

# DIFFERENTIATION WITH RESPECT TO A FUNCTION OF LIMITED VARIATION\*

BY

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Lebesgue, Radon, and Young† have defined integrals with respect to a function of limited variation, and these are generalizations of the Stieltjes integral. The next step which suggests itself is a definition of the corresponding derivative. Such a definition is given in this paper, and the fundamental property of a derivative is proved by means of a modification of Vitali's theorem. The steps taken are parallels of steps given by de la Vallée Poussin‡ in the theory of Lebesgue plane sets. Since this paper was first written, a paper by Young§ has appeared, giving a slightly different definition of the derivative, and an entirely different treatment. The applications at the end of this paper are not given by Young.

**Definition of derived numbers and derivative.** Consider two functions of  $x$ ,  $F(x)$ ,  $\alpha(x)$ , defined in the fundamental interval,  $0 \leq x \leq 1$ . The ratio

$$\frac{F(x + \epsilon) - F(x - \epsilon)}{\alpha(x + \epsilon) - \alpha(x - \epsilon)} = \frac{\Delta F}{\Delta \alpha}$$

may have upper and lower limits as  $\epsilon$  approaches 0. We define these as the upper and lower derived numbers of  $F(x)$  with respect to  $\alpha(x)$ , and we use the notation

$$\bar{D}_\alpha F(x) = \overline{\lim}_{\epsilon \rightarrow 0} \frac{\Delta F}{\Delta \alpha}, \quad D_\alpha F(x) = \underline{\lim}_{\epsilon \rightarrow 0} \frac{\Delta F}{\Delta \alpha}.$$

For  $x$  equal to 0 or 1 it is necessary to add a convention whereby  $F(x)$ ,  $\alpha(x)$  are continued beyond the range  $(0, 1)$ , and have values equal to their values at 0 or 1 respectively. If the two derived numbers are finite and equal,

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† Lebesgue, *Comptes Rendus*, vol. 150 (1910), p. 86; Radon, *Wiener Sitzungsberichte*, vol. 122, section 2a (1913), p. 1295; W. H. Young, *Proceedings of the London Mathematical Society*, vol. 13 (1914), p. 109.

‡ De la Vallée Poussin, these *Transactions*, vol. 16 (1915), p. 435.

§ W. H. Young, *Proceedings of the London Mathematical Society*, vol. 15 (1916), p. 35.

that is, if the relative change ratio  $\Delta F/\Delta\alpha$  possesses a finite limit, this limit is called the  $\alpha$ -derivative of  $F(x)$ , and is denoted by

$$D_\alpha F(x).$$

An interval  $(x - \epsilon, x + \epsilon)$  we shall call a *central interval* with center at  $x$ .

If  $\alpha(x)$ ,  $F(x)$ , are functions of limited variation, we can define absolutely additive functions of sets measurable Borel

$$\int_E d\alpha, \quad \int_E dF,$$

and also the corresponding modular integrals, denoted according to Radon by

$$\int_E |d\alpha|, \quad \int_E |dF|,$$

or according to the writer by

$$\int_E d\omega, \quad \int_E d\Omega,$$

where  $\omega(x)$ ,  $\Omega(x)$ , are the variation functions corresponding to  $\alpha(x)$ ,  $F(x)$ . These modular integrals are additive, finite, and non-negative.

**Absolute continuity.** A function  $F(x)$  is said to be absolutely continuous relative to  $\alpha(x)$ , a function of limited variation, if given any positive  $\epsilon$  we can find  $\delta$  so that

$$\int_e d\Omega(x) < \epsilon,$$

for all sets  $e$  measurable Borel such that

$$\int_e d\omega(x) < \delta,$$

where  $\omega(x)$ ,  $\Omega(x)$  are the variation functions corresponding to  $\alpha(x)$ ,  $F(x)$ . We desire to prove the fundamental proposition:

**THEOREM.** *If  $F(x)$  is absolutely continuous relative to  $\alpha(x)$ , it possesses a finite  $\alpha$ -derivative nearly everywhere ( $\omega$ ) [that is, except for a point set  $e$  for which  $\int_e d\omega(x) = 0$ ]; this  $\alpha$ -derivative is summable ( $\alpha$ ) where it exists, and if  $E$  is any set measurable Borel,*

$$\int_E dF(x) = \int_E D_\alpha F(x) d\alpha(x).$$

In this,  $D_\alpha F(x)$  denotes the  $\alpha$ -derivative of  $F(x)$  where it exists, and any finite value where it does not.

Before we can prove this proposition, it is necessary to prove two lemmas.

LEMMA 1. *Given any positive  $\epsilon$  and a non-decreasing function  $\omega(x)$ , any set  $E$  measurable Borel can be enclosed strictly in a finite or denumerable system  $A$  of disjoint intervals (that is, intervals with no points common to any two) in such a way that*

$$\int_A d\omega < \int_E d\omega + \epsilon.$$

Let  $D$  be the set of points at which  $\omega(x)$  has finite discontinuities; then  $D$  consists at most of a denumerable set of points. Resolve  $\omega(x)$  into  $\omega_1(x)$ , a continuous non-decreasing function, and  $\omega_2(x)$ , a non-decreasing function which is stationary except when  $x$  passes a point of  $D$ . The integral  $\int_E d\omega_1(x)$  will be a continuous additive non-negative function of sets. Hence,\* given any positive  $\epsilon$ , we can enclose  $E$  strictly in a finite or denumerable system  $A_1$  of disjoint intervals so that

$$\int_{A_1} d\omega_1(x) < \int_E d\omega_1(x) + \frac{1}{2}\epsilon.$$

The set  $CE \cdot D$  consists at most of a denumerable set of points  $D'$ . Then, since

$$\int_{D'} d\omega(x) = \int_{D'} d\omega_2(x)$$

is a convergent series of positive terms (considered as the sum of the discontinuities of  $\omega(x)$  at the denumerable set of points  $D'$ ), we can choose a finite  $n$  and the finite set of points  $D_n$  so that

$$\int_{D'} d\omega_2(x) < \int_{D_n} d\omega_2(x) + \frac{1}{2}\epsilon.$$

From  $A_1$  cut out the points belonging to  $D_n$ , which are finite in number. Then  $A = A_1 \cdot CD_n$  still forms a denumerable system of intervals enclosing  $E$  strictly; for the points  $D_n$  belong to  $CE$ . The set  $A$  is the same as  $A_1$  except for the exclusion of a finite set of points, or

$$\int_A d\omega_1(x) = \int_{A_1} d\omega_1(x) < \int_E d\omega_1(x) + \frac{1}{2}\epsilon.$$

Again, from the way in which  $D_n$  was chosen,

$$\int_{CE \cdot D \cdot CD_n} d\omega_2(x) < \frac{1}{2}\epsilon.$$

But

$$\int_e d\omega_2(x) = 0$$

\* De la Vallée Poussin, loc. cit., p. 470.

over any set  $e$  not containing points of  $D$ . Hence

$$\int_{CE \cdot CD_n} d\omega_2(x) < \frac{1}{2}\epsilon, \quad \int_{A_1 \cdot CE \cdot CD_n} d\omega_2(x) < \frac{1}{2}\epsilon,$$

and, since  $A = A_1 \cdot CD_n$ ,

$$\int_{A \cdot CE} d\omega_2(x) < \frac{1}{2}\epsilon, \quad \int_A d\omega_2(x) < \int_E d\omega_2(x) + \frac{1}{2}\epsilon.$$

Hence

$$\int_A d\omega(x) = \int_A d\omega_1(x) + \int_A d\omega_2(x) < \int_E d\omega(x) + \epsilon.$$

The lemma is proved.

The following lemma is a generalization of Vitali's theorem.\*

LEMMA 2. *Given a set  $E$  measurable Borel, and an infinite family  $\Gamma$  of central intervals, such that each point of  $E$  is the center of an infinity of central intervals as small as we please; then a set  $B$  can be found consisting of a finite or denumerable number of disjoint intervals chosen from  $\Gamma$ , such that  $B$  covers nearly all  $E$  (that is, except for a point set of  $\omega$ -measure 0) and such that the  $\omega$ -measure of  $B$  differs from that of  $E$  by as little as we please.*

That is to say, given any positive  $\epsilon$ , we can find  $B$  so that

$$\int_{E \cdot CB} d\omega(x) = 0, \quad \int_B d\omega(x) < \int_E d\omega(x) + \epsilon.$$

By means of Lemma 1, given any positive  $\epsilon$ , we can enclose  $E$  strictly in a denumerable set of disjoint intervals  $A$ , so that

$$\int_A d\omega' < \int_E d\omega' + \epsilon,$$

where  $\omega'(x) = \omega(x) + x$ . In what follows we denote the  $\omega'$ -measure of a set  $E$  simply by  $mE$ . Then  $mE$  is the sum of the  $\omega$ -measure and the usual Lebesgue measure. The intervals  $A$  are not necessarily central intervals. From the family  $\Gamma$  eliminate the intervals which have points in common with  $CA$ . The remaining family  $\Gamma_1$  will possess the same property relative to  $E$ , for each point of  $E$  is interior (strictly) to one or other of the intervals  $A$ .

We affirm that with a finite number of disjoint intervals of  $\Gamma_1$  we can cover a part  $e_1$  of  $E$  such that  $me_1 > kmE$ , where  $k$  is any number less than one-third. For † let the set  $CE$  be enclosed strictly in a denumerable set  $D$  of open intervals so that  $mD$  is arbitrarily close to  $mCE$ . Then  $CD$  is a closed set contained in  $E$ , and given any positive  $\epsilon$  we can make  $mCD > mE - \epsilon$ . By the Heine-

\* See de la Vallée Poussin, *Cours d'Analyse*, second edition, vol. 2, p. 110.

† This part of the proof is due to the Editors of these Transactions.

Borel Theorem\*  $CD$  can be covered by a finite number of intervals  $E_n$  chosen from the family  $\Gamma_1$ . Then

$$mE_n \cong mCD > mE - \epsilon.*$$

From these intervals choose first that one having the greatest  $\omega'$ -measure, and eliminate those which have any point in common with it. Next choose the remaining interval which has the greatest  $\omega'$ -measure, and so on. After a finite number of such steps we shall have chosen a finite number of intervals, and the process will terminate. Each time intervals are eliminated, the  $\omega'$ -measure of the interval retained will be at least one-third of the  $\omega'$ -measure of the interval covered by it and all intervals eliminated as overlapping it. Hence the measure of all intervals retained will be at least  $\frac{1}{3}mE_n$ , or will be greater than  $\frac{1}{3}mE - \epsilon$ . The part of  $E$  not covered, that is  $E - e_1$ , is therefore of measure less than  $mE - \frac{1}{3}mE + \epsilon$ , or less than  $\frac{2}{3}mE + 2\epsilon$ . Hence

$$me_1 > \frac{1}{3}mE - 2\epsilon.$$

Our affirmation is proved, that with a finite number of disjoint intervals belonging to  $\Gamma_1$  we can cover a part  $e_1$  of  $E$  so that

$$me_1 > kmE,$$

where  $k$  is any number less than one-third.

After we have thus chosen  $e_1$ , omit from  $\Gamma_1$  those intervals which overlap  $e_1$ , and let  $\Gamma_2$  be the remainder. Then  $\Gamma_2$  will have the same properties relative to  $E - e_1$  as  $\Gamma_1$  has relative to  $E$ . We can by the same process cover a portion  $e_2$  of  $E - e_1$ , such that

$$me_2 > km(E - e_1),$$

by a finite number of intervals of  $\Gamma_1$ . Continuing the process, we obtain a system  $B$  of disjoint intervals of the family  $\Gamma$ , which are at most denumerable. Moreover the set  $B \cdot E = \sum e_n$ . But

$$me_n > k(mE - me_1 - \dots - me_{n-1}),$$

and the term in the parenthesis is non-negative, and therefore  $\sum me_n$  is convergent. Thus  $me_n$  approaches zero, or

$$m(\sum e_n) = \sum me_n = mE.$$

Then

$$mE \cdot B = mE, \quad mE \cdot CB = 0, \quad \int_{E \cdot CB} d\omega' = 0.$$

But  $\omega'(x) = \omega(x) + x$ , so that

\* Cf. de la Vallée Poussin, *Intégrales de Lebesgue*, pp. 13-15.

$$\int_{E \cdot CB} d\omega \cong \int_{E \cdot CB} d\omega' = 0;$$

and, since

$$\int_{A-E} d\omega \cong \int_{A-E} d\omega' < \epsilon, \quad \int_A d\omega < \int_E d\omega + \epsilon.$$

But  $B$  is contained in  $A$ , or

$$\int_B d\omega < \int_E d\omega + \epsilon.$$

The lemma is proved.

To return to our original proposition, let us prove it first in the case where  $\alpha(x)$  is a non-decreasing function, so that  $\alpha(x) = \omega(x)$ .

In any set  $e$ , if  $\bar{D}_\omega F(x) \cong l$  (this inequality being considered to hold if  $\bar{D}_\omega F(x) = +\infty$ ), we shall show that

$$\int_e dF \cong l \int_e d\omega.$$

When this is proved, it will follow as a corollary, and can also be established directly by parallel reasoning, that a similar conclusion holds if the signs of inequality are reversed, or if  $\bar{D}$  is replaced by  $\underline{D}$ , or both. At any rate, it is sufficient to give the proof for the case first mentioned. Corresponding to every point in  $e$  (which is measurable Borel) we can find an infinity of central intervals as small as we please, such that for each

$$\frac{\Delta F}{\Delta \omega} > l - \epsilon',$$

given any positive  $\epsilon'$ . These form a family  $\Gamma$  having the Vitali property relative to  $e$ . Since  $F(x)$  is absolutely continuous with respect to  $\omega(x)$ , given any positive  $\epsilon$  we can find  $\delta$  so that

$$\int_e d\Omega < \epsilon,$$

for all sets  $e$  for which

$$\int_e d\omega < \delta.$$

Using Lemma 2, we can define a set  $B$  consisting of a denumerable system of disjoint intervals belonging to  $\Gamma$ , such that

$$\int_{e \cdot CB} d\omega = 0,$$

whence

$$\int_{e \cdot CB} d\Omega = 0.$$

and such that

$$\int_{B \cdot C_\epsilon} d\omega < \delta,$$

whence

$$\int_{B \cdot C_\epsilon} d\Omega < \epsilon.$$

Then

$$\left| \int_e d\omega - \int_B d\omega \right| < \delta, \quad \left| \int_e dF - \int_B dF \right| < \epsilon.$$

But for each of the intervals  $B$ ,

$$\frac{\Delta F}{\Delta \omega} > l - \epsilon', \quad \Delta F > l\Delta\omega - \epsilon'\Delta\omega, \quad \int_B dF > l \int_B d\omega - \epsilon' \int_B d\omega,$$

$$\int_e dF > (l - \epsilon') \int_e d\omega - \epsilon - |l|\delta - \epsilon'\delta.$$

In the limit, as  $\epsilon, \epsilon'$  approach 0,  $\delta$  also approaches 0, and

$$\int_e dF \cong l \int_e d\omega.$$

Since  $\int_e dF$  is limited, the  $\omega$ -measure of  $e$  decreases to the limit 0 as  $l$  increases indefinitely. Also, since  $F$  is absolutely continuous with respect to  $\omega$ ,  $\int_e dF$  will also approach the limit 0. Thus the set of points for which  $\bar{D}_\omega F(x) = +\infty$  is of  $\omega$ -measure 0. A similar proof shows that the set for which  $\underline{D}_\omega F(x) = -\infty$  is of  $\omega$ -measure 0.

Take two finite numbers  $m, M$ , and divide the interval between them into sub-intervals by  $m = l_0 < l_1 < \dots < l_n = M$ , where  $\max |l_i - l_{i-1}| < \eta$  a given positive  $\eta$ . Let  $e_i$  be the set in  $E$  for which  $l_{i-1} \leq \bar{D}_\omega F < l_i$ ; then

$$l_{i-1} \int_{e_i} d\omega \leq \int_{e_i} dF \leq l_i \int_{e_i} d\omega,$$

$$\left| \int_{e_i} dF - l_i \int_{e_i} d\omega \right| \leq \eta \int_{e_i} d\omega.$$

By summing up for the sets  $e_i$ , if  $E'$  is the set of points where  $m \leq \bar{D}_\omega F < M$ ,

$$\left| \int_{E'} dF - \sum_i l_i \int_{e_i} d\omega \right| \leq \eta \int_{E'} d\omega;$$

in the limit, as  $\eta$  approaches 0,  $\bar{D}_\omega F$  is summable ( $\omega$ ) in  $E'$ , and

$$\int_{E'} dF = \int_{E'} \bar{D}_\omega F d\omega.$$

It has been proved already that  $\int_{E-\varepsilon'} dF$  approaches 0 as  $M$  increases and  $m$  decreases indefinitely, whence  $\bar{D}_\omega F$  is summable ( $\omega$ ) in  $E$ , and

$$\int_E dF = \int_E \bar{D}_\omega F d\omega.$$

Similarly it can be proved that  $\underline{D}_\omega F$  is summable ( $\omega$ ) in  $E$ , and

$$\int_E dF = \int_E \underline{D}_\omega F d\omega.$$

But  $\underline{D}_\omega F \leq \bar{D}_\omega F$ , or at every point of  $E$  except on a point set of  $\omega$ -measure 0,

$$\underline{D}_\omega F = \bar{D}_\omega F = D_\omega F,$$

and  $F(x)$  has a finite derivative with respect to  $\omega(x)$  nearly everywhere ( $\omega$ ) in  $E$ . Also

$$\int_E dF = \int_E D_\omega F d\omega.$$

More generally, if  $\alpha(x)$  is a function of limited variation, it is absolutely continuous with respect to its variation function  $\omega(x)$ . We may split any set  $E$  measurable Borel into two subsets  $E_1, E_2$ , so that if  $e$  is a variable set,

$$\int_{e \cdot E_1} d\alpha = \int_{e \cdot E_1} d\omega,$$

a non-negative function of sets, and

$$\int_{e \cdot E_2} d\alpha = - \int_{e \cdot E_2} d\omega.$$

We have already proved that

$$\int_{E_1} dF = \int_{E_1} D_\omega F d\omega = \int_{E_1} D_\omega F d\alpha,$$

and by the same proof  $D_\omega \alpha$  exists and is finite nearly everywhere ( $\omega$ ), and

$$\int_{e \cdot E_1} d\omega = \int_{e \cdot E_1} d\alpha = \int_{e \cdot E_1} D_\omega \alpha d\omega;$$

whence  $D_\omega \alpha = \lim \Delta\alpha/\Delta\omega$  exists and equals 1 nearly everywhere ( $\omega$ ) in  $E_1$ . It follows that  $D_\alpha F$  exists and equals  $D_\omega F$  nearly everywhere ( $\omega$ ) in  $E_1$ ; or

$$\int_{E_1} dF = \int_{E_1} D_\alpha F d\alpha.$$

Similarly in  $E_2$ ,  $D_\alpha F$  exists and equals  $-D_\omega F$  nearly everywhere ( $\omega$ ), and



$$\int_{E_2} dF = \int_{E_2} D_\omega F d\omega = - \int_{E_2} D_\alpha F d\alpha = \int_{E_2} D_\alpha F d\alpha.$$

By combining  $E_1$  and  $E_2$ , it is proved that  $D_\alpha F$  exists and is finite in  $E$  except for a set of  $\omega$ -measure 0, that it is summable in  $E$ , and that

$$\int_E dF(x) = \int_E D_\alpha F(x) d\alpha(x).$$

**Applications.** *Reduction of general integral to integral of positive type.* Let  $g(x)$  denote the function

$$g(x) = D_\omega \alpha(x),$$

where this derivative = + 1 or - 1, and

$$g(x) = 0$$

otherwise, that is to say on a set of  $\omega$ -measure 0. On the set  $E_1$ ,  $g(x) = 1$  nearly everywhere ( $\omega$ ), whence, if  $f(x)$  is any function summable ( $\omega$ ),

$$\int_{E_1} f(x) d\alpha(x) = \int_{E_1} f(x) d\omega(x) = \int_{E_1} f(x) g(x) d\omega(x).$$

Similarly,

$$\int_{E_2} f(x) d\alpha(x) = - \int_{E_2} f(x) d\omega(x) = \int_{E_2} f(x) g(x) d\omega(x),$$

whence

$$\int_E f(x) d\alpha(x) = \int_E f(x) g(x) d\omega(x).$$

This proves that any integral with respect to a function of bounded variation  $\alpha(x)$  can be expressed as a single integral of positive type, that is to say, an integral with respect to a non-decreasing function  $\omega(x)$ .

*Integration by Parts.* Let  $F(x) = \alpha^2(x)$ , where  $\alpha(x)$  is a function of limited variation. The latter function may be expressed as the difference of two that are non-decreasing,

$$\alpha(x) = \beta_1(x) - \beta_2(x),$$

whence

$$F(x) = \alpha^2(x) = \beta_1^2(x) + \beta_2^2(x) - 2\beta_1(x)\beta_2(x),$$

$$\Omega(x) = \beta_1^2(x) + \beta_2^2(x) + 2\beta_1(x)\beta_2(x) = \omega^2(x).$$

The function  $\omega(x)$  is limited and less than some finite number  $K$ , so that for any interval

$$\Delta\Omega < 2K \Delta\omega;$$

whence  $F(x) = \alpha^2(x)$  is absolutely continuous with respect to  $\alpha(x)$ . In this case

$$\frac{\Delta F}{\Delta \alpha} = \frac{\alpha^2(x + \epsilon) - \alpha^2(x - \epsilon)}{\alpha(x + \epsilon) - \alpha(x - \epsilon)} = \alpha(x + \epsilon) + \alpha(x - \epsilon).$$

Let  $\bar{\alpha}(x)$  denote

$$\lim_{\epsilon \rightarrow 0} \frac{1}{2} [\alpha(x + \epsilon) + \alpha(x - \epsilon)];$$

then

$$D_\alpha \alpha^2(x) = 2\bar{\alpha}(x).$$

By the fundamental proposition,

$$\int_E d\alpha^2(x) = \int_E 2\bar{\alpha}(x) d\alpha(x).$$

Let  $\alpha_1(x)$ ,  $\alpha_2(x)$  be two functions of limited variation; then

$$\begin{aligned} \int d[(\alpha_1 + \alpha_2)^2] &= \int d\alpha_1^2 + \int d\alpha_2^2 + 2 \int d(\alpha_1 \alpha_2) \\ &= 2 \int (\bar{\alpha}_1 + \bar{\alpha}_2) d(\alpha_1 + \alpha_2) \\ &= 2 \int \bar{\alpha}_1 d\alpha_1 + 2 \int \alpha_2 d\alpha_2 + 2 \int \bar{\alpha}_1 d\alpha_2 + 2 \int \bar{\alpha}_2 d\alpha_1. \end{aligned}$$

So

$$\int_E d(\alpha_1 \alpha_2) = \int_E \bar{\alpha}_1 d\alpha_2 + \int_E \bar{\alpha}_2 d\alpha_1.$$

Transposed and written more fully, the last relation becomes

$$\int_E \bar{\alpha}_1(x) d\alpha_2(x) = \int_E d[\alpha_1(x)\alpha_2(x)] - \int_E \bar{\alpha}_2(x) d\alpha_1(x).$$

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