''_6, \\ C'_6 \sim 8C''_4 - 3C''_6, \end{array} \qquad T_3 \begin{array}{l} C''_4 \sim 3C_4 - C_6, \\ C''_6 \sim C_4. \end{array}$$

Hence the transformation

$$T_1\, T_2\, T_3 \begin{array}{l} C_4 \sim 21C_4 - 8C_6, \\ C_6 \sim 8C_4 - 3C_6, \end{array}$$

leaves K invariant. It is not periodic.†

* Cayley, *Collected Works*, vol. 7, pp. 133–181. See p. 134.

† See Snyder and Sharpe, loc. cit., p. 65. It is readily verified that $T_1\, T_2\, T_3$ is identical with τ^3. The transformation τ^2 is non-existent. That the transformation $F \backsim F'$, etc., lead to a non-periodic transformation which leaves $F = 0$ invariant is mentioned without proof by Cayley, loc. cit., p. 159.

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