A MEMOIR UPON FORMAL INVARIANCY WITH REGARD TO BINARY MODULAR TRANSFORMATIONS. INVARIANTS OF RELATIVITY*

BY

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A readable summary of existing formal concomitant theory can be given, perhaps, by referring to the scope and general trend of my former papers in this field.† The first of these gave the results of an attempt to correlate the modular and the algebraic invariant principles by emphasizing the process of transvection between a given form and the universal covariants of the modular group of binary transformations $G_{(p^2-p)(p^2-1)}$. The chief difficulty encountered was the apparent one that this process is not alone definitive for the construction of complete systems. It is also not the simplest algorism for the generation of modular concomitants. A second paper was an improvement from the latter point of view in that polar and transvectant operations characteristic of the modular theory were developed, but lack of definitiveness remained and yet remains, as each complete system which I have developed contains forms which were constructed by purely empirical methods. In subsequent papers I emphasized what seems to me at present to be the most favorable method upon which to ground a general theory by which it results that a modular covariant is a member of a finite scale of derived modular concomitants constructed by processes theoretically analogous to the important algebraical method of successive convolution.

* Presented to the Society, April 26, 1919, and December 30, 1919.

† References to other papers on this and related subjects, particularly to memoirs by Dickson, are to be found in my article in these Transactions, vol. 20 (1919), and a list of textual references for the present paper is given below. In connection with Section 8 compare my paper in the 1918 volume of the Proceedings of the National Academy of Sciences.

I. Dickson, these Transactions, vol. 12 (1911), p. 75; vol. 14 (1913), p. 299; vol. 15 (1914), p. 497. American Journal of Mathematics, vol. 31 (1909), p. 337.

II. Dickson, *The Madison Colloquium Lectures* (1913).

III. Einstein, Annalen der Physik, vol. 17 (1905).

IV. Glenn, these Transactions, vol. 20 (1919), p. 154.

V. Tolman, *The Theory of Relativity of Motion* (1917).

Fundamental likenesses between the algebraic and the modular theories are not lacking, but it seems to the writer that a theory to comprehend both effectually would not be constructed upon the principles of the algebraic symbolism but rather upon the methods such as characterize general analysis.* The present author thinks, however, that invariant theory has not yet been developed to a sufficient extent to make this synthesis feasible.

The first section of this memoir contains an introduction to a theory of modular covariants appertaining to domains of rationality. The second deals with systems of universal covariants of certain particular modular groups. The third, fourth and fifth are given over to the theory of seminvariants with application to complete systems for forms of orders $1, 2, 4$, moduli 2 and 3. The remainder of the paper deals with complete systems of covariants, with regard to these particular moduli, for the general binary quantic of order four, excepting the last section which treats certain invariant problems in the theory of relativity of motion.

1. Formal covariants belonging to domains

Suppose that the coefficients of the transformation on n variables

$$\tau : x_i = \lambda_i x_1' + \mu_i x_2' + \cdots + \sigma_i x_n' \qquad (i = 1, \cdots, n),$$

are residues modulo p, a prime number, or any other quantities such that the transformation is of a finite period μ, and let the coefficients of the n-ary quantic of order m

$$f = (a_{m0\cdots 0}, a_{m-1\,1\,0\cdots 0}, \cdots \between x_1, \cdots, x_n)^m, \tag{1}$$

be numbers which appertain to any specific domain of rationality R. Then if τ be applied μ times in succession to f, there is produced a closed cycle of forms $f, f_1, \cdots, f_{\mu-1}$ belonging to the domain R_τ obtainable by adjunction of the fundamental quantities of the domain of the coefficients of τ to the domain R. The product

$$F_{+1} = ff_1 \cdots f_{\mu-1},$$

is an absolute covariant of τ appertaining either to R_τ or to a domain included in R_τ. The same statement is true of any symmetric function of the general type

$$\Sigma_{+1}^{(\alpha)} = \Sigma \nu f^{\alpha_0} f_1^{\alpha_1} \cdots f_{\mu-1}^{\alpha_{\mu-1}},$$

these being, of course, all rationally expressible in terms of the complete system consisting of the μ elementary symmetric functions,

$$\Sigma_{+1}, \Sigma_{+1}', \cdots, \Sigma_{+1}^{(\mu-1)}.$$

* E. H. Moore, *An Introduction to a Form of General Analysis, The New Haven Mathematical Colloquium* (1910).

If R_τ is such that one can postulate a theory of conjugacy so each form Σ_{+1} in the domain will have corresponding to it a conjugate form Σ_{-1} in the domain, then

$$P_{\pm 1} = \Sigma_{+1} \pm \Sigma_{-1} \tag{2}$$

is a covariant of τ appertaining, certainly, to a lower domain included in R_τ.

The problem of the complete system of forms P rests upon Hilbert's Lemma, since the forms P are constructed according to a law by which one can locate an arbitrarily chosen form Σ within or without the system of forms P. Thus there exists a finite set $P'_{+1}, P''_{+1}, \cdots; P'_{-1}, P''_{-1}, \cdots$ such that any other form P may be written

$$P = Q'_{+1} P'_{+1} + \cdots + Q'_{-1} P'_{-1} + Q''_{-1} P''_{-1} + \cdots,$$

and the Q's, being symmetric, are forms Σ. The conclusion follows from (2) and repetitions of the course of reasoning.

Consider the problem when the coefficients of τ are residues.

A binary system in the $GF(p^2)$. If τ is binary ($n = 2$) it has two poles, found from

$$\left.\begin{aligned} x_1 &\equiv \lambda_1 x_1 + \mu_1 x_2 \\ x_2 &\equiv \lambda_2 x_1 + \mu_2 x_2 \end{aligned}\right\} \pmod{p},$$

and therefore roots of the quadratic congruence

$$\lambda_2 \rho^2 + (\mu_2 - \lambda_1)\rho - \mu_1 \equiv 0 \pmod{p}.$$

Thus the two poles are, generally, conjugate marks of the $GF(p^2)$, as

$$\rho_{\pm 1} \equiv rx \pm s \pmod{p},$$

where r, s are integral and x is a galoisian imaginary. The linear forms

$$f_{\pm 1} = x_1 - \rho_{\pm 1} x_2, \tag{3}$$

are, accordingly, relative covariants of τ, universal for all transformations τ for which $\mu_2 - \lambda_1$ is congruent to a fixed residue, and appertaining to the domain R of ρ_{+1}, ρ_{-1}, in general the $GF[p^2]$.

Let us now expand the arbitrary binary form

$$f = (a_0, a_1, \cdots, a_m \between x_1, x_2)^m$$

in terms of f_{+1}, f_{-1} as arguments, employing the substitutions inverse to (3),

$$x_1 = (-\rho_{-1} f_{+1} + \rho_{+1} f_{-1})/(\rho_{+1} - \rho_{-1}),$$

$$x_2 = (-f_{+1} + f_{-1})/(\rho_{+1} - \rho_{-1}).$$

The result is of the form

$$f \equiv \phi_m f_{+1}^m + \phi_{m-2} f_{+1}^{m-1} f_{-1} + \cdots + \phi_{-m} f_{-1}^m \pmod{p}. \tag{4}$$

Again if we expand f', the transformed of f by τ, in terms of the arguments

$$f'_{\pm 1} = x'_1 - \rho_{\pm 1} x'_2,$$

we get

(5) $$f' \equiv \phi'_m f'^m_{+1} + \phi'_{m-2} f'^{m-1}_{+1} f'_{-1} + \cdots + \phi'_{-m} f'^m_{-1} \pmod p.$$

But $f' \equiv f$ from the equations of τ, and

$$f'_{\pm 1} \equiv \delta_{\pm 1} f_{\pm 1} \pmod p,$$

where $\delta_{\pm 1}$ are definite marks. Hence

$$f \equiv \sum_{i=0}^{m} \phi'_{m-2i} \delta^{m-i}_{+1} \delta^{i}_{-1} f^{m-i}_{+1} f^{i}_{-1} \pmod p,$$

an expansion which must be formally identical with (4) since (4) is unique. Hence the forms ϕ_{m-2i}, which are linear in $a_0, \cdots, a_m$ and which belong to the $GF(p^2)$ are invariants of τ which appertain to the domain R determined by $f_{\pm 1}$ and $\delta_{\pm 1}$. The invariant relations are

$$\phi'_{m-2i} \equiv \delta^{-m+i}_{+1} \delta^{-i}_{-1} \phi_{m-2i} \pmod p \qquad (i = 0, \cdots, m).$$

THEOREM. *The finite set ϕ_{m-2i} $(i = 0, \cdots, m)$, f_{+1}, f_{-1}, constitutes a complete system of modular concomitants of the form f, in the domain R, under the transformation τ.*

2. Certain universal covariant systems

To determine a complete set of universal covariants of the ternary group

$$G : x \equiv x', \qquad y \equiv y', \qquad z \equiv x' + y' + z' \pmod p,$$

we assume a covariant in the general form

$$F = \sum_{i=0}^{m} f_{m-i} z^i,$$

where

$$f_{m-i} = a_0 x^{m-i} + a_1 x^{m-i-1} y + \cdots + a_m y^{m-i}.$$

When this is transformed into F' by G we must have $F' \equiv F \pmod p$. But

$$F' = \sum_{i=0}^{m} f_{m-i} (- x - y + z)^i \equiv F \pmod p,$$

and as the separate powers of z must accordingly have vanishing coefficients

$$\sum_{i=0}^{m} f_{m-i} (x + y)^i (- 1)^i \equiv 0 \pmod p.$$

This is equivalent to the statement that F is annuled by being made simultaneous with $x + y + z = 0$. Hence

$$F \equiv (x + y + z) F_1 \pmod p .$$

By the theory of § 1, therefore, F is necessarily a power of the covariant

$$\psi = z[z + x + y][z + 2(x + y)] \cdots [z + (p - 1)(x + y)]$$
$$\equiv z^p - z(x + y)^{p-1} \pmod p .$$

There is a single exception to this theory, viz. when F does not contain z and hence is a binary form in the covariants x, y. Hence,

THEOREM. *A complete system of universal covariants of G is composed of* x, y, ψ.

A similar method will give the system for the group $G_1 : x \equiv x' + y'$, $y \equiv y' \pmod p$, previously derived,* although by a different method, by Dickson. That is, a covariant of G_1 may be assumed in the form

$$F = \sum_{i=0}^{m} a_i x^{m-i} y^i .$$

Hence

$$F' \equiv \sum_{i=0}^{n} a_i (x - y)^{m-i} y^i \equiv F \pmod p ,$$

and as the coefficients of separate powers of x must vanish identically

$$\sum_{i=0}^{m} a_i (-1)^{m-i} y^{m-i} y^i \equiv 0 \pmod p .$$

Hence F is annuled when made simultaneous with $x + y = 0$ and so

$$F \equiv (x + y) F_1 \pmod p .$$

Thus a complete system for G_1 consists of y and

$$(6) \quad \psi_1 = x(x + y)(x + 2y) \cdots (x + \overline{p - 1}\, y) \equiv x^p - xy^{p-1} \pmod p .$$

Consider the universal covariants of the simultaneous binary modular groups

$$G_1 : x_1 = x_1' + x_2', \qquad x_2 = x_2'; \qquad G_2 : y_1 = y_1' + y_2', \qquad y_2 = y_2' .$$

These evidently consist of

$$x_2,\ y_2,\ \psi_1 = x_1^p - x_1 x_2^{p-1}, \qquad \psi_2 = y_1^p - y_1 y_2^{p-1},$$

and other covariants which are properly simultaneous. To determine the latter we note that such a form is necessarily a homogeneous doubly binary form in the sets x_1, x_2; y_1, y_2, and, employing Gordan's series, we may write this quantic as follows:

*Bulletin of the American Mathematical Society, vol. 20 (1913), p. 132.

$$F = a_x^m b_y^n = a_x^m b_x^n]_{y^n} + \sum_{i=1}^{n} \frac{\binom{m}{i}\binom{n}{i}}{\binom{m+n-i+1}{i}} (xy)^i (a_x^m, b_x^n)_{y^{n-i}}^i \quad (n \leqq m).$$

Since the variables y_1, y_2 are cogredient to x_1, x_2 all terms of this expansion are covariantive. Likewise $a_x^m b_x^n$ is a covariant of G_1 and so it is expressible as a polynomial in x_2 and ψ_1. The polarizing operation indicated by $a_x^m b_x^n]_{y^n}$ gives therefore a polynomial in

$$x_2, \psi_1, \text{ and } (xy) = x_1 y_2 - x_2 y_1, y_2,$$

inasmuch as

$$\left(y_1 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial x_2}\right)\psi_1 \equiv (xy) x_2^{p-2} \pmod{p}. \tag{7}$$

Every term in $\sum$ contains (xy) as a factor whereas the other factor is a doubly binary form similar to F and of the type $\alpha_x^{m-1}\beta_y^{n-1}$ to which a similar process of reduction by expansion can be applied in turn. We have now proved the following theorem:

THEOREM. *A complete system of universal covariants of the simultaneous groups* (*mod p*) G_1, G_2 *is composed of*

$$x_2, y_2, \psi_1, \psi_2, (xy).$$

3. SEMINVARIANCY

If the form

$$f_m = (a_0, a_1, \cdots, a_m \between x_1, x_2)^m$$

be transformed by the substitution $G_t : x_1 = x_1' + t x_2', x_2 = x_2'$, the transformations

$$\Gamma_t : a_j' \equiv \binom{m}{j} t^j a_0 + \binom{m-1}{j-1} t^{j-1} a_1 + \cdots + a_j \quad (j = 0, \cdots, m),$$

are induced upon the coefficients.

Any homogeneous quantic in $a_0, \cdots, a_m$ which remains absolutely unaltered modulo p under Γ_t is called a seminvariant of f_m under G_t.

The principles explained in § 1 give an important class of seminvariants. For if we select for the polynomial f any one of the N possible linear quantics in the a's

$$f = \alpha_0 a_0 + \alpha_1 a_1 + \cdots + \alpha_m a_m,$$

where α_i ($i = 0, \cdots, m$) takes any value $0, 1, \cdots, p-1$, and where the α's are selected so the different f forms are essentially different modulo p, then, if f is not itself a seminvariant, the successive application of the transformation Γ_1 of period p, to f, gives p forms

$$f, f_1, \cdots, f_{p-1},$$

and any elementary symmetric function

$$S_r = \Sigma \nu f f_1 \cdots f_r \qquad (r = 0 \text{ to } p - 1), \tag{8}$$

which is not congruent to zero is a seminvariant. Some of these functions S will be shown to be reducible in terms of other seminvariants but they must all be taken into account in the problem of the construction of a complete seminvariant system.

4. An algorism for the construction of seminvariants, invariants, and covariants

If

$$K_\nu \doteq C_0 x_1^\nu + C_1 x_1^{\nu-1} x_2 + \cdots + C_\nu x_2^\nu$$

is any formal covariant modulo p of f_m the copied form constructed from the coefficients C_i upon the model of S_r of the preceding section is a seminvariant of f_m.

Various methods of constructing covariants are available. Polarization of K_ν by the modular operators

$$w = x_1^p \frac{\partial}{\partial x_1} + x_2^p \frac{\partial}{\partial x_2},$$

$$E = a_0^p \frac{\partial}{\partial a_0} + a_1^p \frac{\partial}{\partial a_1} + \cdots + a_m^p \frac{\partial}{\partial a_m},$$

produces covariants if $\nu \not\equiv 0 \pmod{p}$. Substitution in the conjugate to a seminvariant $S(a_0, \cdots)$ under the permutations $s = (a_0 a_m)(a_1 a_{m-1}) \cdots$, according to Γ_t and reduction of the powers of the residue t by Fermat's theorem gives a covariant in non-homogeneous form. It can be made homogeneous by the replacement $t = x_1/x_2$. This process yields a covariant only on condition that S satisfies a definite pair of necessary and sufficient conditions as given in one of my former papers. When this principle is applied in the case of the seminvariant leading coefficient of K_ν where ν is a number of the form

$$\nu = (sp - \nu')(p - 1) = \sigma(p - 1)$$

there results a covariant

$$[K_\nu] = [C_0 + (K_\nu)] x_1^{p-1} + \sum_{r=0}^{p-2} \sum_{s=0}^{\sigma-1} C_{\nu - s(p-1) - r} x_1^r x_2^{p-r-1}, \tag{9}$$

where (K_ν) is the pure invariant

$$(K_\nu) = C_{p-1} + C_{2(p-1)} + \cdots + C_{(\sigma-1)(p-1)}. \tag{10}$$

If we form the product of any two covariants

$$F_{m'} = (C_0, C_1, \cdots, C_{m'} \between x_1, x_2)^{m'}, \qquad G_n = (D_0, D_1, \cdots, D_n \between x_1, x_2)^n$$

in which $m' + n$ is a number of the form $\sigma(p-1)$ and construct the formulas analogous to (9), (10) for the result we obtain the concomitants of f_m,

$$(11) \qquad (F_{m'} G_n) = \sum_{i=0}^{j(p-1)} \sum_{j=1}^{\sigma-1} C_i D_{j(p-1)-i},$$

$$(12) \qquad \begin{aligned} [F_{m'} G_n] = [C_0 D_0 + (F_{m'} G_n)] x_1^{p-1} \\ + \sum_{r=0}^{p-2} \sum_{s=0}^{\sigma-1} \sum_{i=0}^{t} C_i D_{(\sigma-s)(p-1)-r-i} x_1^r x_2^{p-1-r} \\ \{t = (\sigma - s)(p-1) - r\}. \end{aligned}$$

In connection with a single covariant K_ν we designate the derived set of $\mu = p + \nu + 2$ concomitants

$$(13) \qquad (K_\nu),\ w^\lambda[K_\nu] \quad (\lambda = 0, 1, \cdots, p-1), \qquad w^t K_\nu \quad (t = 0, 1, \cdots, \nu),$$

as the *μ-adic scale* for K_ν.

The processes (11), (12), (13) constitute the methods to which we referred in the introduction as being theoretically analogous to symbolical convolution in the algebraic invariant theory.*

5. Seminvariant systems

Complete system modulo 2 of the set f_1, f_2. The algorism described above relates in its complete generality to simultaneous systems. Consider the typical problem of the seminvariant system (mod 2) of a linear form and a quadratic,

$$f_1 = a_0 x_1 + a_1 x_2, \qquad f_2 = b_0 x_1^2 + b_1 x_1 x_2 + b_2 x_2^2.$$

The induced transformations Γ_t under G_t are these;

$$a_0' = a_0, \qquad a_1' = a_0 t + a_1,$$

$$b_0' = b_0, \qquad b_1' = b_1, \qquad b_2' = b_0 t^2 + b_1 t + b_2,$$

and, if we put $b_0 = 0$, G_1, Γ_1 become

$$a_1 = a_1' + a_0'; \qquad b_2 = b_2' + b_1',$$

$$a_0 = a_0', \qquad\qquad b_1 = b_1',$$

and these are the transformations of the simultaneous groups G_1, G_2 of § 2. Hence, if

$$S(b_0, b_1, b_2, a_0, a_1)$$

is any seminvariant of the system f_1, f_2,

$$S(0, b_1, b_2, a_0, a_1) \equiv F(a_0, b_1, \psi_1, \psi_2, C) \pmod 2,$$

* The notation here adopted was first given in the Proceedings of the National Academy of Sciences, vol. 5 (1919), pp. 107–110.

where F is a polynomial in its arguments and

$$\psi_1 = a_1^2 + a_0 a_1, \qquad \psi_2 = b_2^2 + b_1 b_2, \qquad C = a_0 b_2 + a_1 b_1.$$

However the expressions ψ_2, C are merely fragments of seminvariants which remain after one makes $b_0 = 0$. We now require these in their entirety. Apply the transformation of period 2

$$b_0 = b_0', \qquad b_1 = b_1', \qquad b_2 = b_0' + b_1' + b_2'$$

to $f = b_2$ and we get $f' = b_0 + b_1 + b_2$. The sum $f + f'$ is the reducible seminvariant $b_0 + b_1$ whereas

$$s = ff' = (b_0 + b_1 + b_2) b_2 = \psi_2 + b_0 b_2.$$

Again the seminvariant leading coefficient of $[f_1 f_2]$ is $a_0 b_0 + (f_1 f_2)$, where $(f_1 f_2)$ is the invariant

$$(14) \qquad (f_1 f_2) = a_0 (b_1 + b_2) + a_1 (b_0 + b_1) = a_0 b_1 + C + b_0 a_1$$
$$= a_0 b_1 + \rho.$$

Therefore

$$S(b_0, b_1, b_2, a_0, a_1) = F(a_0, b_0, b_1, \psi_1, s, \rho) + b_0 F_1(a_0, a_1, b_0, b_1, b_2).$$

But F_1 is a seminvariant and can be reduced in the form

$$F_1 = \Phi(a_0, b_0, b_1, \psi_1, s, \rho) + b_0 \Phi_1(a_0, a_1, b_0, b_1, b_2),$$

and the process can be repeated. Hence,

THEOREM. *A complete system of seminvariants of the set f_1, f_2, modulo 2, consists of*

$$a_0, \; b_0, \; b_1, \; \psi_1, \; s, \; \rho.$$

The expressions $k = b_0 s$, $q = s + b_0 b_1 + b_0^2$ are invariants, as are

$$Q' = a_0^2 + \psi_1, \qquad L' = a_0 \psi_1,$$
$$E(f_1 f_2) = a_0 (b_1^2 + b_2^2) + a_1 (b_0^2 + b_1^2).$$

We may note the following syzygies:

$$(15) \qquad \begin{aligned} &\rho^2 + (a_0 b_0 + a_0 b_1) \rho + a_0^2 s + (b_0^2 + b_1^2) \psi_1 \equiv 0, \\ &a_0^2 + \psi_1 + Q' \equiv 0, \\ &b_0^3 + b_0^2 b_1 + b_0 q + k \equiv 0, \\ &a_0^3 + L' + a_0 Q' \equiv 0, \end{aligned}$$

and we now see that the general seminvariant S can be reduced, by means of these congruences, to the finite form

$$S \equiv \phi_0 + \phi_1 b_0 + \phi_2 b_0^2 + a_0(\psi_0 + \psi_1 b_0 + \psi_2 b_0^2) + a_0^2(\chi_0 + \chi_1 b_0 + \chi_2 b_0^2), \tag{16}$$

where ϕ_i, ψ_i, χ_i are polynomials in the pure invariants $q, k, b_1, L', Q', (f_1 f_2)$.

A complete system modulo 3 *of* f_4. I shall refer informally at this juncture to an important fact which is to be employed later on in the paper viz. that every form f_m of order $> p^2 - 1$ is reducible modulo p in terms of the universal covariants* of $G_{(p^2-p)(p^2-1)}$ i.e.

$$L = x_1^p x_2 - x_1 x_2^p, \qquad Q = (x_1^{p^2} x_2 - x_1 x_2^{p^2})/L, \tag{17}$$

and covariants of f_m. Thus we need to consider in detail only those covariants of order $< p^2$.

For the sake of constructing seminvariants by the principle of copied forms we first derive those of the first degree (modulo 3) for the quantics of orders $2, 4, 6, 8 = 3^2 - 1$.

The linear seminvariant of f_2 mod 3 is a_0; those of a quartic $f_4 = a_0 x_1^4 + \cdots$ may be found by applying the induced transformations Γ_1 to the general expression

$$S = \alpha a_0 + \beta a_1 + \gamma a_2 + \delta a_3 + \epsilon a_4,$$

assumed to be a seminvariant, making $S' \equiv S \pmod 3$ and solving for $\alpha, \beta, \gamma, \delta, \epsilon$. The induced transformations are

$$\begin{gathered} a_0' = a_0, \qquad a_1' = a_0 + a_1, \qquad a_2' = a_2, \\ a_3' = a_0 + 2a_2 + a_3, \qquad a_4' = a_0 + a_1 + a_2 + a_3 + a_4, \end{gathered} \tag{18}$$

and S reduces to $\alpha a_0 + \gamma a_2$. Hence the linear seminvariants are

$$a_0, \ a_2 = (f_4). \tag{19}$$

For f_6 the induced transformations are

$$\begin{aligned} &a_0' = a_0, \qquad a_1' = a_1, \qquad a_2' = 2a_1 + a_2, \qquad a_3' = 2a_0 + a_1 + a_2 + a_3, \\ &a_4' = a_1 + a_4, \qquad a_5' = 2a_1 + a_2 + 2a_4 + a_5, \\ &a_6' = a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6, \end{aligned}$$

and the linear seminvariants prove to be

$$a_0, \ a_1, \ a_2 + a_4 = (f_6). \tag{20}$$

In the case of the octic f_8 we find, for Γ_1,

$$\begin{aligned} &a_0' = a_0, \qquad a_1' = 2a_0 + a_1, \qquad a_2' = a_0 + a_1 + a_2, \qquad a_3' = 2a_0 + a_3, \\ &a_4' = a_0 + 2a_1 + 2a_3 + a_4, \qquad a_5' = 2a_0 + 2a_1 + 2a_2 + a_3 + a_4 + a_5, \\ &a_6' = a_0 + a_3 + a_6, \qquad a_7' = 2a_0 + a_1 + 2a_3 + a_4 + 2a_6 + a_7, \\ &a_8' = a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8, \end{aligned}$$

* Compare II.

and for the seminvariants

$$a_0, \qquad V = a_1 - a_3, \qquad a_2 + a_4 + a_6 = (f_8).$$

Seminvariants of the second and third degrees of f_4. The algebraic invariants i, J, and the hessian, H, of f_4, which would also be modular concomitants, are reducible modulo 3; $H \equiv 2a_2 f_4 \pmod{3}$.

The seminvariant constructed on the model of V for the octic f_4^2 is

$$P = a_0 a_1 + 2a_1 a_2 + 2a_0 a_3,$$

while

$$q = (f_4^2) = a_1^2 + a_2^2 + a_3^2 + 2a_0 a_2 + 2a_0 a_4 + 2a_1 a_3 + 2a_2 a_4.$$

The transformations of (18) which affect a_0, $-a_2$, a_3 but not a_1 or a_4, form the particular ternary group G of § 2 and hence the following quantic is a seminvariant

$$N = a_3^3 - (a_0^2 + a_0 a_2 + a_2^2) a_3.$$

If we take $f = a_1$ in the theory of § 1, then $f' = a_0 + a_1$, $f'' = a_0 - a_1$ (by (18)), from which we obtain

$$\beta = a_1^3 - a_0^2 a_1 = -ff'f'',$$

or this seminvariant is obtained as a case of the universal covariant ψ_1 of the group G_1 (cf. (6)).

Selection of a_4 as f and application of (18) gives

$$\gamma_0 = a_4 (a_0 + a_1 + a_2 + a_3 + a_4)(a_0 - a_1 + a_2 - a_3 + a_4) = ff'f'',$$

whereas, if we take $f = a_0 + a_4$ this method gives the invariant

$$X = (a_0 + a_4)(a_0 + a_1 - a_2 + a_3 - a_4)(a_0 - a_1 - a_2 - a_3 - a_4) + a_0^2 a_2.$$

Lastly, if we select $f = a_1 + a_4$ so that $-f'' = a_2 - a_3 + a_4$,

$$f' = a_0 + a_1 - a_2 - a_3 - a_4,$$

the seminvariant $-ff'f''$ is congruent to $2\gamma_0 + D + H + a_2 P$, where

$$H = a_0^2 a_4 + a_1^2 a_2 - a_0 a_1 a_3,$$

$$D = a_0 a_2 a_3 - a_1 a_2 a_4 - a_1^2 a_3 + a_1 a_3^2 + a_0 a_1 a_4 - a_0 a_3 a_4.$$

We shall show that γ_0 can be eliminated by means of a syzygy between a_0, γ_0, q, X. Note that D, β, P are skew. I desire to enunciate and to treat at some length the following

THEOREM. *The nine seminvariants β, D, H, N, P, q, X, a_0, $a_2 = (f_4)$ compose a complete system for f_4 modulo 3.*

Suppose that

$$S = S(a_0, a_1, a_2, a_3, a_4)$$

is an arbitrary seminvariant modulo 3 of f_4. Then it will ordinarily be possible to separate S into two seminvariantive portions

$$S = S_1 + S_2$$

where S_2 contains a_4 but S_1 does not. Since S_1 is not affected by the fifth congruence of (18) we write

$$S_1 = S_1(a_0, a_1, a_2, a_3)$$

and note that S_1' is invariantive under the binary groups G_1, G_2 of § 2, i.e.

$$(21) \qquad \begin{aligned} &G_1 : a_0' = a_0, \qquad a_1' = a_0 + a_1, \\ &G_2 : a_0' - a_2' = a_0 - a_2, \qquad a_3' = a_0 - a_2 + a_3, \end{aligned}$$

therefore S_1 is a polynomial in a_0, a_2, β, N, P, since

$$-P = \begin{vmatrix} a_0 & a_1 \\ a_0 - a_2 & a_3 \end{vmatrix}.$$

With reference to S_2 we next assume that $a_0 = a_2 = 0$, momentarily, and note that the group (18) is then equivalent to G (§ 2). Thus $S_2(0, a_1, 0, a_3, a_4)$ is a polynomial in a_1, a_3 and

$$\gamma_0' = a_4^3 - (a_1 + a_3)^2 a_4 = \gamma_0]_{a_0=a_2=0}.$$

The coefficients of the different powers of γ_0' in this expression are polynomials in a_1, a_3 and of such a nature as to be expressible in terms of functions of a_1, a_3 which are fragments of seminvariants, which remain when we make $a_0 = a_2 = 0$. Four seminvariants which reduce to expressions in a_1, a_3 when $a_0 = a_2 = 0$ are β, q, D, N and we here make an assumption,* to be treated in a later paragraph from another point of view, viz., that the coefficients of the different powers of γ_0' in the expression for $S_2(0, a_1, 0, a_3, a_4)$ are all rationally expressible in terms of

$$(22) \quad \beta_1 = a_1^3, \qquad N_1 = a_3^3, \qquad q_1 = a_1^2 + 2a_1 a_3 + a_3^2, \qquad D_1 = a_1 a_3^2 - a_1^2 a_3,$$

these being the only ones of the nine seminvariants which reduce to expressions in a_1, a_3 when $a_0 = a_2 = 0$ and $a_4 \neq 0$. Under this assumption we now have

$$(23) \quad S_2(a_0, a_1, a_2, a_3, a_4) \equiv \phi(\beta, D, N, q, \gamma_0) + a_0 \phi_0 + a_2 \phi_2 \pmod 3,$$

where ϕ_0, ϕ_2 are polynomials in a_0, a_1, a_2, a_3, a_4, not both free from a_4, and such that $a_0 \phi_0 + a_2 \phi_2$ is a seminvariant. If, then, ϕ_0, ϕ_4 are both seminvariants they are of the type of the original S and the reductions can be applied to them in turn. If they are not separately seminvariantive

$$\psi = a_0 \phi_0 + a_2 \phi_2$$

is a seminvariant every term of which contains either a_0 or a_2 as a factor and which is not free from a_4. Moreover, under (18),

$$\psi' = a_0(\phi_0 + \delta\phi_0) + a_2(\phi_2 + \delta\phi_2) \equiv \psi,$$

where the expressions $\delta\phi_i$ $(i = 0, 2)$ are increments. Hence

$$\frac{\delta\phi_0}{a_2} = -\frac{\delta\phi_2}{a_0}, \tag{24}$$

i.e. $\delta\phi_0$ contains the factor a_2 and hence ϕ_0 is a function of both a_3, a_4 combined in the form of a difference, or of $a_1 - a_3$, whereas ϕ_2, in order that $\delta\phi_2$ may contain a_0 as a factor, must be a function of a_1 only or contain a power of a_1 as a factor. It follows, since

$$H = \begin{vmatrix} a_0 & -a_2 \\ a_1^2 & a_0 a_4 - a_1 a_3 \end{vmatrix},$$

that*

$$\psi = ZM,$$

where Z is a polynomial in H, P with numerical coefficients, whereas M is a seminvariant of arbitrary form.

The reader has observed, however, that this proof is definitely intuitional at two points, i.e. at the conclusions which are starred. To reduce a possible doubt as to these inferences to a minimum I set for myself the task of computing and reducing in terms of the nine fundamental seminvariants the 285 seminvariants of f_4 modulo 3 constructed as elementary symmetric functions of f, f', f'' where f is a chosen linear expression of the form

$$f = \alpha_0 a_0 + \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3 + \alpha_4 a_4,$$

the α's being the numbers $0, 1, 2$ in definite sequence, and f', f'' the two functions transformed of f by (18) (compare § 3). This problem has a certain ulterior significance as it is not impossible that a theory of complete systems might be formulated from the point of view of the symmetric function theory.§ I have no suggestions to make in this connection at present, however; the point here being that all of the 285 seminvariants so calculated proved to be reducible in terms of β, D, H, N, P, q, X, a_0, a_2.

It would not be relevant to reproduce the computations for these various cases, but I give the results for two types f which are typical.

$$f = a_3 + 2a_4, \qquad f' = 2a_1 + a_2 + 2a_4, \qquad f'' = a_0 + a_1 + 2a_3 + 2a_4. \tag{25}$$

$$\Sigma f = a_0 + a_2, \qquad \Sigma ff' = 2(q + P) + a_2^2,$$

$$\Sigma ff'f'' = 2\gamma_0 + 2a_2 q + Ha_2^3 - a_0 a_2^2 + D.$$

§ Compare the third reference of I.

$$(26)\; f = a_0 + a_3 + a_4, \qquad f' = a_1 + 2a_3 + a_4, \qquad f'' = a_0 + 2a_1 + 2a_2 + a_4.$$

$$\Sigma f = 2(a_0 + a_2), \qquad \Sigma ff' = a_0^2 + 2q + P + a_0 a_2 + a_2^2,$$

$$\Sigma ff'f'' = D + a_2 P + 2H + \gamma_0 + (a_0 + 2a_2)(P + 2q) + 2a_0 a_2 (a_0 + a_2).$$

Congruential syzygies. If the course of development similar to that followed in § 5 is to be followed it is now required to derive a sufficient number of syzygies which can be employed as reduction moduli to reduce the general seminvariant S to a polynomial *in finite form* (cf. (16)) in the nine seminvariants of the complete system. This again is a formal problem of much difficulty, but by simple combinations we get the two below, which one can easily verify:

$$(27) \quad \left.\begin{array}{l} a_0^3 + 2a_0 q + \gamma_0 + 2X + 2a_0 a_2 (a_0 + 2a_2) \equiv 0 \\ a_0^4 + 2a_0^2 q + 2a_0 X + k + 2a_0^2 a_2 (a_0 + 2a_2) \equiv 0 \end{array}\right\} (\mathrm{mod}\ 3),$$

where k is the invariant $a_0 \gamma_0$.

We next represent by S' the result of placing $a_1 = 0$ in a seminvariant S and by S'' the result of substituting zero for both a_1 and a_3 in S. Then

$$q'' = (a_0 + a_2 + a_4)^2 + 2a_0^2 + 2a_4^2,$$

$$\gamma_0'' = (a_0 + a_2 + a_4)^2 a_4,$$

$$R'' = (a_0 - a_2 - a_4)(a_2 + a_4) a_4,$$

where

$$R = (a_1 + a_4)(a_2 - a_3 + a_4)(a_0 + a_1 - a_2 - a_3 - a_4) = -ff'f'',$$

then

$$R'' + \gamma_0'' = a_0^2 a_4,$$

$$2a_0^2 (q'' + a_2^2) = a_0^2 a_2^2 + a_0^3 a_2 + a_0^3 a_4 + a_0^2 a_2 a_4,$$

hence

$$2a_0^2 (q'' + a_2^2) = a_0^2 a_2^2 + a_0^3 a_2 + (a_0 + a_2)(R'' + \gamma_0''),$$

and, in consequence,

$$\Theta' = a_0^3 a_2 + (a_0 + a_2)(R' + \gamma_0') + a_0^2 (q' + 2a_2^2)$$

is a function which vanishes when we place $a_3 = 0$.

Furthermore

$$q' = a_2^2 + a_3^2 + 2a_0 a_2 + 2a_0 a_4 + 2a_2 a_4,$$

$$P' = 2a_0 a_3,$$

$$N' = a_3^3 - a_2^2 a_3 - (a_0^2 + a_0 a_2) a_3,$$

$$= a_3^3 - a_2^2 a_3 + (a_0 + a_2) P',$$

$$\gamma_0' = a_0^2 a_4 + a_2^2 a_4 + 2a_0 a_2 a_4 + 2a_0 a_4^2 + 2a_2 a_4^2 + 2a_3^2 a_4 + a_4^3,$$

$$R' = a_0 a_2 a_4 - a_0 a_3 a_4 + a_0 a_4^2 - a_2^2 a_4 + a_2 a_4^2 + a_3^2 a_4 - a_4^3,$$

and thus we get

$$\begin{aligned}\Theta' &\equiv 2a_0 a_3 [a_0 a_4 + 2a_0 a_3 + a_2 a_4] \\ &\equiv 2P' [q' + 2P' + a_0 a_2 + a_2^2] + 2a_0 N' + (a_0^2 + a_0 a_2) P'.\end{aligned}$$

Hence a function which vanishes when $a_1 = 0$ is

$$\begin{aligned}\Theta = P^2 + 2(q + a_0 a_2 + a_2^2) P + 2a_0 N + (a_0^2 + a_0 a_2) P \\ + 2a_0^2 (q + 2a_2^2) + 2a_0^3 a_2 + 2(a_0 + a_2)(R + \gamma_0),\end{aligned}$$

and so Θ is congruent to a seminvariant which contains β as a factor. We find by actual evaluation $\Theta = (2a_0 + a_2)\beta$. Also the seminvariant R is reducible in the form

$$R \equiv 2\gamma_0 + H + D + a_2 P \pmod{3},$$

therefore we have derived the following syzygy of degree 4:

$$\begin{aligned}P^2 + 2(q + 2a_0^2 + a_2^2) P + 2a_0 N + 2a_0^2 (q + 2a_2^2) \\ + 2a_0^3 a_2 + 2(a_0 + a_2)(D + a_2 P + H) + (a_0 + 2a_2)\beta = 0.\end{aligned} \tag{28}$$

Some of the seminvariants involved are of even and others of odd weight, hence the relation is equivalent to two syzygies obtained by equating the terms of even and odd weights to zero separately. Thus, finally, we get

$$\begin{aligned}P^2 + 2a_0^2 (q + 2a_2^2) + 2a_0^3 a_2 + 2(a_0 + a_2) H = 0, \\ P(q + 2a_0^2 + a_2^2) + a_0 N + (a_0 + a_2)(D + a_2 P) + (2a_0 + a_2)\beta = 0.\end{aligned} \tag{29}$$

The seminvariant N is connected with a third degree invariant

$$\tau = a_3^3 - a_1^3 + a_1 a_2^2 - a_2^2 a_3,$$

by the linear relation

$$N = \tau + \beta + (a_0 + a_2) P,$$

and we therefore see that the equivalents of five of the nine seminvariants, i.e., a_2, q, X, D, N, are irreducible invariants. The arbitrary seminvariant S can be reduced, as a result of this fact, by the relations (27) and the first syzygy of (29), to a polynomial of indefinite order in two seminvariants β, H with coefficients which are of degree three in a_0 and linear in P, in the form

$$\begin{aligned}S = \sum_{i=1}^{\nu} [\Theta_{0i} + \Theta_{1i} a_0 + \Theta_{2i} a_0^2 + \Theta_{3i} a_0^3 \\ + (\Delta_{0i} + \Delta_{1i} a_0 + \Delta_{2i} a_0^2 + \Delta_{3i} a_0^3) P] \beta^{r_i} H^{s_i},\end{aligned} \tag{30}$$

where $r_1 = s_1 = 0$, and θ_{hi} and Δ_{hi} $(i = 1, \cdots, \nu;\ h = 0, \cdots, 3)$ are polynomials with numerical coefficients in the pure invariants

$$a_2, D, q, \tau, X, k. \tag{30_1}$$

There doubtless exist two syzygies, one headed by a low power of each seminvariant β, H, of degree higher than four, and involving additional members of a complete system of pure invariants, by means of which S could be reduced to a form finite and numerical as to powers of β, H also.

6. Covariant systems of the set f_1, f_2, and for the binary quartic f_4, modulo 2

I shall prove that a fundamental system of covariants of f_4 modulo 2 is furnished by construction of a simultaneous system of a certain pair of covariants of f_4, one a quadratic and the other linear. For this reason I select for the next topic to be treated the problem of the covariant system modulo 2 of the set f_1, f_2 (cf. § 5).

The formula (16) for the arbitrary seminvariant leading coefficient of a simultaneous covariant of f_1, f_2 suggests the form of the covariants requisite to reduce the simultaneous covariant of general type. That is we require covariants of the various possible orders led by $a_0^i\, b_0^j$ $(i, j = 0, 1, 2)$ and by invariants.

When the modulus is 2 the concomitant scale (13) of § 4 is not complete without the addition of the two forms*

$$\{K_\nu\} = C_0\, x_1^2 + (K_\nu)\, x_1\, x_2 + C_\nu\, x_2^2,$$

$$\{\bar{K}_\nu\} = C_0\, x_1^3 + I_1\, x_1^2\, x_2 + I_2\, x_1\, x_2^2 + C_\nu\, x_2^3,$$

where I_1, I_2 are definite linear expressions in $C_1, \cdots, C_{\nu-1}$ such that

$$I_1 + I_2 \equiv (K_\nu) \pmod 2 .$$

There exist no covariants of f_1 led by invariants, no quadratic covariant of any system of forms (mod 2) led by an invariant I, other than IQ, and the only possible invariant leading coefficient for a covariant C of the set f_1, f_2, which is a polynomial in q, k, b_1, L', Q', contains $b_1\, q + k$ as a factor. The first two of these statements are easily verified; it will be sufficient to prove the latter for a cubic covariant in view of the form of $\{\bar{K}_\nu\}$. If I is the assumed leading coefficient it can be reduced, by virtue of the congruence

$$b_0\, q = b_0 (s + b_0\, b_1 + b_0^2) = b_0^3 + b_0^2\, b_1 + k, \tag{31}$$

to the form

$$I = \psi_0 + \psi_1\, b_0 + \psi_2\, b_0^2$$

where ψ_0, ψ_1, ψ_2 are seminvariants. Apply the induced transformations under $x_2 = x_1' + x_2'$, $x_1 = x_1'$ to

$$C = I x_1^3 + S_1\, x_1^2\, x_2 + S_2\, x_1\, x_2^2 + (\psi_0 + \psi_1\, b_0 + \psi_2\, b_0^2)\, x_2^3$$

* Compare IV.

and equate the result to C and there results

$$\delta S_1 = \delta S_2 = I, \qquad \delta S_1 + \delta S_2 + S_1 + S_2 + I = 0,$$

$$I = \delta S_1 + \delta S_2 + S_1 + S_2 + I + \psi_0 + (b_0 + b_1 + b_2)\psi_1 + (b_0 + b_1 + b_2)^2 \psi_2. \tag{32}$$

Hence

$$I = S_1 + S_2 = (b_1 + b_2)[\psi_1 + (b_1 + b_2)\psi_2].$$

But an invariant with the factor $b_1 + b_2$ has the factor

$$b_1 q + k = (b_0 + b_1)(b_0 + b_2)(b_1 + b_2), \tag{33}$$

which was to be proved.

The following cubic covariant which I constructed by empirical methods is led by $b_1 q + k$ and the product of it by an arbitrary invariant of f_1, f_2 is the general cubic with an invariant leader of the stated type:

$$\begin{aligned} C = (b_0 + b_1)(b_0 + b_2)(b_1 + b_2)x_1^3 & \\ + (b_0^3 + b_1^3 + b_2^3 + b_0 b_1^2 + b_1 b_2^2 + b_0^2 b_2) x_1^2 x_2 & \\ + (b_0^3 + b_1^3 + b_2^3 + b_0^2 b_1 + b_1^2 b_2 + b_0 b_2^2) x_1 x_2^2 & \\ + (b_0 + b_1)(b_0 + b_2)(b_1 + b_2) x_2^3. & \end{aligned}$$

The form C is reducible in terms of concomitants from the sets (34), (35), (36).

An invariant leading coefficient which is properly simultaneous and therefore not merely a polynomial in q, k, b_1, L', Q' must contain $(f_1 f_2)$ as a factor, this being the only remaining invariant in the set from which ϕ_0 is formed. We shall construct a cubic covariant B led by $(f_1 f_2)$. The most general form of ϕ_0 is

$$\phi_0 = \lambda + (f_1 f_2)\mu,$$

where λ is a polynomial in q, k, b_1, L', Q' and μ is an invariant. Assuming that ϕ_0 leads a cubic covariant D', $D' - \mu B$ is led by λ and so $\lambda = (b_1 q + k)\nu$ where ν is an invariant. Thus

$$D' = \nu C + \mu B.$$

I now give a list of cubic covariants needed, with the seminvariant (or invariant) leader juxtaposed.

$$\begin{gathered} a_0,\ Qf_1; \qquad a_0^2,\ QEf_1; \qquad b_0 + b_1,\ Q[f_2]; \\ b_0^2 + b_1^2,\ Q[Ef_2]; \qquad a_0 b_0,\ f_1 f_2; \qquad a_0 b_0^2,\ f_1 Ef_2; \\ a_0^2 b_0,\ f_2 Ef_1; \qquad a_0^2 b_0^2,\ Ef_1 Ef_2; \qquad (f_1 f_2),\ Q[f_1 f_2] + f_1 f_2 = B. \end{gathered} \tag{34}$$

The quadratic and linear covariants required are given in the following lists respectively:

$$
\begin{aligned}
&a_0,\ uf_1; \quad a_0^2,\ f_1^2; \quad b_0,\ f_2; \quad b_0^2,\ Ef_2; \quad a_0 b_0,\ \{f_1 f_2\}; \\
&a_0^2 b_0,\ \{Ef_1 \cdot f_2\}; \quad a_0 b_0^2,\ \{f_1\, Ef_2\}; \quad a_0^2 b_0^2,\ \{Ef_1\, Ef_2\}.
\end{aligned} \tag{35}
$$

$$
\begin{aligned}
&a_0,\ f_1; \quad a_0^2,\ Ef_1; \quad b_0 + b_1,\ [f_2]; \quad b_0^2 + b_1^2,\ E[f_2]; \\
&a_0 b_0 + (f_1 f_2),\ [f_1 f_2]; \quad a_0 b_0^2 + (f_1\, Ef_2),\ [f_1\, Ef_2]; \\
&a_0^2 b_0 + (Ef_1 \cdot f_2),\ [Ef_1 f_2]; \quad a_0^2 b_0^2 + (Ef_1\, Ef_2),\ [Ef_1\, Ef_2].
\end{aligned} \tag{36}
$$

Observe next that the cubic covariant of the set f_1, f_2 of arbitrary form, i.e. (see (16))

$$
\begin{aligned}
K = Sx_1^3 + \cdots = [\phi_0 + b_1 \phi_1 + b_1^2 \phi_2 + \phi_1 (b_0 + b_1) \\
+ \phi_2 (b_0^2 + b_1^2) + a_0(\psi_0 + \cdots) + \cdots] x_1^3 + \cdots
\end{aligned}
$$

is reducible in the form

$$
\begin{aligned}
K \equiv \mu B + \nu C + \phi_1 Q[f_2] + \phi_2 Q[Ef_2] + \psi_0 Qf_1 + \psi_1 f_1 f_2 \\
+ \psi_2 f_1\, Ef_2 + \chi_0 QEf_1 + \chi_1 f_2\, Ef_1 + \chi_2\, Ef_1\, Ef_2 + K' L \pmod 2,
\end{aligned} \tag{37}
$$

where K' is an invariant.

Moreover the arbitrary quadratic is reduced by the formula ($\phi_0 = 0$)

$$
\begin{aligned}
K = \phi_1 f_2 + \phi_2\, Ef_2 + \psi_0\, wf_1 + \psi_1 \{f_1 f_2\} + \psi_2 \{f_1\, Ef_2\} \\
+ \chi_0 f_1^2 + \chi_1 \{Ef_1 f_2\} + \chi_2 \{Ef_1\, Ef_2\}, \pmod 2.
\end{aligned} \tag{38}
$$

As a preliminary to the reduction of linear covariants we write the formula S as follows:

$$
\begin{aligned}
S = \{&\phi_0 + b_1 \phi_1 + b_1^2 \phi_2 + \psi_1 (f_1 f_2) + \psi_2 (f_1\, Ef_2) \\
&+ \chi_1 (Ef_1 f_2) + \chi_2 (Ef_1\, Ef_2)\} + \phi_1 (b_0 + b_1) + \phi_2 (b_0^2 + b_1^2) + \psi_0 a_0 \\
&+ \psi_1 [a_0 b_0 + (f_1 f_2)] + \psi_2 [a_0 b_0^2 + (f_1\, Ef_2)] + \chi_0 a_0^2 \\
&+ \chi_1 [a_0^2 b_0 + (Ef_1 f_2)] + \chi_2 [a_0^2 b_0^2 + (Ef_1\, Ef_2)],
\end{aligned}
$$

and since there are no linear covariants led by invariants the invariant expression in the brace is congruent to zero.

Therefore the arbitrary linear covariant K is reduced in the form

$$
\begin{aligned}
K \equiv \phi_1 [f_2] + \phi_2 E[f_2] + \psi_0 f_1 + \psi_1 [f_1 f_2] + \psi_2 [f_1\, Ef_2] + \chi_0\, Ef_1 \\
+ \chi_1 [Ef_1 f_2] + \chi_2 [Ef_1\, Ef_2] \pmod 2.
\end{aligned} \tag{39}
$$

We have thus completed the proof of the following

THEOREM. *A fundamental system of covariants of the set f_1, f_2 under the modular group G_6* (mod 2) *consists of nineteen quantics, viz.,*

$$
(40)\qquad
\begin{gathered}
f_1,\ f_2,\ Q,\ L,\ Ef_1,\ Ef_2,\ wf_1,\ [f_2],\ [Ef_2],\\
[f_1 f_2],\ [f_1\, Ef_2],\ [Ef_1 f_2],\ [Ef_1\, Ef_2],\ (f_1 f_2),\\
q,\ k,\ b_1,\ L',\ Q'.
\end{gathered}
$$

It is an elementary problem to show that the remaining forms involved in the sets (34), (35), (36) are reducible in terms of the quantics (40). Note that it is not stated that the six invariants given form a complete system of pure invariants. The form K' in (37) might involve invariants not reducible in terms of the six.

A theorem concerning modular systems under $G_{(p^2-p)(p^2-1)}$. I proved in previous papers,* under rather strong restrictions as to the point of generality, that a binary form f_m of order m ($m \geqq p^2$) can be expanded covariantively, i.e. given a typical representation in which the denominator function is numerical, in terms of L and Q as argument forms, in such a way that the coefficient forms are first degree formal covariants modulo p of orders $\leqq p^2 - 1$. This expansion was given under the notation

$$
(41)\qquad f_m = Q^\alpha \phi_1 + Q^{\beta-1} L\chi + \cdots + QL^{\gamma-1}\psi + L^\delta w \pmod p,
$$

where ϕ_1 is any one of a number of covariants of a definite type, called principal covariants, each led by a_0. Such an expansion, though not unique, is such that the total number of coefficients involved in the forms $\phi_1,\ \chi,\ \cdots$ is at least as great as $m+1$. For brevity write $\phi_1,\ \chi,\ \cdots$ under the notation ϕ_i ($i = 1, 2, \cdots$), and let

$$
\phi_i = \lambda_{i0}\, x_1^{\nu_i} + \lambda_{i1}\, x_1^{\nu_i - 1} x_2 + \cdots + \lambda_{i\nu_i} x_2^{\nu_i}.
$$

THEOREM. *A fundamental system modulo p of f_m is a simultaneous system of the set ϕ_i ($i = 1, 2, \cdots$).*

Suppose that the equation (congruence) (41) is transformed by the general modular substitutions

$$
(42)\qquad x_1 = \lambda_1 x_1' + \mu_1 x_2', \qquad x_2 = \lambda_2 x_1' + \mu_2 x_2'.
$$

Since $L' \equiv (\lambda\mu)^{-1} L$, $Q' \equiv Q \pmod p$ we get

$$
(43)\qquad f_m' = (a_0', \cdots \between x_1', x_2')^m \equiv Q^\alpha \phi_1' + Q^{\beta-1} L\phi_2' + \cdots \pmod p,
$$

and

$$
\phi_i' = \lambda_{i0}' x_1'^{\nu_i} + \lambda_{i1}' x_1'^{\nu_i - 1} x_2' + \cdots + \lambda_{i\nu_i}' x_2'^{\nu_i},
$$

where the λ_{ij}' are linear combinations of $\lambda_{i0},\ \cdots,\ \lambda_{i\nu_i}$ and the latter are linear forms in $a_0,\ \cdots,\ a_m$. Suppose next that we do not transform (41) as a congruence but that f_m is first transformed by (42) into f_m' and then that the latter is expanded by (41). The result is

* Annals of Mathematics, vol. 19 (1918), p. 201.

$$f'_m = Q^\alpha \phi''_1 + Q^{\beta-1} L'\chi'' + \cdots + L'^{\delta} w'' \pmod p, \tag{44}$$

where

$$\phi''_i = \lambda''_{i0} x_1'^{\nu_i} + \lambda''_{i1} x_1'^{\nu_i - 1} x'_2 + \cdots \qquad (i = 1, 2, \cdots),$$

and λ''_{ij} is the same function of $a'_0, \cdots, a'_m$ that λ_{ij} is of $a_0, \cdots, a_m$; provided (41) satisfies any set of necessary and sufficient conditions whatever which make this expansion unique. Moreover, under these conditions, we get ϵ different sets of induced transformations on the a's and a''s, ϵ being the number of different forms ϕ''_i, viz.,

$$\lambda''_{ij} \equiv \lambda'_{ij} (\lambda\mu)^{\alpha_i} \qquad (i = 1, 2, \cdots, \epsilon). \tag{45}$$

These ϵ sets are together equivalent to the ordinary single set constituting the induced group under (42). Hence, since (45) gives ϵ induced groups for the ϵ forms ϕ_i, and these groups are together equivalent to the transformations on the a's, a concomitant under (42) and (45) i.e. a simultaneous concomitant of the forms ϕ_i $(i = 1, \cdots, \epsilon)$, is a concomitant of f_m and conversely. Hence the theorem is proved.

A complete system of the binary quartic modulo 2. When $m = 4$ (41) is unique without condition and its explicit form is

$$\begin{aligned} &f_4 \equiv Lf_1 + Qf_2 \pmod 2; \\ &f_1 = (a_0 + a_2 + a_3) x_1 + (a_1 + a_2 + a_4) x_2, \\ &f_2 = a_0 x_1^2 + (a_1 + a_2 + a_3) x_1 x_2 + a_4 x_2^2. \end{aligned} \tag{46}$$

Combination of this conclusion with the results numbered (40) gives at once the following

THEOREM. *A fundamental system of formal covariants modulo* 2 *of the binary quartic* f_4 *is composed of nineteen quantics, viz.,* (cf. (46))

$$\begin{aligned} &f_1, f_2, Q, L, Ef_1, Ef_2, wf_1, [f_2], [Ef_2], [f_1 f_2], \\ &[f_1 Ef_2], [Ef_1 f_2], [Ef_1 Ef_2], (f_1 f_2), q, k, (f_4), L', Q'. \end{aligned} \tag{47}$$

In this theory

$$\begin{aligned} q &= a_0^2 + a_0 a_1 + a_0 a_2 + a_0 a_3 + a_0 a_4 + a_1 a_4 + a_2 a_4 + a_3 a_4 + a_4^2, \\ k &= a_0 (a_0 + a_1 + a_2 + a_3 + a_4) a_4, \\ L' &= (a_0 + a_2 + a_3)(a_1 + a_2 + a_4)(a_0 + a_1 + a_3 + a_4), \\ Q' &= a_0 a_2 + a_0 a_4 + a_1 a_2 + a_2 a_4 + a_1 a_3 + a_2 a_3 + a_3 a_4 \\ &\qquad + a_2^2 + a_3^2 + a_4^2 + a_0 a_1 + a_0^2 + a_1^2. \end{aligned} \tag{48}$$

7. Covariants of f_4 modulo 3

The problem of the determination of fundamental covariants led by the seminvariants of the complete system given in § 5 may now be treated.

There exists no quadratic covariant led by an invariant, but $a_0 + a_2$ leads

$$[f_4] = [a_0 + (f_4)]x_1^2 + (a_1 + a_3)x_1 x_2 + [(f_4) + a_4]x_2^2.$$

If we substitute from the induced transformations under $x_1 = x_1' + tx_2'$, $x_2 = x_2'$ in the conjugate to P under the permutations $(a_0 a_4)(a_1 a_3)(a_2)$ and reduce the powers, of the residue t by Fermat's theorem, there results the remarkable skew-covariant

$$[\bar{f}_4] = Px_1^2 + (2a_0 a_2 + a_2 a_4 + a_1^2 - a_3^2)x_1 x_2 + (a_1 a_4 + a_2 a_3 + 2a_3 a_4)x_2^2. \tag{49}$$

From $H + (f_4)^3$ we obtain, by this method,

$$\begin{aligned}[\bar{\bar{f}}_4] = [H + (f_4)^3]x_1^2 &+ [a_0 a_1 a_4 + a_2^2 a_3 + 2a_1^2 a_3 + a_1 a_2^2 + 2a_1 a_3^2 \\ &+ a_0 a_3 a_4 + a_0 a_2 a_3 + a_1 a_2 a_4]x_1 x_2 \\ &+ [(f_4)^3 + a_0 a_4^2 + a_2 a_3^2 - a_1 a_3 a_4]x_2^2.\end{aligned} \tag{50}$$

The algebraical covariant $T_{(3)}$ of order six and degree three with numerical coefficients reduced modulo 3 is a formal covariant whose leader is the seminvariant

$$a_0^2 a_3 + a_0 a_1 a_2 + 2a_1^3.$$

An equivalent covariant which is simpler, in that its leading coefficient is $-\beta$, has the form

$$T = T_{(3)} + [\bar{\bar{f}}_4],$$

and

$$\begin{aligned}T = -\beta x_1^6 &+ (a_0 a_2 a_4 + 2a_0^2 a_2 + 2a_0 a_1^2 + 2a_0 a_3^2 + a_0 a_1 a_3 \\ &+ 2a_0 a_2^2 + a_0^2 a_4)x_1^5 x_2 + (2a_0 a_2 a_3 + 2a_0 a_3 a_4 + a_1^2 a_3 \\ &+ a_1 a_2 a_4 + 2a_1 a_2^2 + 2a_1 a_3^2 + a_1^3)x_1^4 x_2^2 + (a_0 a_3^2 + 2a_1^2 a_4 \\ &+ a_0 a_1 a_3 + 2a_1 a_3 a_4 + 2a_0 a_2^2 + a_1^2 a_2 + a_2^2 a_4 + 2a_2 a_3^2)x_1^3 x_2^3 \\ &+ (a_0 a_1 a_4 + a_1 a_2 a_4 + 2a_1 a_3^2 + a_2^2 a_3 + 2a_0 a_2 a_3 \\ &+ a_1^2 a_3 - a_3^3)x_1^2 x_2^4 + (2a_0 a_4^2 + 2a_1 a_3 a_4 + a_2^2 a_4 + a_3^2 a_4 \\ &+ 2a_0 a_2 a_4 + a_2 a_4^2 + a_1^2 a_4)x_1 x_2^5 + (a_3^3 - a_3 a_4^2)x_2^6.\end{aligned} \tag{51}$$

From this one obtains

$$(T) = D + 2\tau \equiv D + 2N + \beta + (a_0 + a_2)P \pmod 3,$$

and

$$
\begin{aligned}
[T] = [-\beta + (T)]x_1^2 + (a_0 a_2^2 + 2a_0 a_1 a_3 + a_0^2 a_4 + 2a_0 a_1^2 \\
+ 2a_0^2 a_2 + a_1 a_3 a_4 + a_1^2 a_2 + 2a_2 a_3^2 + 2a_2^2 a_4 + a_3^2 a_4 \\
+ a_2 a_4^2 + 2a_0 a_4^2) x_1 x_2 + [a_3^3 - a_3 a_4^2 + (T)] x_2^2.
\end{aligned} \tag{52}
$$

Note that $2N + (a_0 + a_2) P + D$, and not β, is the leading coefficient of $[T]$. We find, in addition,

$$
\begin{aligned}
[f_4^2] = [a_0^2 + (f_4^2)]x_1^2 + 2(a_0 a_1 + a_0 a_3 + a_1 a_2 + a_1 a_4 + a_2 a_3 \\
+ a_3 a_4) x_1 x_2 + [(f_4^2) + a_4^2] x_2^2.
\end{aligned} \tag{53}
$$

$$
[Ef_4] = [a_0^3 + (f_4)^3]x_1^2 + (a_1^3 + a_3^3) x_1 x_2 + [(f_4)^3 + a_4^3] x_2^2, \tag{54}
$$

$$
\begin{aligned}
[f_4[\bar{f}_4]] = [a_0 P + 2D + 2\tau] x_1^2 + (2a_0^2 a_2 + 2a_0 a_1^2 + 2a_1^2 a_4 \\
+ 2a_0 a_2^2 + a_2^2 a_4 + a_0 a_3^2 + a_2 a_4^2 \\
+ a_3^2 a_4) x_1 x_2 + [2D + 2\tau + a_1 a_4^2 + a_2 a_3 a_4 + 2a_3 a_4^2] x_2^2.
\end{aligned} \tag{55}
$$

The number of these concomitants which can be constructed by thus superimposing these symbols, one upon another, is infinite. We shall note particularly those given explicitly above, and

$$
[f_4^2[\bar{f}_4]],\ [Ef_4[\bar{f}_4]],\ [f_4 \cdot T],\ \cdots,
$$

whose respective seminvariant leaders are

$$
a_0^2 P + I_1,\ a_0^3 P + I_2,\ - a_0 \beta + I_3,\ \cdots,
$$

where I_1, I_2, I_3 are invariants.

It is apparent from the foregoing theory that we have brought a complete system of covariants into view, so to speak, for a covariant led by the arbitrary seminvariant (30) can be derived, part by part, as in the case of covariants (49) to (55), and the parts added, the sum furnishing a reduction similar to that in (37). The formidable calculations required, I have hopes of completing and making the subject of another paper. The main portions of the general problem which remain to be solved are; (1) the determination of the two syzygies described in § 5, needed for the reduction of formula (30) to finite form as regards powers of β and H; (2) determination of all covariants led by invariants which are polynomials in the invariants (30_1); and (3), calculation, from bracket symbols, of the covariants needed to reduce the covariants of the various orders led by the arbitrary seminvariant (30). These can be formed from products of powers of the covariants (49) to (55), combined with f_4. Meanwhile I desire to prove the following theorem, which is of individual importance and is a necessary preliminary to the solution of the complete problem described above.

THEOREM. *A complete set of quadratic covariants led by seminvariants of the general form*

$$
(56) \qquad \sum_{i=1}^{2} [\,\theta_{0i} + \theta_{1i}\,a_0 + \theta_{2i}\,a_0^2 + \theta_{3i}\,a_0^3 + (\Delta_{0i} + \Delta_{1i}\,a_0 + \Delta_{2i}\,a_0^2 + \Delta_{3i}\,a_0^3)\,P\,]\,\beta^{i-1}
$$

is composed of

$$
(57) \qquad \begin{gathered}
[f_4],\ [f_4^2],\ [Ef_4],\ [\bar{f}_4],\ [f_4[\bar{f}_4]],\ [f_4^2[\bar{f}_4]],\ [Ef_4[\bar{f}_4]],\\
[T],\ [f_4\,T],\ [f_4^2\,T],\ [Ef_4\,T],\ [T[\bar{f}_4]],\ [f_4\,T[\bar{f}_4]],\\
[f_4^2\,T[\bar{f}_4]],\ [Ef_4\,T[\bar{f}_4]].
\end{gathered}
$$

The formula (56) can be written

$$
(58) \qquad \begin{aligned}
&\{\theta_{01} - \theta_{11}(f_4) - \theta_{21}(f_4^2) - \theta_{31}(Ef_4) - \Delta_{11}(2D + 2\tau) - \Delta_{21}(f_4^2[\bar{f}_4])\\
&\quad - \Delta_{31}(Ef_4[\bar{f}_4]) + \theta_{02}(T) + \theta_{12}(f_4\,T) + \theta_{22}(f_4^2\,T) + \theta_{23}(Ef_4\,T)\\
&\quad + \Delta_{02}(T[\bar{f}_4]) + \Delta_{12}(f_4\,T[\bar{f}_4]) + \Delta_{22}(f_4^2\,T[\bar{f}_4])\\
&\quad + \Delta_{23}(Ef_4\,T[\bar{f}_4])\} + \theta_{11}(a_0 + (f_4)) + \theta_{21}(a_0^2 + (f_4^2))\\
&\quad + \theta_{31}(a_0^3 + (Ef_4)) + \Delta_{01}\,P + \Delta_{11}(a_0\,P + 2D + 2\tau)\\
&\quad + \Delta_{21}(a_0^2\,P + (f_4^2[\bar{f}_4])) + \Delta_{31}(a_0^3\,P + (Ef_4[\bar{f}_4]))\\
&\quad + \theta_{02}(+\beta - (T)) + \theta_{12}[\,a_0\,\beta - (f_4\,T)\,] + \theta_{22}[\,a_0^2\,\beta - (f_4^2\,T)\,]\\
&\quad + \theta_{32}[\,a_0^3\,\beta - (Ef_4\,T)\,] + \Delta_{02}[\,\beta P - (T[\bar{f}_4])\,] + \Delta_{12}[\,a_0\,\beta P\\
&\quad - (f_4\,T[\bar{f}_4])\,] + \Delta_{22}[\,a_0^2\,\beta P - (f_4^2\,T[\bar{f}_4])\,]\\
&\quad + \Delta_{32}[\,a_0^3\,\beta P - (Ef_4\,T[\bar{f}_4])\,].
\end{aligned}
$$

Since there are no quadratic covariants led by pure invariants the brace expression in (58) is congruent to zero. The coefficients of the indeterminate invariants θ_{ij}, Δ_{ij} are, in order, respectively, the leading coefficients of the covariants (57), and, by an argument similar to that given in connection with (37), the theorem is therefore proved. Note that the order 2 is the least possible order for a covariant modulo 3 of any form of even order; thus we see that every covariant of this order is rationally and integrally expressible in terms of the fifteen covariants (57) and the six invariants (30_1).

8. ADDENDA CONCERNING INVARIANTS OF VELOCITY AND ACCELERATION IN THE THEORY OF RELATIVITY OF MOTION

The applications of invariant theory noted in this paragraph are grounded upon an algorithm similar to that given in § 1 and may be treated, appropriately perhaps, in a paper which gives mention of the general theory and algorithm. For this will not only avoid repetitions, but it will emphasize the relevancy of this research also.

Observations for measurements of velocity, acceleration, time, or distance, made by two observers situated upon two different worlds or systems of reference in space, which are in relative motion with a constant velocity v, give sets of values of these quantities which are connected by linear equations. These equations furnish a means of transferring measurements from one system to another, and three different varieties of transformations of this description are to be considered.

Suppose that the line connecting the two systems S, S' is taken as the axis of abscissas for both systems and that the y- and z-axes upon one system S are respectively parallel to the y'- and z'-axes upon S'. Then, if t and t' are the time variables and c is the velocity of light we can employ the Einstein transformations

$$(59)\quad \Delta: t = \mu(c^2 t' + vx')/c, \qquad x = \mu(vt' + x')c, \qquad y = y', \qquad z = z',$$
$$(\mu = 1/\sqrt{c^2 - v^2}),$$

a known transformation upon the velocities

$$\Omega: \quad \dot{x}' = (\dot{x} - v) \bigg/ \left(1 - \frac{v\dot{x}}{c^2}\right),$$
$$\dot{y}' = \dot{y}k^{-1} \bigg/ \left(1 - \frac{v\dot{x}}{c^2}\right),$$
$$\dot{z}' = \dot{z}k^{-1} \bigg/ \left(1 - \frac{v\dot{x}}{c^2}\right) \qquad (k = c\mu),$$

where $\dot{x} = \partial x/\partial t$, $\dot{x}' = \partial x'/\partial t'$, etc., and one upon the accelerations given by*

$$A: \quad \ddot{x} = c^{-3}\mu^3(c^2 - v\dot{x})^3\ddot{x}',$$
$$(60)\qquad \ddot{y} = c^{-2}\mu^2(c^2 - v\dot{x})^2(\ddot{y}' - v\mu c^{-1}\dot{y}\ddot{x}'),$$
$$\ddot{z} = c^{-2}\mu^2(c^2 - v\dot{x})^2(\ddot{z}' - v\mu c^{-1}\dot{z}\ddot{x}'),$$

in which $\ddot{x} = \partial^2 x/\partial t^2$.

We can reduce the equations Ω to homogeneous form by introducing two fictitious time variables τ, τ' connected with t and t' by functional equations

$$t = f(\tau), \qquad t' = f_1(\tau').$$

For then

$$\frac{\partial x}{\partial t} = \frac{\partial x}{\partial \tau}\frac{\partial \tau}{\partial t}, \qquad \frac{\partial x'}{\partial t'} = \frac{\partial x'}{\partial \tau'}\frac{\partial \tau'}{\partial t'}, \qquad \cdots,$$

* Compare V, p. 48.

and, by taking ρ to be a proportionality factor, we can get the following form,

$$
\text{(61)} \qquad \begin{gathered} V : \dot{t} = \rho\,(c^2 \dot{t}' + v\dot{x}')/c^2, \qquad \dot{x} = \rho\,(v\dot{t}' + \dot{x}'), \\ \dot{y} = (\mu c)^{-1}\,\rho\dot{y}', \qquad \dot{z} = (\mu c)^{-1}\,\rho\dot{z}', \\ [\,\dot{x} = \partial x/\partial\tau, \qquad \dot{x}' = \partial x'/\partial\tau', \qquad \text{etc.}; \qquad dt'/dt = \mu\,(c^2 - v\partial x/\partial t)/c\,]. \end{gathered}
$$

*Invariants of V appertaining to an irrational domain.** The components $\dot{y}'$, $\dot{z}'$, are relative covariants. For this reason we are able to treat V as if it were a binary transformation V_1 affecting $\dot{t}$ and $\dot{x}$. Then V_1 has two poles which are the roots of the linear forms

$$
\text{(62)} \qquad \lambda_{\pm 1} = c\dot{t} \pm \dot{x}.
$$

Thus $\lambda_{\pm 1}$ are universal covariants of V_1 for an arbitrary velocity v, the invariant relations being

$$
\lambda'_{\pm 1} = \left(\frac{c - v}{c + v}\right)^{\pm 1/2} \lambda_{\pm 1}.
$$

Consider a binary form f in the variables $\dot{t}$, $\dot{x}$, whose coefficients a_i are homogeneous polynomials in the velocities $\dot{y}$, $\dot{z}$, with coefficients arbitrary functions of the quantities left unaltered by V, these being of the same order in $\dot{y}$, $\dot{z}$,

$$
f = a_0\,\dot{t}^m + m a_1\,\dot{t}^{m-1}\,\dot{x} + \cdots + a_m\,\dot{x}^m.
$$

When f is expanded in terms of $\lambda_{\pm 1}$ as argument forms by means of the substitutions

$$
\dot{t} = (\lambda_{+1} + \lambda_{-1})/2c, \qquad \dot{x} = c\,(\lambda_{+1} - \lambda_{-1})/2c,
$$

there results a form f' whose coefficients are invariants of a non-absolute type, linear in $a_0, \cdots, a_m$;

$$
\text{(63)} \qquad f' = \sum_{i=0}^{m} \binom{m}{i} \psi_{m-2i}\,\lambda_{+1}^{m-i}\,\lambda_{-1}^{i}/(2c)^m.
$$

The explicit form of the typical invariant is

$$
\text{(64)} \quad \binom{m}{i}\psi_{m-2i} = \sum_{j=0}^{m}\sum_{t=0}^{i}(-1)^t \binom{m}{j}\binom{m-j}{i-t}\binom{j}{t} a_j\,c^j \qquad (i = 0, \cdots, m),
$$

the invariant relations being

$$
\psi'_{m-2i} = \left(\frac{c + v}{c - v}\right)^{(m-2i)/2} \psi_{m-2i}.
$$

A general theory of the invariants derived in this way,† which the author has developed, shows that systems of absolute concomitants of f can be

* Cf. these Transactions, vol. 18 (1917), p. 443, and vol. 20 (1919), p. 203.

† Annals of Mathematics, vol. 20 (1918), p. 125.

derived by forming properly chosen products of powers of the invariants ψ_{m-2i}; that is, systems which are absolute except for a power of the proportionality factor ρ occurring in the invariant relations.

Concomitants belonging to a rational domain. The general form of such a product is

$$I_r = \phi_{m-2i_1}^{r_{i_1}} \phi_{m-2i_2}^{r_{i_2}} \cdots \lambda_{+1}^{s_1} \lambda_{-1}^{s_2},$$

and the following product is conjugate to I_r

$$I_{-r} = \phi_{-(m-2i_1)}^{r_{i_1}} \phi_{-(m-2i_2)}^{r_{i_2}} \cdots \lambda_{+1}^{s_2} \lambda_{-1}^{s_1}.$$

According to the theory $I_r \pm I_{-r}$ is an absolute concomitant provided the positive integral exponents satisfy the linear diophantine equation

$$\begin{aligned} s_2 + r_0 m + r_1(m-2) + r_2(m-4) + \cdots \\ = \cdots r_{m-2}(m-4) + r_{m-1}(m-2) + r_m m + s_1 \end{aligned} \tag{65}$$

and a complete system is furnished by the finite complete set of irreducible solutions of (65).

In the case of a binary quartic ($m = 4$) I have found that the number of irreducible solutions is 12, as shown in the table below.

	r_0	r_1	r_2	r_3	r_4	s_1	s_2
ν	0	1	0	1	0	0	0
ξ	1	0	0	0	1	0	0
o	0	0	0	0	0	1	1
π	0	0	1	0	0	0	0
$\rho_{\pm 1}$	0	1	0	0	0	2	0
	0	0	0	1	0	0	2
$\sigma_{\pm 1}$	1	0	0	2	0	0	0
	0	2	0	0	1	0	0
$\tau_{\pm 1}$	1	0	0	1	0	2	0
	0	1	0	0	1	0	2
$v_{\pm 1}$	1	0	0	0	0	4	0
	0	0	0	0	1	0	4

The fundamental system is

$$\begin{aligned} &\nu = \psi_2 \psi_{-2}, \quad \xi = \psi_4 \psi_{-4}, \quad o = c^2\left(\frac{\partial t}{\partial \tau}\right)^2 - \left(\frac{\partial x}{\partial \tau}\right)^2 = \lambda_{+1}\lambda_{-1}, \\ &\pi = \psi_0, \quad \rho_{\pm 1} = \psi_2 \lambda_{+1}^2 \pm \psi_{-2}\lambda_{-1}^2, \\ &\sigma_{\pm 1} = \psi_4 \psi_{-2}^2 \pm \psi_{-4}\psi_2^2, \quad \tau_{\pm 1} = \psi_4 \psi_{-2} \lambda_{+1}^2 \pm \psi_{-4}\psi_2 \lambda_{-1}^2, \\ &v_{\pm 1} = \psi_4 \lambda_{+1}^4 \pm \psi_{-4}\lambda_{-1}^4, \end{aligned} \tag{66}$$

and is given explicitly by the following functions:

$$\psi_0 = a_0 - 2a_2 c^2 + a_4 c^4,$$

$$\psi_{\pm 2} = a_0 \pm 2a_1 c \mp 2a_3 c^3 - a_4 c^4,$$

$$\psi_{\pm 4} = a_0 \pm 4a_1 c + 6a_2 c^2 \pm 4a_3 c^3 + a_4 c^4.$$

Invariants of acceleration

Two of the three poles of the transformations (60) are coincident. For this reason we cannot derive three linearly independent linear universal covariants of this transformation, but only two, and thus a ternary form in the component accelerations $\ddot{x}$, $\ddot{y}$, $\ddot{z}$, cannot be expanded covariantively in the form analogous to f' above, and no analogue of the covariant theory of velocities exists, in the ternary realm. The two linear universal covariants are

$$\ddot{x}, \; \dot{z}\ddot{y} - \dot{y}\ddot{z},$$

and the second one can be employed to reduce the transformation (60) to a canonical form in which both quantities $\ddot{x}_1$, $\ddot{y}_1$ are relative covariants. The invariant problem then becomes binary.

Let

$$\dot{z}\ddot{y} - \dot{y}\ddot{z} = \ddot{y}_1,$$

this notation being adopted for uniformity. Then $\ddot{y}_1 = \mu^2 \delta^2 \ddot{y}_1'$, where $c\delta = c^2 - v\dot{x}$. Accordingly the transformation (60) is equivalent to the following:

$$\theta : \begin{cases} \ddot{x} = \mu^3 \delta^3 \ddot{x}', \\ \ddot{y}_1 = \mu^2 \delta^2 \ddot{y}_1', \\ \ddot{z} = \mu^2 \delta^2 (- v\mu c^{-1} \dot{z}\ddot{x}' + \ddot{z}'). \end{cases} \tag{67}$$

The substitutions on $\ddot{x}$, $\ddot{z}$, form a binary transformation θ_1 and the linear universal covariants of θ_1 are

$$\begin{aligned} f_{+1} &= 2\mu^2 \delta^2 [- \mu v c^{-1} \dot{z}\ddot{x} + (1 - \mu\delta)\ddot{z}] \equiv C\ddot{x} + D\ddot{z}, \\ f_{-1} &= - 2\mu^3 v c^{-1} \delta^2 \dot{z}\ddot{x} \equiv C\ddot{x}. \end{aligned} \tag{68}$$

Then, if $f_m = (a_0, \cdots, a_m \between \ddot{x}, \ddot{z})^m$ is a binary quantic with constant coefficients, the expansion of f_m in terms of f_{+1}, f_{-1} as arguments has for coefficients linear invariants of f_m under θ_1, the totality of $m + 1$ furnishing a complete system of the non-absolute type.

In a paper published in 1917 I gave the invariants of this type* for the general transformation

$$x_1 = \alpha_1 x_1' + \alpha_2 x_2',$$

$$x_2 = \beta_0 x_1' + \beta_1 x_2',$$

* These Transactions, vol. 18 (1917), p. 443.

and the orders $m = 1, 2, 3$. From these we get the invariants of θ_1 by the particularizing substitutions

$$\alpha_1 = \mu^3 \delta^3, \qquad \alpha_2 = 0, \qquad \beta_0 = -\mu^3 \delta^2 c^{-1} v\dot{z}, \qquad \beta_1 = \mu^2 \delta^2. \tag{69}$$

The sets for the orders $m = 1$, $m = 2$, for example are as follows:

$$m = 1$$

$$\begin{aligned} \phi_1' &= a_1, \\ \phi_{-1}' &= (1 - \mu\delta) a_0 + \mu v c^{-1} \dot{z} a_1. \end{aligned} \tag{70}$$

$$m = 2$$

$$\begin{aligned} \phi_2' &= a_2, \\ \phi_0' &= (1 - \mu\delta) a_1 + \mu v c^{-1} \dot{z} a_2, \\ \phi_{-2}' &= (1 - \mu\delta)^2 a_0 + 2\mu v c^{-1} \dot{z} (1 - \mu\delta) a_1 + \mu^2 v^2 c^{-2} \dot{z}^2 a_2. \end{aligned}$$

For these invariants of f_m under θ_1 we find the following general formula:

$$\phi_{m-2r}' = \sum_{s=0}^{r} \binom{r}{s} (1 - \mu\delta)^{r-s} (\mu v c^{-1} \dot{z})^s a_{m-r+s} \qquad (r = 0, \cdots, m). \tag{71}$$

We also give the corresponding invariant relations, viz.,

$$f_{+1}' = (\mu^2 \delta^2)^{-1} f_{+1}, \qquad f_{-1}' = (\mu^3 \delta^3)^{-1} f_{-1}, \tag{72}$$

$$(\phi_{m-2r}')' = (\mu \cdot \delta)^{2m+r} \phi_{m-2r}'. \tag{73}$$

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