PROJECTIVE TRANSFORMATIONS IN FUNCTION SPACE*

BY

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The Fredholm transformation†

$$\phi'(x) = \phi(x) + \int_0^1 \gamma(x, y)\phi(y)\,dy, \tag{1}$$

considered as a geometric transformation of a point $\phi(x)$ of function space‡ into a point $\phi'(x)$ of the same space, is an analogue of the transformation

$$x_i' = x_i + \sum_{j=1}^{n} c_{ij}\, x_j, \qquad (i = 1, 2, \cdots, n), \tag{1_n}$$

which takes a point $(x_1, x_2, \cdots, x_n)$ of n-space into a point $(x_1', x_2', \cdots, x_n')$ of the same space. This latter transformation is a special case of the general projective transformation

$$x_i' = \frac{a_i + b_i\, x_i + \sum_{j=1}^{n} c_{ij}\, x_j}{d + \sum_{j=1}^{n} e_j\, x_j}, \qquad (i = 1, 2, \cdots, n), \tag{2_n}$$

the specialization being characterized geometrically by the fact that the transformation (1_n) leaves invariant the origin and all points at infinity.

The purpose of the present paper is the study of an analogue in function space of the transformation (2_n), viz.,

$$\phi'(x) = \frac{\alpha(x) + \beta(x)\phi(x) + \int_0^1 \gamma(x, y)\phi(y)\,dy}{\delta + \int_0^1 \epsilon(y)\phi(y)\,dy}, \tag{2}$$

* Containing results presented to the Society, December 27, 1916, and September 4, 1917.

† Treated by Fredholm in his famous memoir, Acta mathematica, vol. 27.

‡ An introduction to geometry of function space has been given by Kowalewski, Sitzungsberichte der mathematisch-naturwissenschaftlichen Klasse der Kaiserlichen Akademie der Wissenschaften zu Wien, vol. 120, pp. 77 and 1435. See also a paper by Ingold, these Transactions, vol. 13 (1912), pp. 319–341. In the present paper as in both of the papers cited, the point of view is that of non-homogeneous coordinates; that is, $\phi(x)$ (not identically zero) and $c\phi(x)$ represent distinct points unless $c = 1$.

which we shall call the *projective functional transformation*. The principal features of the well known theory relative to the transformation (1) are extended to the transformation (2).

An extension of the notion of the Fredholm determinant is defined, (§ 1); and a unique inverse of (2) is obtained in terms of this new determinant and four suitably defined first minors, upon the assumption that the determinant is different from zero, (§ 6). A product theorem for transformations (2) is obtained, (§ 5); and from it follows readily the result that the totality of transformations (2) with non-vanishing determinants form a group. We call it the group of non-singular projective transformations.

In Sections 8–10, the *infinitesimal* projective transformation is considered. Infinitesimal transformations in function space have been studied by Kowalewski,* who termed *regular* an infinitesimal transformation of form

$$\delta\phi = \mathrm{P}(\phi)\,\delta t,$$

where the coefficient $\mathrm{P}(\phi)$ of the independent infinitesimal δt is an integral-power-series of the type defined by E. Schmidt.† The most general regular infinitesimal transformation possessing the characteristic projective property of transforming lines of function space into lines was then found to be expressible in the form

$$\delta\phi(x) = [\lambda(x) + \mu(x)\phi(x) + \int_0^1 \gamma(x, y)\phi(y)\,dy - \phi(x)\int_0^1 \rho(y)\phi(y)\,dy]\,\delta t. \tag{3}$$

In Sections 9–10 of the present paper, the class of all finite transformations which can be generated (in the sense of Lie) by infinitesimal transformations of form (3) is shown to be essentially identical with the group of non-singular projective transformations (2), and the equations of transformation from one form to the other are obtained.

1. The bordered Fredholm determinant

The Fredholm determinant

$$D = 1 + \sum_{n=1}^{\infty} \frac{1}{n!}\int_0^1 \cdots \int_0^1 ds_1 \cdots ds_n |\gamma(s_i, s_j)| \quad (i, j = 1, 2, \cdots, n),$$

which is involved in the inversion of the transformation (1), is the functional analogue of the determinant

* Loc. cit.

† *Über die Auflösung der nichtlinearen Integralgleichung und die Verzweigung ihrer Lösungen*, Mathematische Annalen, vol. 65 (1908).

$$D_n = \begin{vmatrix} 1+c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & 1+c_{22} & \cdots & c_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ c_{n1} & c_{n2} & \cdots & 1+c_{nn} \end{vmatrix} = |d_{ij} + c_{ij}| \quad (i, j = 1, 2, \cdots, n)$$

of the transformation (1_n). In fact the Fredholm determinant D can be obtained as the limit, for increasing n, of a sequence of determinants of the form D_n.*

In a similar way we shall need, for the consideration of (2), the functional analogue of the determinant

$$B_n = \begin{vmatrix} 1+c_{11} & c_{12} & \cdots & c_{1n} & a_1 \\ c_{21} & 1+c_{22} & \cdots & c_{2n} & a_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ c_{n1} & c_{n2} & \cdots & 1+c_{nn} & a_n \\ e_1 & e_2 & \cdots & e_n & d \end{vmatrix} = \begin{vmatrix} d_{ij} + c_{ij} & a_i \\ e_j & d \end{vmatrix}$$

$$(i, j = 1, 2, \cdots, n),$$

which is the determinant of the transformation $(2n)$ in case $b_i = 1$.

This functional analogue, relative to continuous functions $\gamma(x, y)$, $\alpha(x)$, $\epsilon(y)$, and a constant δ, we define by the series

$$B = \delta + \sum_{n=1}^{\infty} \frac{1}{n!} \int_0^1 \cdots \int_0^1 ds_1 \cdots ds_n \begin{vmatrix} \gamma(s_i, s_j) & \alpha(s_i) \\ \epsilon(s_j) & \delta \end{vmatrix} \quad (i, j = 1, 2, \cdots, n);$$

the convergence of which can be established by methods similar to those commonly used (for instance by Fredholm) in proving the convergence of the series for D. It is, so to speak, a *bordered Fredholm determinant*†; and it can in fact be obtained as the limit of a sequence of bordered determinants of form B_n by a line of reasoning similar to that used by Kowalewski in obtaining D from D_n.

When it is desired to exhibit the functions upon which B depends, we shall use the notations

$$B = B\begin{bmatrix} \gamma(x, y) & \alpha(x) \\ \epsilon(y) & \delta \end{bmatrix} = B\begin{bmatrix} \gamma & \alpha \\ \epsilon & \delta \end{bmatrix}.$$

We note for future use the obvious identity

$$(4) \qquad B\begin{bmatrix} \gamma(x, y)\beta(y)/\beta(x) & \alpha(x)/\beta(x) \\ \epsilon(y)\beta(y) & \delta \end{bmatrix} = B\begin{bmatrix} \gamma(x, y) & \alpha(x) \\ \epsilon(y) & \delta \end{bmatrix},$$

* Cf. Kowaleswki, *Einführung in die Determinantentheorie*, Leipzig, 1909.

† Professor T. H. Hildebrandt has kindly called my attention to the fact that the idea of a bordered Fredholm determinant is not new. It is to be found for the case $\delta = 0$ in the works of Hilbert (*Integralgleichungen*, p. 11), and Landsberg, Mathematische Annalen, vol. 69, pp. 234, 235.

relative to any continuous function $\beta(x)$ which does not vanish on the interval of integration.

The determinants B admit the following

PRODUCT THEOREM. *The product of two bordered Fredholm determinants*

$$B\begin{bmatrix} \gamma' & \alpha' \\ \epsilon' & \delta' \end{bmatrix} \cdot B\begin{bmatrix} \gamma'' & \alpha'' \\ \epsilon'' & \delta'' \end{bmatrix}$$

is equal to a bordered Fredholm determinant

$$B\begin{bmatrix} \gamma & \alpha \\ \epsilon & \delta \end{bmatrix},$$

in which

$$\gamma(x, y) = \gamma'(x, y) + \gamma''(x, y) + \int_0^1 \gamma'(x, s)\gamma''(s, y)\,ds + \alpha'(x)\epsilon''(y),$$

$$\alpha(x) = \alpha''(x) + \int_0^1 \gamma'(x, t)\alpha''(t)\,dt + \alpha'(x)\delta'',$$

$$\epsilon(y) = \epsilon'(y) + \int_0^1 \epsilon'(s)\gamma''(s, y)\,ds + \delta'\epsilon''(y),$$

$$\delta = \int_0^1 \epsilon'(s)\alpha''(s)\,ds + \delta'\delta''.$$

This theorem can be established by a method similar to that used by Kowalewski for the product theorem for Fredholm determinants, that is by applying a limiting process to the determinant relation

$$B_n' \cdot B_n'' = B_n,$$

where B_n' and B_n'' are nth order determinants, the limits of which are the bordered Fredholm determinants which occur as factors in the theorem.*

For the application below, relative to the transformation (2), we shall need to use a slight modification of this theorem, the modification being characterized by the fact that certain of the argument functions appear as fractions, the denominators of which are supposed to be different from zero on the interval of integration.

MODIFIED PRODUCT THEOREM. *The product of two bordered Fredholm determinants*

$$(5)\quad B\begin{bmatrix} \gamma'(x, y)/\beta'(x) & \alpha'(x)/\beta'(x) \\ \epsilon'(y) & \delta' \end{bmatrix} \cdot B\begin{bmatrix} \gamma''(x, y)/\beta''(x) & \alpha''(x)/\beta''(x) \\ \epsilon''(y) & \delta'' \end{bmatrix}$$

is equal to a bordered Fredholm determinant

* This proof, while simple in outline, involves somewhat long and complicated details which will be omitted. An alternative proof developed by Professor Hildebrandt is, I believe, soon to be published.

$$B\begin{bmatrix} \gamma(x,y)/\beta(x) & \alpha(x)/\beta(x) \\ \epsilon(y) & \delta \end{bmatrix},$$

in which

$$\gamma(x,y) = \beta'(x)\gamma''(x,y) + \gamma'(x,y)\beta''(y) + \int_0^1 \gamma'(x,s)\gamma''(s,y)\,ds + \alpha'(x)\epsilon''(y),$$

$$\alpha(x) = \beta'(x)\alpha''(x) + \int_0^1 \gamma'(x,t)\alpha''(t)\,dt + \alpha'(x)\delta'',$$

$$\epsilon(y) = \epsilon'(y)\beta''(y) + \int_0^1 \epsilon'(s)\gamma''(s,y)\,ds + \delta'\epsilon''(y),$$

$$\delta = \int_0^1 \epsilon'(s)\alpha''(s)\,ds + \delta'\delta'', \qquad \beta(x) = \beta'(x)\beta''(x).$$

If the first factor of (5) be replaced by

$$B\begin{bmatrix} \gamma'(x,y)\beta''(y)/\beta'(x)\beta''(x) & \alpha'(x)/\beta'(x)\beta''(x) \\ \epsilon'(y)\beta''(y) & \delta \end{bmatrix},$$

this substitution being permissible on account of (4), the modified theorem becomes an obvious corollary of the product theorem previously stated.

2. The first minors of the bordered Fredholm determinant

In the inversion of the Fredholm transformation (1), use is made of the so-called *first minor* of D, defined by the series

$$D_1(x,y) = -\gamma(x,y) - \sum_{n=1}^{\infty}\frac{1}{n!}\int_0^1\cdots\int_0^1 ds_1\cdots ds_n \begin{vmatrix} \gamma(x,y) & \gamma(x,s_j) \\ \gamma(s_i,y) & \gamma(s_i,s_j) \end{vmatrix} \qquad (i,j=1,2,\cdots,n).$$

For the inversion of (2) we shall need *four first-minors* of B which we define as follows:

$$B_1(x,y) = -\begin{vmatrix} \gamma(x,y) & \alpha(x) \\ \epsilon(y) & \delta \end{vmatrix} - \sum_{n=1}^{\infty}\frac{1}{n!}\int_0^1\cdots\int_0^1 ds_1\cdots ds_n \begin{vmatrix} \gamma(x,y) & \gamma(x,s_j) & \alpha(x) \\ \gamma(s_i,y) & \gamma(s_i,s_j) & \alpha(s_i) \\ \epsilon(y) & \epsilon(s_j) & \delta \end{vmatrix} \qquad (i,j=1,2,\cdots,n).$$

$$A(x) = -\alpha(x) - \sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\int_0^1\cdots\int_0^1 ds_1\cdots ds_n \begin{vmatrix} \gamma(x,s_i) & \alpha(x) \\ \gamma(s_i,s_j) & \alpha(s_i) \end{vmatrix} \qquad (i,j=1,2,\cdots,n).$$

$$E(y) = -\epsilon(y) - \sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\int_0^1\cdots\int_0^1 ds_1\cdots ds_n \begin{vmatrix} \gamma(s_i,y) & \gamma(s_i,s_j) \\ \epsilon(y) & \epsilon(s_j) \end{vmatrix} \qquad (i,j=1,2,\cdots,n).$$

$$D = 1 + \sum_{n=1}^{\infty}\frac{1}{n!}\int_0^1\cdots\int_0^1 ds_1\cdots ds_n\,|\gamma(s_i,s_j)| \qquad (i,j=1,2,\cdots,n),$$

the last one being the ordinary Fredholm determinant. The convergence of the series defining B_1, A, and E can be established by the usual methods.

The following fundamental identities relative to the bordered Fredholm determinant and its first minors can be verified from the definitions:

$$\text{I} \qquad B_1(x,y) + B\gamma(x,y) + \int_0^1 B_1(x,s)\gamma(s,y)\,ds + A(x)\,\epsilon(y) = 0,$$

$$\text{I}' \qquad B_1(x,y) + B\gamma(x,y) + \int_0^1 \gamma(x,s)B_1(s,y)\,ds + \alpha(x)E(y) = 0,$$

$$\text{II} \qquad B\alpha(x) + \int_0^1 B_1(x,t)\alpha(t)\,dt + A(x)\delta = 0,$$

$$\text{II}' \qquad A(x) + \int_0^1 \gamma(x,t)A(t)\,dt + \alpha(x)D = 0,$$

$$\text{III} \qquad E(y) + \int_0^1 E(s)\gamma(s,y)\,ds + D\epsilon(y) = 0,$$

$$\text{III}' \qquad B\epsilon(y) + \int_0^1 \epsilon(s)B_1(s,y)\,ds + \delta E(y) = 0,$$

$$\text{IV} \qquad \int_0^1 E(s)\alpha(s)\,ds + D\delta = B,$$

$$\text{IV}' \qquad \int_0^1 \epsilon(s)A(s)\,ds + \delta D = B.$$

It is of some interest, though not essential for what follows, to note that B and its first minors can be expressed in rather simple form in terms of D and its first and second minors.*

3. Notation, and conditions on the transformation (2)

By $\mathfrak{C}_n$ we will denote the class of all real functions of n real variables which are continuous when each argument is restricted to the unit interval $(0-1)$. The transformation (2) will be restricted as follows:

(*a*) α, β, *and* ϵ *belong to* $\mathfrak{C}_1$; γ *to* $\mathfrak{C}_2$; *and* δ *is a real constant.*

(*b* β *is nowhere zero on the unit interval.*

$$* B = D\left[\delta - \int_0^1 \epsilon(s)\alpha(s)\,ds\right] - \int_0^1\int_0^1 \epsilon(s)D_1(s,t)\alpha(t)\,ds\,dt,$$

$$B_1(x,y) = D_1(x,y)\left[\delta - \int_0^1 \epsilon(s)\alpha(s)\,ds\right] - \int_0^1\int_0^1 \epsilon(s)D_2(x,s;y,t)\alpha(t)\,ds\,dt,$$
$$+ D\alpha(x)\epsilon(y) + \alpha(x)\int_0^1 \epsilon(s)D_1(s,y)\,ds + \int_0^1 D_1(x,t)\alpha(t)\,dt\cdot\epsilon(y),$$

$$A(x) = -D\alpha(x) - \int_0^1 D_1(x,t)\alpha(t)\,dt, \qquad E(y) = -D\epsilon(y) - \int_0^1 \epsilon(s)D_1(s,y)\,ds.$$

From conditions (a) it follows that every function ϕ of $\mathfrak{C}_1$ is transformed into a function ϕ' of $\mathfrak{C}_1$, with the exception of those functions ϕ which satisfy the equation $\delta + \int_0^1 (y)\phi(y)\,dy = 0$.*

In much of what follows it will be unnecessary to write the arguments of functions. To make possible this abbreviation, the following convention with regard to products will be observed. In writing the product of two functions, the arguments may be omitted if and only if: (1), the arguments are all distinct; or (2), the last argument of the first factor and the first argument of the second factor are identical, while all other arguments are distinct. The former case will be distinguished from the latter by a dot between the two factors. For example

$$\alpha(x)\epsilon(y) = \alpha\cdot\epsilon, \quad \text{while} \quad \alpha(x)\epsilon(x) = \alpha\epsilon$$

$$\gamma'(x, y)\gamma''(s, t) = \gamma'\cdot\gamma'', \text{ while } \gamma'(x, s)\gamma''(s, y) = \gamma'\gamma''.$$

The operation of definite integration from 0 to 1 of the product of two functions having a common argument will be indicated by the letter J, it being understood that the variable of integration is the common argument. Thus

$$\int_0^1 \gamma(x, y)\phi(y)\,dy = J\gamma\phi, \qquad \int_0^1 \epsilon(y)\phi(y)\,dy = J\epsilon\phi,$$

$$\int_0^1 \gamma'(x, s)\gamma''(s, y)\,ds = J\gamma'\gamma''.$$

With these conventions, the transformation (2) can be written in the form

$$\phi' = \frac{\alpha + \beta\phi + J\gamma\phi}{\delta + J\epsilon\phi}.$$

I will often be desirable to look upon this transformation as consisting of a single operation upon the function ϕ. In such a case we will use

$$\begin{pmatrix} \beta + \gamma & \alpha \\ \epsilon & \delta \end{pmatrix}$$

as a symbol for the operator, and will write the transformation in the form

$$\phi' = \begin{pmatrix} \beta + \gamma & \alpha \\ \epsilon & \delta \end{pmatrix}\phi.$$

The symbol for the operator may also for the sake of brevity be used as a symbol for the transformation.

* This class of exceptional elements might be called the "vanishing lineoid," upon consideration of the analogous situation in ordinary projective geometry.

4. The determinant of the transformation (2)

The theory of the projective transformation

$$\begin{pmatrix} \beta + \gamma & \alpha \\ \epsilon & \delta \end{pmatrix} \tag{2}$$

depends largely upon the bordered Fredholm determinant

$$B\begin{bmatrix} \bar\gamma & \bar\alpha \\ \epsilon & \delta \end{bmatrix},$$

the functions $\bar\gamma$ and $\bar\alpha$ having the definitions

$$\bar\gamma(x, y) = \gamma(x, y)/\beta(x), \qquad \bar\alpha(x) = \alpha(x)/\beta(x).$$

This bordered Fredholm determinant will be called *the determinant* of the projective transformation (2). From § 2 we have the following identities relative to this determinant and its first minors:

(I) $B_1 + B\bar\gamma + JB_1\bar\gamma + A\cdot\epsilon = 0,$ (I′) $B_1 + B\bar\gamma + J\bar\gamma B_1 + \bar\alpha\cdot E = 0,$

(II) $B\bar\alpha + JB_1\bar\alpha + A\delta = 0,$ (II′) $A + J\bar\gamma A + \bar\alpha D = 0,$

(III) $E + JE\bar\gamma + D\epsilon = 0,$ (III′) $B\epsilon + J\epsilon B_1 + \delta E = 0,$

(IV) $JE\bar\alpha + D\delta = B,$ (IV′) $J\epsilon A + \delta D = B.$

5. The product of two projective transformations

If

$$\begin{pmatrix} \beta' + \gamma' & \alpha' \\ \epsilon' & \delta' \end{pmatrix}, \qquad \begin{pmatrix} \beta'' + \gamma'' & \alpha'' \\ \epsilon'' & \delta'' \end{pmatrix}, \tag{6}$$

are any two projective transformations, there is a projective transformation

$$\begin{pmatrix} \beta + \gamma & \alpha \\ \epsilon & \delta \end{pmatrix}, \tag{7}$$

such that, for any ϕ belonging to C_1,

$$\begin{pmatrix} \beta' + \gamma' & \alpha' \\ \epsilon' & \delta' \end{pmatrix}\begin{pmatrix} \beta'' + \gamma'' & \alpha'' \\ \epsilon'' & \delta'' \end{pmatrix}\phi = \begin{pmatrix} \beta + \gamma & \alpha \\ \epsilon & \delta \end{pmatrix}\phi.$$

The transformation (7) will be called the product of the transformations (6). The coefficients of the product transformation are given by the formulas*:

* It will be noticed that the transformation symbols obey a law of multiplication similar to the law of multiplication of determinants, provided the products

$$\gamma'\gamma'', \quad \gamma'\alpha'', \quad \epsilon'\gamma'', \quad \epsilon'\alpha'', \quad \alpha'\epsilon'',$$

be interpreted respectively as

$$J\gamma'\gamma'', \quad J\gamma'\alpha'', \quad J\epsilon'\gamma'', \quad J\epsilon'\alpha'', \quad \alpha'\cdot\epsilon''.$$

$$\beta = \beta'\beta'', \qquad \gamma = \beta'\gamma''+\gamma'\beta''+J\gamma'\gamma''+\alpha'\cdot\epsilon'', \qquad \alpha = \beta'\alpha''+J\gamma'\alpha''+\alpha'\delta'',$$
$$\epsilon = \epsilon'\beta''+J\epsilon'\gamma''+\delta'\epsilon'', \qquad \delta = J\epsilon'\alpha''+\delta'\delta''.$$

From these relations there follows, as an immediate consequence of the Modified Product Theorem of Section 1, the important

THEOREM I. *The determinant of the product of two projective transformations is equal to the product of their determinants.*

It can be verified without difficulty that the multiplication of transformations is associative:

$$\left[\begin{pmatrix}\beta' + \gamma' & \alpha' \\ \epsilon' & \delta'\end{pmatrix}\begin{pmatrix}\beta'' + \gamma'' & \alpha'' \\ \epsilon'' & \delta''\end{pmatrix}\right]\begin{pmatrix}\beta''' + \gamma''' & \alpha''' \\ \epsilon''' & \delta'''\end{pmatrix}$$
$$= \begin{pmatrix}\beta' + \gamma' & \alpha' \\ \epsilon' & \delta'\end{pmatrix}\left[\begin{pmatrix}\beta'' + \gamma'' & \alpha'' \\ \epsilon'' & \delta''\end{pmatrix}\begin{pmatrix}\beta''' + \gamma''' & \alpha''' \\ \epsilon''' & \delta'''\end{pmatrix}\right].$$

It is not in general commutative.

6. THE INVERSION OF THE TRANSFORMATION (2)

Let us consider a transformation

$$\begin{pmatrix}\beta + \gamma & \alpha \\ \epsilon & \delta\end{pmatrix} \tag{2}$$

with determinant B and first minors B_1, A, E, D; and let us suppose $B \neq 0$. In connection with it we shall consider the transformation

$$\begin{pmatrix}\beta' + \gamma' & \alpha' \\ \epsilon' & \delta'\end{pmatrix} \tag{2'}$$

where

$$\begin{gathered}\beta'(x) = \frac{1}{\beta(x)}, \qquad \gamma'(xy) = \frac{B_1(xy)}{B\beta(y)}, \qquad \alpha'(x) = \frac{A(x)}{B}, \\ \delta' = \frac{D}{B}, \qquad \epsilon'(y) = \frac{E(y)}{B\beta(y)}.\end{gathered} \tag{8}$$

According to the rule for multiplication developed in the preceding section, we find

$$\begin{pmatrix}\beta' + \gamma' & \alpha' \\ \epsilon' & \delta'\end{pmatrix}\begin{pmatrix}\beta + \gamma & \alpha \\ \epsilon & \delta\end{pmatrix}$$
$$= \begin{pmatrix}\beta'\beta + (\beta'\gamma + \gamma'\beta + J\gamma'\gamma + \alpha'\cdot\epsilon) & \beta'\alpha + J\gamma'\alpha + \alpha'\delta \\ \epsilon'\beta + J\epsilon'\gamma + \delta'\epsilon & J\epsilon'\alpha + \delta'\delta\end{pmatrix}.$$

But upon substitution of the definitional values of the primed functions and application of the identities I–IV, it is seen that the coefficients of the product transformation have the values:

$$
(9)\qquad \begin{aligned} &\beta'\beta = 1, \quad \beta'\gamma + \gamma'\beta + J\gamma'\gamma + \alpha'\cdot\epsilon = 0, \quad \beta'\alpha + J\gamma'\alpha + \alpha'\delta = 0,\\ &\qquad\qquad \epsilon'\beta + J\epsilon'\gamma + \delta'\epsilon = 0, \qquad\qquad J\epsilon'\alpha + \delta'\delta = 1. \end{aligned}
$$

Hence

$$
(10)\qquad \begin{pmatrix} \beta' + \gamma' & \alpha' \\ \epsilon' & \delta' \end{pmatrix}\begin{pmatrix} \beta + \gamma & \alpha \\ \epsilon & \delta \end{pmatrix} = \begin{pmatrix} 1 + 0 & 0 \\ 0 & 1 \end{pmatrix};
$$

that is, the product of the transformations (2′) and (2) is the identity transformation; or in other words, (2′) *is the inverse transformation of* (2).

The identity (10) is to be interpreted in the sense that for any function ϕ of C_1,

$$
\begin{pmatrix} \beta' + \gamma' & \alpha' \\ \epsilon' & \delta' \end{pmatrix}\begin{pmatrix} \beta + \gamma & \alpha \\ \epsilon & \delta \end{pmatrix}\phi = \phi.
$$

Hence it states that a necessary consequence of

$$
(2)\qquad \phi' = \begin{pmatrix} \beta + \gamma & \alpha \\ \epsilon & \delta \end{pmatrix}\phi
$$

is

$$
(11)\qquad \begin{pmatrix} \beta' + \gamma' & \alpha' \\ \epsilon' & \delta' \end{pmatrix}\phi' = \phi.
$$

Therefore, if there exists a solution of the equation (2) for ϕ in terms of ϕ', that solution is unique and given by (11). To establish the existence of the solution, we have but to substitute the value of ϕ from (11) in (2) obtaining

$$
\phi' = \begin{pmatrix} \beta + \gamma & \alpha \\ \epsilon & \delta \end{pmatrix}\begin{pmatrix} \beta' + \gamma' & \alpha' \\ \epsilon' & \delta' \end{pmatrix}\phi',
$$

and to verify by means of the identities (I′)–(IV′) that

$$
(10')\qquad \begin{pmatrix} \beta + \gamma & \alpha \\ \epsilon & \delta \end{pmatrix}\begin{pmatrix} \beta' + \gamma' & \alpha' \\ \epsilon' & \delta' \end{pmatrix} = \begin{pmatrix} 1 + 0 & 0 \\ 0 & 1 \end{pmatrix}.
$$

Incidentally the verification furnishes a system of identities symmetrical to (9) as follows:

$$
(9')\qquad \begin{aligned} &\beta\beta' = 1, \quad \beta\gamma' + \gamma\beta' + J\gamma\gamma' + \alpha\cdot\epsilon' = 0, \quad \beta\alpha' + J\gamma\alpha' + \alpha\delta' = 0,\\ &\qquad\qquad \epsilon\beta' + J\epsilon\gamma' + \delta\epsilon' = 0, \qquad\qquad J\epsilon\alpha' + \delta\delta' = 1. \end{aligned}
$$

We have then proven

THEOREM II. *If the determinant of the transformation*

$$
(2)\qquad \phi' = \frac{\alpha + \beta\phi + J\gamma\phi}{\delta + J\epsilon\phi}
$$

is different from zero, the equation (2) *has a unique solution for* ϕ *in terms of* ϕ', *namely*

$$\phi = \frac{\alpha' + \beta'\phi' + J\gamma'\phi'}{\delta' + J\epsilon'\phi'}, \tag{2'}$$

where α', β', γ', δ', ϵ', *are given by* (8).

COROLLARY 1. *If the determinants of the transformations* (2) *and* (2′) *be denoted by* B *and* B' *respectively, then*

$$B'B = 1.$$

This follows from the identity (10), Theorem I, and the easily verifiable fact that the determinant of the identity transformation is equal to unity.

COROLLARY 2. *If* ϕ, ϕ', δ, δ', ϵ, *and* ϵ' *have the same meanings as in Theorem II, then*

$$[\delta' + J\epsilon'\phi'][\delta + J\epsilon\phi] = 1. \tag{12}$$

For

$$\delta' + J\epsilon'\phi' = \delta' + J\epsilon'\left[\frac{\alpha + \beta\phi + J\gamma\phi}{\delta + J\epsilon\phi}\right],$$

and therefore

$$\begin{aligned}[\delta' + J\epsilon'\phi'][\delta + J\epsilon\phi] &= \delta'[\delta + J\epsilon\phi] + J\epsilon'[\alpha + \beta\phi + J\gamma\phi] \\ &= [\delta'\delta + J\epsilon'\alpha] + J[\epsilon'\beta + J\epsilon'\gamma + \delta'\epsilon]\phi \\ &= 1,\end{aligned}$$

the final equality being a consequence of the identities (9).

7. THE GROUP OF NON-SINGULAR PROJECTIVE TRANSFORMATIONS

A projective transformation whose determinant is different from zero will be said to be *non-singular*. We have established the following properties of the class of all non-singular projective transformations:

1. The product of any two transformations of the class is a transformation of the class. (§ 5.)
2. The multiplication of transformations of the class is associative. (§ 5.)
3. The class contains the identity transformation.
4. Every transformation of the class has a unique inverse in the class. (§ 6.)

Hence, *the totality of non-singular projective transformations form a group.*

If either or both of the conditions $B = 1$, $\beta = 1$, be imposed on the non-singular transformations, a sub-group is evidently obtained. We note also that upon imposing the simultaneous conditions $\beta = \delta = 1$, $\alpha = \epsilon = 0$, we obtain as a sub-group the ordinary Fredholm group.*

* A more elaborate investigation of the subgroups of the projective group from the standpoint of the infinitesimal transformation will be reserved for a later paper.

8. The infinitesimal projective transformation

To obtain a projective transformation which changes all points of $\mathfrak{C}_1$ by an infinitesimal amount, we may give to each coefficient of the identity transformation

$$\begin{pmatrix} 1 + 0 & 0 \\ & 0 & 1 \end{pmatrix}$$

an infinitesimal increment. In view of the fractional character of the projective transformation however, it is obvious that no generality is lost by assuming that the constant term in the denominator remains equal to unity. The remaining coefficients may have the forms

$$\alpha(x) = \lambda(x)\,\delta t, \qquad \beta(x) = 1 + \mu(x)\,\delta t,$$
$$\gamma(x, y) = \nu(x, y)\,\delta t, \qquad \epsilon(y) = \rho(y)\,\delta t\,;$$

where δt is an arbitrary infinitesimal; λ, μ, ρ are any functions of $\mathfrak{C}_1$; and ν is any function of $\mathfrak{C}_2$. The resulting transformation is

$$\phi' = \frac{\phi + (\lambda + \mu\phi + J\nu\phi)\,\delta t}{1 + J\rho\phi\,\delta t}. \tag{13}$$

Upon expanding the right hand member of (13) as a power series in δt, and dropping the terms of higher order than the first, the infinitesimal increment $\phi' - \phi$ or $\delta\phi$ may be written

$$\delta\phi = [\lambda + \mu\phi + J\nu\phi - \phi J\rho\phi]\,\delta t. \tag{3}$$

This is precisely the "regular infinitesimal projective transformation" obtained by Kowalewski.*

Suppose now that the coefficient functions λ, μ, ν, ρ in the infinitesimal transformation (3) have been in some way definitely prescribed. Upon continuous application, the infinitesimal transformation will then generate, in the sense of Lie, a one-parameter family of finite transformations which take the point $\phi(x)$ into a point $\phi'(x;\, t)$ defined by the equation

$$\begin{aligned} \frac{\partial\phi'(x;\, t)}{\partial t} = \lambda(x) + \mu(x)\phi'(x;\, t) + \int_0^1 \nu(x,\, y)\phi'(y;\, t)\,dy \\ - \phi'(x;\, t)\int_0^1 \rho(y)\phi'(y;\, t)\,dy \end{aligned} \tag{14}$$

with the initial condition $\phi'(x;\, 0) = \phi(x)$. The solution of this integro-differential equation can be expressed in the form of a power series in t, of which the coefficients are Schmidt integral-power-series. Such solutions

* Loc. cit., cf. our introduction.

have been obtained by Kowalewski for a general type of equations which includes (14).

It will be our purpose in the next section to show that *the solution of* (14) *can always be expressed in the form* (2) *by proper choice of* $\alpha, \beta, \gamma, \delta, \epsilon$.

9. The finite transformations generated by an infinitesimal projective transformation (3)

We are to determine $\alpha(x;\, t)$, $\beta(x;\, t)$, $\gamma(x, y;\, t)$, $\delta(t)$, $\epsilon(y;\, t)$, so that

$$(15)\qquad \phi'(x;\, t) = \frac{\alpha(x;\, t) + \beta(x;\, t)\phi(x) + \int_0^1 \gamma(x, y;\, t)\phi(y)\,dy}{\delta(t) + \int_0^1 \epsilon(y;\, t)\phi(y)\,dy}$$

is the solution of the equation

$$(16)\qquad \frac{\partial\phi'(x;\, t)}{\partial t} = \lambda(x) + \mu(x)\phi'(x;\, t) + \int_0^1 \nu(x, y)\phi'(y;\, t)\,dy - \phi'(x;\, t)\int_0^1 \rho(y)\phi'(y;\, t)\,dy,$$

satisfying the initial condition $\phi'(x;\, 0) = \phi(x)$.

Differentiating (15) with respect to t, we get* for $\partial\phi'/\partial t$ the expression

$$(17)\qquad \frac{[\delta + J\epsilon\phi]\left[\dfrac{\partial\alpha}{\partial t} + \dfrac{\partial\beta}{\partial t}\phi + J\dfrac{\partial\gamma}{\partial t}\phi\right] - [\alpha + \beta\phi + J\gamma\phi]\left[\dfrac{\partial\delta}{\partial t} + J\dfrac{\partial\epsilon}{\partial t}\phi\right]}{[\delta + J\epsilon\phi]^2}.$$

This fraction is to be made identical with the right member of (16). To this end it is desirable to replace ϕ in (17) by its value

$$(18)\qquad \phi = \frac{\alpha' + \beta'\phi' + J\gamma'\phi'}{\delta' + J\epsilon'\phi'}$$

obtained by the inversion of (15). This substitution is of course legitimate only if the determinant of (15) is different from zero; it will be made, subject to later consideration of the determinant of (15). Some labor may be saved in the substitution by making use of the identity

$$(12)\qquad [\delta' + J\epsilon'\phi'][\delta + J\epsilon\phi] = 1.$$

By use of (12) and (15), we may write (17) in the form

---

* The dependence of functions upon the parameter t will be disregarded in applying the rules regarding the omission of variables from functional notations, as laid down in § 3.

$$[\delta' + J\epsilon'\phi']\left[\frac{\partial\alpha}{\partial t} + \frac{\partial\beta}{\partial t}\phi + J\frac{\partial\gamma}{\partial t}\phi - \phi'\left(\frac{\partial\delta}{\partial t} + J\frac{\partial\epsilon}{\partial t}\phi\right)\right];$$

and then replacing ϕ by its value from (18), we have

$$[\delta' + J\epsilon'\phi']\left[\frac{\partial\alpha}{\partial t} - \phi'\frac{\partial\delta}{\partial t}\right] + \frac{\partial\beta}{\partial t}[\alpha' + \beta'\phi' + J\gamma'\phi']$$
$$+ J\frac{\partial\gamma}{\partial t}[\alpha' + \beta'\phi' + J\gamma'\phi'] - \phi' J\frac{\partial\epsilon}{\partial t}[\alpha' + \beta'\phi + J\gamma'\phi'].$$

Arranging the expression thus obtained so that terms involving ϕ' in the same manner are grouped together, we get finally

$$\left[\frac{\partial\alpha}{\partial t}\delta' + \frac{\partial\beta}{\partial t}\alpha' + J\frac{\partial\gamma}{\partial t}\alpha'\right] + \left[\frac{\partial\beta}{\partial t}\beta' - \frac{\partial\delta}{\partial t}\delta' - J\frac{\partial\epsilon}{\partial t}\alpha'\right]\phi'$$
$$+ J\left[\frac{\partial\alpha}{\partial t}\cdot\epsilon' + \frac{\partial\beta}{\partial t}\gamma' + \frac{\partial\gamma}{\partial t}\beta' + J\frac{\partial\gamma}{\partial t}\gamma'\right]\phi' - \phi' J\left[\frac{\partial\delta}{\partial t}\epsilon' + \frac{\partial\epsilon}{\partial t}\beta' + J\frac{\partial\epsilon}{\partial t}\gamma'\right]\phi'.$$

In order that this last expression be identical with the right member of (16), it is evidently sufficient that $\alpha, \beta, \gamma, \delta, \epsilon$, satisfy the following system of equations:

$$\frac{\partial\alpha}{\partial t}\delta' + \frac{\partial\beta}{\partial t}\alpha' + J\frac{\partial\gamma}{\partial t}\alpha' = \lambda, \qquad \frac{\partial\beta}{\partial t}\beta' - \frac{\partial\delta}{\partial t}\delta' - J\frac{\partial\epsilon}{\partial t}\alpha' = \mu,$$

$$\frac{\partial\alpha}{\partial t}\cdot\epsilon' + \frac{\partial\beta}{\partial t}\gamma' + \frac{\partial\gamma}{\partial t}\beta' + J\frac{\partial\gamma}{\partial t}\gamma' = \nu, \qquad \frac{\partial\delta}{\partial t}\epsilon' + \frac{\partial\epsilon}{\partial t}\beta' + J\frac{\partial\epsilon}{\partial t}\gamma' = \rho.$$

To these four conditions we annex arbitrarily a fifth

$$\frac{\partial\delta}{\partial t}\delta' + J\frac{\partial\epsilon}{\partial t}\alpha' = 0.$$

This system S of five linear integral equations can be solved for $\partial\alpha/\partial t$, $\partial\beta/\partial t$, $\partial\gamma/\partial t$, $\partial\delta/\partial t$, $\partial\epsilon/\partial t$, by use of the identities (9). For, multiplying the first equation by δ, the second by α, integrating the third multiplied by $\alpha(y)$ with respect to y, and adding the results we get

$$\frac{\partial\alpha}{\partial t}[\delta'\delta + J\epsilon'\alpha] + \frac{\partial\beta}{\partial t}[\alpha'\delta + \beta'\alpha + J\gamma'\alpha] + J\frac{\partial\gamma}{\partial t}[\alpha'\delta + \beta'\alpha + J\gamma'\alpha]$$
$$= \lambda\delta + \mu\alpha + J\nu\alpha.$$

Reference to (9) shows that the coefficient of $\partial\alpha/\partial t$ in this equation is unity while the coefficients of $\partial\beta/\partial t$ and $\partial\gamma/\partial t$ are zero. By other similar combinations of equations of the system S, we obtain the four equations

$$\frac{\partial\beta}{\partial t}\beta'\beta = \mu\beta,$$

$$\frac{\partial\alpha}{\partial t}\cdot[\delta'\epsilon + \epsilon'\beta + J\epsilon'\gamma] + \frac{\partial\beta}{\partial t}[\alpha'\cdot\epsilon + \beta'\gamma + \gamma'\beta + J\gamma'\gamma] + \frac{\partial\gamma}{\partial t}\beta'\beta$$
$$+ J\frac{\partial\gamma}{\partial t}[\alpha'\cdot\epsilon + \beta'\gamma + \gamma'\beta + J\gamma'\gamma] = \lambda\cdot\epsilon + \mu\gamma + \nu\beta + J\nu\gamma,$$

$$\frac{\partial\delta}{\partial t}[J\epsilon'\alpha + \delta'\delta] + J\frac{\partial\epsilon}{\partial t}[\beta'\alpha + J\gamma'\alpha + \alpha'\delta] = J\rho\alpha,$$

$$\frac{\partial\delta}{\partial t}[\epsilon'\beta + J\epsilon'\gamma + \delta'\epsilon] + \frac{\partial\epsilon}{\partial t}\beta'\beta + J\frac{\partial\epsilon}{\partial t}[\beta'\gamma + \gamma'\beta + J\gamma'\gamma + \alpha'\cdot\epsilon]$$
$$= \rho\beta + J\rho\gamma.$$

Applying the identities (9), we obtain finally the system of equations

$$\frac{\partial\alpha}{\partial t} = \mu\alpha + J\nu\alpha + \lambda\delta, \qquad \frac{\partial\beta}{\partial t} = \mu\beta,$$
$$(19)\qquad \frac{\partial\gamma}{\partial t} = \mu\gamma + \nu\beta + J\nu\gamma + \lambda\cdot\epsilon, \qquad \frac{\partial\delta}{\partial t} = J\rho\alpha, \qquad \frac{\partial\epsilon}{\partial t} = \rho\beta + J\rho\gamma.$$

From the way in which it has been obtained, the system of equations (19) is evidently a consequence of the system S. That S is likewise a consequence of (19) can easily be verified by substitution and use of (9′). Our problem is therefore reduced to the finding of a set of functions α, β, γ, δ, ϵ which satisfies the system of integro-differential equations (19).

If there is such a set of functions, analytic in t, and regular at $t = 0$, they can be written in the form

$$\alpha(x;t) = \alpha_0 + \frac{\alpha_1}{1!}t + \frac{\alpha_2}{2!}t^2 + \cdots, \quad \beta(x;t) = \beta_0 + \frac{\beta_1}{1!}t + \frac{\beta_2}{2!}t^2 + \cdots,$$
$$(20)\quad \gamma(x,y;t) = \gamma_0 + \frac{\gamma_1}{1!}t + \frac{\gamma_2}{2!}t^2 + \cdots, \quad \delta(t) = \delta_0 + \frac{\delta_1}{1!}t + \frac{\delta_2}{2!}t^2 + \cdots,$$
$$\epsilon(y;t) = \epsilon_0 + \frac{\epsilon_1}{1!}t + \frac{\epsilon_2}{2!}t^2 + \cdots,$$

where

$$\alpha_i = \left.\frac{\partial^i\alpha}{\partial t^i}\right]_{t=0}, \quad \beta_i = \left.\frac{\partial^i\beta}{\partial t^i}\right]_{t=0}, \quad \gamma_i = \left.\frac{\partial^i\gamma}{\partial t^i}\right]_{t=0}, \quad \delta_i = \left.\frac{\partial^i\delta}{\partial t^i}\right]_{t=0}, \quad \epsilon_i = \left.\frac{\partial^i\epsilon}{\partial t^i}\right]_{t=0}.$$

The initial condition will be satisfied if we take

$$(21)\qquad \alpha_0 = \gamma_0 = \epsilon_0 = 0, \qquad \beta_0 = \delta_0 = 1.$$

The remaining coefficients in (20) are then uniquely determined by the recursion formulas

$$
\begin{aligned}
&\alpha_i = \mu\alpha_{i-1} + J\nu\alpha_{i-1} + \lambda\delta_{i-1}, \qquad \beta_i = \mu\beta_{i-1}, \\
(22) \qquad &\gamma_i = \mu\gamma_{i-1} + \nu\beta_{i-1} + J\nu\gamma_{i-1} + \lambda\cdot\epsilon_{i-1}, \\
&\delta_i = J\rho\alpha_{i-1}, \qquad \epsilon_i = \rho\beta_{i-1} + J\rho\gamma_{i-1},
\end{aligned}
$$

obtained by repeated differentiation of equations (19) with respect to t.

The functions given in (20) with coefficients thus determined, *formally* satisfy the equations (19). The convergence of the series in (20) is easily established by comparison with the dominating series

$$
(23) \qquad e^{at} = 1 + \frac{a}{1!}t + \frac{a^2}{1!}t^2 + \cdots,
$$

where

$$
a = \tilde{\lambda} + \tilde{\mu} + \tilde{\nu} + \tilde{\rho};
$$

$\tilde{\lambda}$, $\tilde{\mu}$, $\tilde{\nu}$, and $\tilde{\rho}$ representing respectively the maximum of $|\lambda|$, $|\mu|$, $|\nu|$, and $|\rho|$ on the unit interval.

Each coefficient numerator α_i, β_i, γ_i, δ_i, ϵ_i, of (20) is dominated by the corresponding coefficient numerator a^i of (23), as may be seen from the recursion formulas (22). Hence, since the series (23) converges for every finite value of t, each of the series (20) converges for every finite value of t, and for any fixed t, uniformly in x and y. From the latter fact it follows that for any fixed t, the functions $\alpha(x;\, t)$, $\beta(x;\, t)$, and $\epsilon(y;\, t)$, belong to $\mathfrak{C}_1$, while $\gamma(x, y;\, t)$ belongs to $\mathfrak{C}_2$. Furthermore, $\beta(x;\, t)$ is never zero since, as is obvious from its expansion,

$$
\beta(x\,;\,t) = e^{\mu(x)t}.
$$

We have then obtained the result that the transformations generated by the infinitesimal projective transformation (3) constitute a one-parameter family of projective transformations (15). In order to completely validate this result, and at the same time to show that the transformations (15) are non-singular, it is necessary to prove that the determinant of (15) is different from zero for every value of t. To this end, denote this determinant by $B(t)$. Then $B(t+\delta t)$ is the product of $B(t)$ and the determinant of the infinitesimal transformation (13). But this latter determinant turns out to be equal to

$$
1 + \int_0^1 \nu(xx)\,dx\cdot\delta t \text{ plus terms of higher order in } \delta t.
$$

Therefore

$$
B(t+\delta t) = B(t)\left[1 + \int_0^1 \nu(xx)\,dx\cdot\delta t \text{ plus higher terms}\right],
$$

and

$$
\frac{1}{B(t)}\frac{dB}{dt} = \int_0^1 \nu(xx)\,dx.
$$

Hence

$$\log B(t) = t\int_0^1 \nu(xx)\,dx,$$

that is

$$B(t) = e^{t\int_0^1 \nu(xx)\,dx}, \tag{24}$$

a non-vanishing function of t.

We have proven

THEOREM III. *The finite transformations generated by a regular infinitesimal projective transformation*

$$\delta\phi = [\lambda + \mu\phi + J\nu\phi - \phi J\rho\phi]\,\delta t$$

constitute a one-parameter family of non-singular projective transformations. For any value of the parameter t, the coefficient functions of the generated transformation are given by (20), (21), *and* (22), *and the determinant is given by* (24).

10. The infinitesimal transformation which generates a given non-singular projective transformation

In this section we consider the converse question. Given a non-singular projective transformation

$$\phi' = \frac{\alpha + \beta\phi + J\gamma\phi}{\delta + J\epsilon\phi}; \tag{25}$$

can it be generated by a regular infinitesimal projective transformation? From the developments of Section 9 it follows that this question is to be answered in the affirmative if equations (20) can be solved for λ, μ, ν, ρ, and t, in terms of α, β, γ, δ, and ϵ. We will show that this inversion can be made if α, β, γ, δ, and ϵ be suitably restricted.

The first such restriction will be that $\delta \not\equiv 0$;* and in view of this restriction, there will be no loss of generality in assuming that $\delta = 1$, since the transformation (25) is not changed in effect if each coefficient is multiplied by a constant different from zero. It will be convenient to take full advantage of this permissible factor of proportionality by multiplying the left side of each equation of (20) by a parameter s to be determined. These equations can then be written in the form

$$\alpha s = \lambda t + \frac{\alpha_2}{2!}t^2 + \cdots, \qquad \beta s = 1 + \mu t + \frac{\beta_2}{2!}t^2 + \cdots,$$

$$\gamma s = \nu t + \frac{\gamma_2}{2!}t^2 + \cdots, \qquad s = 1 + \frac{\delta_2}{2!}t^2 + \cdots, \qquad \epsilon s = \rho t + \frac{\epsilon_2}{2!}t^2 + \cdots,$$

the leading coefficients α_0, α_1, β_0, β_1, etc., having been replaced by their explicit values. We will show that this system can be satisfied by a proper choice of λ, μ, ν, ρ, and s, the parameter t having the value unity.

* This assumption means geometrically that the origin $[\phi \equiv 0]$ does not lie on the "vanishing lineoid."

Putting $t = 1$, we may write the system of equations to be solved, in the form

$$\alpha s = \lambda - \sum_{i=2}^{\infty} A_i(\lambda\mu\nu\rho), \qquad \beta s = 1 + \mu - \sum_{i=2}^{\infty} B_i(\lambda\mu\nu\rho),$$
$$\gamma s = \nu - \sum_{i=2}^{\infty} C_i(\lambda\mu\nu\rho), \qquad s = 1 + \sum_{i=2}^{\infty} D_i(\lambda\mu\nu\rho), \qquad \epsilon s = \rho - \sum_{i=2}^{\infty} E_i(\lambda\mu\nu\rho),$$

where A_i, B_i, C_i, D_i, and E_i are homogeneous integral-power-forms* of order i in the argument functions λ, μ, ν, ρ.

Upon substituting the value of s from the fourth equation into the remaining four equations, and letting $\beta - 1 = \bar{\beta}$, we may write the resulting system in the form

$$\tag{26} \begin{aligned} \lambda &= \alpha + \sum_{i=2}^{\infty} A_i(\lambda\mu\nu\rho) + \alpha \sum_{i=2}^{\infty} D_i(\lambda\mu\nu\rho), \\ \mu &= \bar{\beta} + \sum_{i=2}^{\infty} B_i(\lambda\mu\nu\rho) + \beta \sum_{i=2}^{\infty} D_i(\lambda\mu\nu\rho), \\ \nu &= \gamma + \sum_{i=2}^{\infty} C_i(\lambda\mu\nu\rho) + \gamma \sum_{i=2}^{\infty} D_i(\lambda\mu\nu\rho), \\ \rho &= \epsilon + \sum_{i=2}^{\infty} E_i(\lambda\mu\nu\rho) + \epsilon \sum_{i=2}^{\infty} D_i(\lambda\mu\nu\rho). \end{aligned} \qquad (\beta = 1 + \bar{\beta}),$$

We will try to satisfy this system of equations by functions λ, μ, ν, ρ, defined by integral power series

$$\tag{27} \begin{aligned} \lambda &= \sum_{i=1}^{\infty} L_i(\alpha\bar{\beta}\gamma\epsilon), \qquad \mu = \sum_{i=1}^{\infty} M_i(\alpha\bar{\beta}\gamma\epsilon), \\ \nu &= \sum_{i=1}^{\infty} N_i(\alpha\bar{\beta}\gamma\epsilon), \qquad \rho = \sum_{i=1}^{\infty} R_i(\alpha\bar{\beta}\gamma\epsilon). \end{aligned}$$

Substituting these series for λ, μ, ν, ρ, in (26), and equating integral-power-forms of the same order on the two sides of each equation of (26), we find

$$L_1(\alpha\bar{\beta}\gamma\epsilon) = \alpha, \quad M_1(\alpha\bar{\beta}\gamma\epsilon) = \bar{\beta}, \quad N_1(\alpha\bar{\beta}\gamma\epsilon) = \gamma, \quad R_1(\alpha\bar{\beta}\gamma\epsilon) = \epsilon,$$
$$L_2 = A_2, \qquad M_2 = B_2 + D_2, \qquad N_2 = C_2, \qquad R_2 = E_2;$$

while L_i, M_i, N_i, and R_i are for any i uniquely determined in terms of L's, M's, N's, and R's of lower subscript. The series (27) thus determined *formally* satisfy equations (26).

* Cf. E. Schmidt, Mathematische Annalen, vol. 65 (1908), p. 375. The integral-power-forms used in the present paper are slightly different from those defined by Schmidt, in that one of the argument functions ν is a function of two variables. This peculiarity however causes no essential difficulty.

To prove the convergence we consider the system of ordinary analytic equations

$$(\tilde{26}) \qquad \begin{aligned} l &= a + \sum_{i=2}^{\infty} \tilde{A}_i(lmnr) + a \sum_{i=2}^{\infty} \tilde{D}_i(lmnr), \\ m &= \bar{b} + \sum_{i=2}^{\infty} \tilde{B}_i(lmnr) + b \sum_{i=2}^{\infty} \tilde{D}_i(lmnr), \\ n &= c + \sum_{i=2}^{\infty} \tilde{C}_i(lmnr) + c \sum_{i=2}^{\infty} \tilde{D}_i(lmnr), \\ r &= e + \sum_{i=2}^{\infty} \tilde{E}_i(lmnr) + e \sum_{i=2}^{\infty} \tilde{D}_i(lmnr); \end{aligned} \qquad (b = 1 + \bar{b}).$$

where $\tilde{A}_i$, $\tilde{B}_i$, $\tilde{C}_i$, $\tilde{D}_i$, $\tilde{E}_i$, are homogeneous polynomials of degree i, obtained by replacing the argument functions λ, μ, ν, ρ, in A_i, B_i, C_i, D_i, E_i, by real parameters l, m, n, r; removing all integration operators; and making all coefficients positive.

Since the series (20) were found to converge for all continuous functions λ, μ, ν, ρ, it follows that the series $(\tilde{26})$ converge for all finite values of the arguments l, m, n, r, a, $\bar{b}$, c, e. Hence by the ordinary analytic implicit function theory, it follows that the equations $(\tilde{26})$ determine l, m, n, r as power series in a, $\bar{b}$, c, e, converging for values of these arguments of sufficiently small absolute value.

Furthermore from the relation of the series $(\tilde{26})$ to the series (26) it follows that the solution power series for l, m, n, r, will be

$$(\tilde{27}) \qquad \begin{aligned} l &= \sum_{i=1}^{\infty} \tilde{L}_i(a\bar{b}ce), \qquad & m &= \sum_{i=1}^{\infty} \tilde{M}_i(a\bar{b}ce), \\ n &= \sum_{i=1}^{\infty} \tilde{N}_i(a\bar{b}ce), \qquad & r &= \sum_{i=1}^{\infty} \tilde{R}_i(a\bar{b}ce); \end{aligned}$$

obtained from the series in (27) by replacing α, $\bar{\beta}$, γ, ϵ by a, b, c, e, removing all integration operators and making all signs positive. Hence if a domain of convergence of the series $(\tilde{27})$ is defined by

$$|a| \leqq a_0, \qquad |\bar{b}| \leqq b_0, \qquad |c| \leqq c_0, \qquad |e| \leqq e_0;$$

then the series (27) converge uniformly in x and y for argument functions α, $\bar{\beta}$, γ, ϵ, satisfying the conditions

$$\begin{aligned} |\alpha| &< a_0, \qquad & |\bar{\beta}| &< b_0, \\ |\gamma| &< c_0, \qquad & |\epsilon| &< e_0. \end{aligned}$$

Since the identity transformation has the coefficients $\alpha = \gamma = \epsilon = 0$,

$\beta\,[=\bar{\beta}+1]=\delta=1$, the result we have obtained may be stated in the form

THEOREM IV. *Any non-singular projective transformation*

$$\begin{pmatrix} \beta+\gamma & \alpha \\ \epsilon & \delta \end{pmatrix},$$

which does not differ too greatly from the identity transformation, can be generated by a regular infinitesimal projective transformation. The coefficients of the infinitesimal transformation are given by (27).

11. A SUGGESTED GENERALIZATION

A generalized Fredholm transformation which includes as special cases the transformations (1) and (1_n) has been developed by E. H. Moore.* The essence of the generalization lies in the fact that the class of functions $\phi(x)$, $(0 \leqq x \leqq 1)$, upon which the transformation (1) operates, and the class of functions $x(i)$, $(i=1,2,\cdots,n)$, upon which (1_n) operates are replaced by a class of real (or complex) valued functions $\mu(p)$, of which the argument p ranges over a general, absolutely unconditioned class of elements. By the postulation of certain properties for this class of functions μ, and for a class of kernel functions $k(rs)$, and for a generalized operator J, a generalized Fredholm transformation

$$\mu' = \mu + Jk\mu$$

is obtained. The theory of this transformation which can be developed along the same lines as the classic Fredholm theory, includes as special instances the theories of the transformations (1) and (1_n).

With this generalization in mind, the possibility of defining an analogously generalized projective transformation

$$\mu' = \frac{\alpha + \beta\mu + Jk\mu}{\delta + J\epsilon\mu},$$

of which (2) and (2′) will be special instances can hardly be overlooked. If we restrict ourselves to the case in which $\beta = 1$, then sufficient restrictions for the classes of functions to which μ, k, α, and ϵ can respectively belong, and sufficient restrictions for the operator J, are immediately suggested by Moore's generalization of the Fredholm theory. Under these restrictions the developments of Sections 1–7 are extensible to the generalized transformation.

* Bulletin of the American Mathematical Society, vol. 18 (1911–12), pp. 334–362.

The course to be pursued in the generalization of Sections 8–10 is suggested by the recent work of Hildebrandt on generalized differential equations.*

UNIVERSITY OF SASKATCHEWAN,
January, 1918.

* *On a theory of linear differential equations in general analysis*, these Transactions, vol. 18 (1917), pp. 73–96.