CONCERNING SIMPLE CONTINUOUS CURVES

BY

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1. Introduction

Various definitions of simple continuous arcs and closed curves have been given.† The definitions of arcs usually contain the requirement that the point-set in question should be bounded. In attempting to prove that every interval $t$ of an open curve as defined in a recent paper‡ is a simple continuous arc, while I found it easy to prove that $t$ satisfies all the other requirements of Janiszewski’s definition (modified as indicated below) it was only by a rather lengthy and complicated argument that I succeeded in proving that it satisfies the requirement of boundedness. In Lennes’ definition the requirement of boundedness is superfluous.§ However I found it difficult to prove that $t$ satisfies a certain one of the other requirements of this definition, namely that the point-set in question should contain no proper connected subset that contains both $A$ and $B$. In the present paper I will give a definition|| of a simple continuous arc which stipulates neither that the set $M$ should be bounded nor that it should contain no proper connected subset containing both $A$ and $B$. I will show that, in a euclidean space of two dimensions, every point-set that satisfies this definition is an arc in the sense of Jordan. It is easy to prove¶ that every interval of an open curve satisfies this definition.

* Presented to the Society, October 26, 1918.


|| Definition 1 below.

¶ See proof of Theorem 3 below.
In connection with certain problems where the boundedness of the point-set in question is not presupposed, but where relatively more information is at hand concerning connectedness, it seems likely that Definition 1 may be more useful than that of Janiszewski. On the other hand Janiszewski’s definition (or a modification of it*) may be of more use in certain cases where one is concerned with sets that are known in advance to be bounded (or can easily be proved to be bounded) but concerning which less is known in advance with regard to connectedness.

In the latter part of § 2, a very simple characterization of a simple closed curve is given. It is defined merely as a closed connected and bounded point-set which is disconnected by the omission of any two of its points.†

In § 3 the problem of defining simple continuous arcs, closed and open curves and rays is approached from a different point of view. With the use of the notion of the boundary of a point-set M with respect to a point-set that contains M, conditions are given which a point-set must satisfy in order that it should be a simple continuous curve (that is to say one of the four types of curves mentioned above). This classification of the general notion simple continuous curve having been given, it is easy to so particularize it as to obtain a characterization of any given one of the four special types of simple continuous curves. A definition of an open curve from this point of view can be obtained from that of a closed curve by the mere substitution of the word “bounded” in place of the word “proper.”

DEFINITION 1. If A and B are two distinct points, a simple continuous arc from A to B is a closed, connected set of points M containing A and B such that (1) M — A and M — B are connected, (2) if P is any point of M

* Cf. Definition 2 below.
† In this connection see the article by Sierpinski referred to above. His definition does not require explicitly even that M itself should be connected. It requires boundedness however and also certain positive information concerning the relation of the endpoints A and B to the set M. In my Definition 2 the phrase “ except the points A and B ” is to be interpreted not as meaning that the points A and B do not fulfill the requirements indicated but merely as leaving the question open whether they do or do not fulfill these requirements. Sierpinski definitely stipulates that M is not the sum of two closed point-sets having only A, or only B, in common and each consisting of more than one point. If this stipulation were omitted his definition would not completely characterize a simple continuous arc and would indeed apply to some sets that are not connected, e.g., to a set composed of three distinct points.
‡ Lennes defines a simple closed curve as “ the set of points consisting of two continuous arcs, each connecting a pair of distinct points A and B and having no other point in common.” This definition presupposes a previous definition of a simple continuous arc. Janiszewski (loc. cit., p. 137) defines “ Une ligne simple fermée Γ ” as “ un continu qui peut être décomposé en deux continus ε, et ε, n’ayant en commun que deux points M et N arbitrairement donnés sur Γ.”
distinct from $A$ and from $B$ then $M - P$ is the sum of two mutually exclusive connected point-sets neither of which contains a limit point of the other one.

If for a point $O$ of a connected point-set $M$ the set $M - O$ is the sum of two mutually exclusive point-sets $M_1$ and $M_2$ neither of which contains a limit point of the other one then $M_1$ and $M_2$ will be called sects* (of $M$) from $O$, and $M$ will be said to be disconnected by the omission of $O$ and will be said to be separated by the omission of $O$ into the two sets $M_1$ and $M_2$. If $P$ is a point of $M$ distinct from $O$ then in case there is only one sect of $M$ from $O$ that contains $P$ that sect will be called the sect $OP$. If at the same time there is only one sect from $P$ that contains $O$ the set of all those points of $M$ that are common to the sects $OP$ and $PO$ will be called the segment $OP$ (of $M$) while the set of points consisting of all the points of the segment $OP$ together with its endpoints ($O$ and $P$) will be called the interval $OP$ (of $M$).

Theorem 1. In a Euclidean space of two dimensions every set of points $M$ that satisfies the requirements of Definition 1 is an arc in the sense of Jordan.

Proof. Suppose $M$ is a set of points satisfying all the requirements of Definition 1. If $O$ is any point of $M$ distinct from $A$ and from $B$ it is easy to see that there are only two sects of $M$ from $O$. One of these sects contains $A$ and the other one contains $B$. For suppose that one of them contains both $A$ and $B$. Let $OC$ denote the other one. Let $K$ denote the set of all points $X$ such that one sect from $X$ contains neither $A$ nor $B$ but has at least one point in common with $OC$. For every point $P$ of $K$ let $K_P$ denote that sect from $P$ which contains neither $A$ nor $B$ and let $K_P$ denote the other sect from $P$.

The points of $OC$ can† be arranged in a well-ordered sequence $\beta$. Let $P_1$ be the first point in the sequence $\beta$. Let $P_2$ be the first point of $\beta$ which lies in $K_{P_1}$. Let $P_3$ be the first point of $\beta$ which is common to $K_{P_1}$ and $K_{P_2}$. Let $P_4$ be the first point of $\beta$ which is common to $K_{P_1}$, $K_{P_2}$, and $K_{P_3}$. This process may be continued. It follows that the sequence $\beta$ contains a well-ordered subsequence $\alpha$ such that if $T$ is a subset of the elements of $\alpha$ there is an element of $\alpha$ which follows, in $\alpha$, all the elements of $T$ if, and only if, there exists a point which belongs to $K_P$ for every point $P$ of $T$ and, if there does exist such a point, then the first element of $\alpha$ that follows all the elements of $T$ is the first element of $\alpha$ which follows all the elements of $T$ is the first element of $\beta$ which belongs to $K_P$ for every point $P$ of $T$. Suppose that $X$ and $Y$ are distinct points of $\alpha$ and that $Y$ precedes $X$ in $\alpha$. Then $X$ is in $K_Y$ and therefore is not in $\overline{K_Y}$ and $Y$ hence $\overline{K_Y} + Y$ must

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* The term "sect" is used by Halsted with a somewhat different meaning. Cf. G. B. Halsted, Rational Geometry, Wiley and Sons, New York, 1904.

lie wholly in $K_X$ or wholly in $\overline{K}_X$ (otherwise $\overline{K}_Y + Y$ would not be connected and consequently $K_Y + Y + \overline{K}_Y$ would not be connected). But $\overline{K}_Y$ contains $A$ and $K_X$ does not. Hence $\overline{K}_Y + Y$ is a subset of $\overline{K}_X$. Therefore $K_X$ contains no point of $\overline{K}_Y + Y$. It follows that $K_X$ is a subset of $K_Y$. But of any two points in $\alpha$ one of them precedes the other one in $\alpha$. Hence if $X$ and $Y$ are two points of $\alpha$ either $K_X$ contains $K_Y$ or $K_Y$ contains $K_X$.

If $X$ and $Y$ are distinct points of $K$ and $K_X$ contains $K_Y$ then $K_Y$ does not contain $K_X$. For since $X + K_X$ is closed, $X \neq Y$ and $Y$ is a limit point of $K_Y$, therefore $Y$ is in $K_X$. But $Y$ is not in $K_Y$. Hence $K_Y$ does not contain $K_X$.

If, for two distinct points $X$ and $Y$ of the set $K$, the sect $K_X$ contains the sect $K_Y$ then $X$ will be said to precede $Y$ in $K$. In view of the results established above it is clear that if $X$ and $Y$ are two distinct points in $K$ then (1) if $X$ precedes $Y$, $Y$ does not precede $X$, (2) if $X$ precedes $Y$ and $Y$ precedes $Z$ then $X$ precedes $Z$, (3) if $X$ and $Y$ are both elements of $\alpha$ then either $X$ precedes $Y$ or $Y$ precedes $X$.

The ray $OC$ is unbounded. For suppose it is bounded. Then the family of all sets $K_X + X$ for all points $X$ of $\alpha$ is a family of closed, bounded point-sets such that of every two of them one contains the other one. It follows by a theorem established in a recent paper* that there exists at least one point $W$ which belongs to $K_X + X$ for every point $X$ of $\alpha$. Let $W_1$ denote the first such point $W$ in the sequence $\beta$. Then $W_1$ is an element of $\alpha$ that follows all the elements of $\alpha$. Thus the supposition that $OC$ is bounded leads to a contradiction.

Suppose that $X$ and $Y$ are two distinct points of $K$ that do not belong to $O + OC$. Since the connected set $O + OC$ contains a point of $K_X$ but does not contain $X$ therefore $O + OC$ is a subset of $K_X$. Similarly $O + OC$ is a subset of $K_Y$. Thus $K_X$ and $K_Y$ have at least one point in common. Suppose now that $K_X$ is not a subset of $K_Y$. Then since $X + K_X$ is connected and contains at least one point of $K_Y$, $K_X$ must contain $Y$. Hence $X + K_X$ does not contain $Y$. But $K_X + X$ is connected and contains the point $A$ in common with $K_Y$. Hence $K_X + X$ is a subset of $K_Y$ and therefore $K_Y + Y$ is a subset of $K_X$. Thus it is proved that if $X$ and $Y$ are two distinct points of $K - OC - O$ then either $X$ precedes $Y$ or $Y$ precedes $X$.

Let $H$ denote the set of points composed of $K$ together with the set of all points $[Y]$ such that $Y$ belongs to $K_X$ for some point $X$ of $K$.

Since $M$ is connected either $H$ contains a limit point of $M - H$ or $M - H$ contains a limit point of $H$. Suppose that $M - H$ contains a point $Z$ which

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is a limit point of $H$. Then $Z$ is the sequential limit point of some infinite sequence of distinct points $Z_1, Z_2, Z_3, \cdots$ belonging to $H$. It follows that there exists an infinite sequence of distinct points $X_1, X_2, X_3, \cdots$ belonging to $K - O - OC$ and an infinite sequence of distinct positive integers $n_1, n_2, n_3, \cdots$ such that, for every $m$, (1) $Z_{n_m}$ belongs to $K_{X_m}$ and (2) $X_{m+1}$ precedes $X_m$ in $K$. Let $K^*$ denote the point-set $X_1 + X_2 + X_3 + \cdots$. It is clear that (1) if $Y$ is a point of $H$ there exists a positive integer $m$ such that $K_{X_m}$ contains $Y$, (2) $M - H$ contains every limit point of $K^*$. Every limit point of $H$ which lies in $M - H$ is a limit point of $K^*$. Suppose $M - H$ contains two points $D$ and $E$ which are limit points of $H$. Then it can be shown that there exist in $K^*$ two sequences of points $D_1, D_2, D_3, \cdots$ and $E_1, E_2, E_3, \cdots$ such that (1) $D$ and $E$ are sequential limit points of $D_1, D_2, D_3, \cdots$ and of $E_1, E_2, E_3, \cdots$ respectively, (2) for every $i$, $E_i$ is preceded by $D_i$ and $D_i$ is preceded by $E_{i+1}$ in $K$. There exist five regions§ $R_1, R_2, R_3, R_4$, and $R_5$ such that (1) $R_3$ contains $D$ and $R_4$ contains $E$, (2) $R'_3$ is a subset of $R_2$, $R'_4$ is a subset of $R_1$, and $R'_1$, $R'_2$, and $R'_5$ have no point in common. There exists a positive integer $m$ such that for every positive integer $m$, the point $D_{n+m}$ is in $R_3$ and the point $E_{n+m}$ is in $R_4$. It is clear that the intervals $E_{n+1} D_{n+1}, E_{n+2} D_{n+2}, E_{n+3} D_{n+3}, \cdots$ of $M$ are all closed connected subsets of $K$ and that no two of them have a point in common. For every $m$ the interval $E_{n+m} D_{n+m}$ contains a closed connected subset $t_m$ that contains at least one point on the boundary of $R_1$ and at least one point on the boundary of $R_4$ but contains no point without $R_1$ or within $R_4$. No two point-sets belonging to the infinite sequence $t_1, t_2, t_3, \cdots$ have a point in common.† It follows that there exists (1) an infinite sequence of positive integers $n_1, n_2, n_3, \cdots$ such that, for every $j$, $n_{j+1} > n_j$ and (2) a closed connected set of points $t$ and a sequence of closed connected point-sets $k_{n_1}, k_{n_2}, k_{n_3}, \cdots$ such that, for every $j$, $k_{n_j}$ is a subset of $t$ and such that (a) each of the point-sets

§ In my paper On the foundations of plane analysis situs, loc. cit. (this paper will be referred to as F.A.), the notion point is undefined and region is also undefined except in so far as it is understood that every region is some sort of collection of points. A considerable part of the present argument holds good not only for a euclidean space of two dimensions but for any space satisfying Axioms 1' and 4 of F.A. The whole of this argument holds good for every space satisfying the set of Axioms $\Sigma_1$ of F.A. Every such space is, however, in one to one continuous correspondence with an ordinary euclidean space of two dimensions; cf. my paper Concerning a set of postulates for plane analysis situs, these Transactions, vol. 20 (1919), pp. 169–178. If one desires to read the present paper without special reference to any particular system of axioms he may think of the word region as applying to any bounded connected domain in a euclidean space of two dimensions.

† Up to this point the present proof holds good for every space satisfying Axioms 1' and 4 of F.A. It therefore holds good for all euclidean spaces (of however many dimensions) as well as for many other spaces including certain spaces that are neither metrical, descriptive, nor separable (cf. F.A., p. 131). If the statement in the next sentence (which can be easily established for euclidean space of two dimensions) can be proved to hold true in euclidean space of any number of dimensions then the present proof holds good for any such space.
sets \( t, k_1, k_2, k_3, \ldots \) is a subset of \( R' \setminus R_4 \) but contains at least one point on the boundary of \( R_1 \) and at least one point on the boundary of \( R_4 \). \( \)) if \( P_n, P_{n_2}, \ldots \) is a sequence of points such that, for every \( j \), \( P_n \) belongs to \( k_j \), then \( t \) contains every limit point of the point-set \( P_n + P_{n_2} + P_{n_3} + \cdots \). \( \)) if \( n_1, n_2, n_3, \ldots \) is an infinite sequence of distinct integers belonging to the set \( n_1, n_2, n_3, \ldots \) and \( P \) is a point of \( t \) there exists an infinite sequence of points \( P_n, P_{n_2}, P_{n_3}, \ldots \) such that, for every \( j \), \( P_{n_j} \) belongs to \( k_{n_j} \) and such that \( P \) is the sequential limit point of the sequence \( P_n, P_{n_2}, P_{n_3}, \ldots \). \( \)) if \( t \) is a point of \( t \), \( T \) is the sequential limit point of some sequence of points \( T_{n_1}, T_{n_2}, T_{n_3}, \ldots \) such that, for every \( j \), \( T_{n_j} \) belongs to \( k_{n_j} \). Of the two sectors of \( M \) from \( T \) one must contain an infinite subsequence \( T_{n_{j_1}}, T_{n_{j_2}}, T_{n_{j_3}}, \ldots \) of the sequence \( T_{n_1}, T_{n_2}, T_{n_3}, \ldots \). For every positive integer \( m \) the connected point-set \( k_{n_{j_1}}, k_{n_{j_2}}, k_{n_{j_3}}, \ldots \) contains \( T_{n_{j_1}} \) but does not contain \( T \). It follows that that sector from \( T \) which contains the points \( T_{n_{j_1}}, T_{n_{j_2}}, T_{n_{j_3}}, \ldots \) contains also the point-sets \( k_{n_{j_1}}, k_{n_{j_2}}, k_{n_{j_3}}, \ldots \). But every point of \( t \) is a limit point of \( k_{n_{j_1}} + k_{n_{j_2}} + \cdots \). Hence that sector contains every point of \( t - T \). For every point \( T \) of \( t \) let \( \overline{R_T} \) denote that sector from \( T \) which contains \( t - T \) and let \( R_T \) denote the other sector from \( T \). If \( T_1 \) and \( T_2 \) are two distinct points of \( t \), \( \overline{R}_{T_1} \) contains the point \( T_2 \) of the connected point-set \( R_{T_2} + T_2 \) and \( T_1 \) does not belong to \( R_{T_1} + T_2 \). It follows that \( R_{T_1} \) is a subset of \( \overline{R}_{T_1} \). Hence \( R_{T_1} \) and \( R_{T_2} \) have no point in common. Hence there do not exist more than two points \( X \) belonging to \( t \) such that \( R_X \) contains \( A \) or \( B \). Let \( t_0 \) denote \( t, t - X_1, \) or \( t - (X_1 + X_2) \) according as there are no such points \( X \) or there is only one such point \( X_1 \) or there are two such points \( X_1 \) and \( X_2 \). For each point \( T \) of \( t_0 \) the sector \( R_T \) is unbounded (cf. above) and contains a point \( T \) in \( R_T^* \). Hence it contains at least one point \( P_T \) on the boundary of \( R_T \). Consider the set \( L \) of all \( P_T \)'s for all points \( T \) of \( t_0 \). There is a one to one correspondence between the point-sets \( L \) and \( t_0 \). It follows that \( L \) is uncountable. But no point of any sector \( R_{T_1} \) is a limit point of \( M - R_{T_1} \). Hence no point of \( L \) is a limit point of \( L \). But every uncountable set of points \( \alpha \) contains at least one point which is a limit point of \( \alpha \). Thus the supposition that \( M - H \) contains more than one limit point of \( H \) has led to a contradiction.

It is clear that \( H \) cannot contain more than one limit point of \( M - H \). For no point of \( OC \) is a limit point of \( M - OC \) and if \( X \) and \( Y \) are two points of \( H \) not belonging to \( OC \) there exist two points \( \overline{X} \) and \( \overline{Y} \) belonging to \( K \) such that \( \overline{X} + K_{\overline{X}} \) contains \( X \) and \( K_{\overline{Y}} + \overline{Y} \) contains \( Y \). But either \( K_{\overline{X}} \) contains \( K_{\overline{Y}} + \overline{Y} \) or \( K_{\overline{X}} \) contains \( \overline{K}_{\overline{X}} + \overline{X} \). In the first case \( Y \) is not a limit point of \( \overline{K}_{\overline{X}} \) and therefore is not a limit point of \( M - H \) which is a subset of \( \overline{K}_{\overline{X}} \). In the second case \( X \) is not a limit point of \( M - H \).

It follows that there exists one and only one point \( O \) which belongs to one of the sets \( H \) and \( M - H \) and is a limit point of the other one.
Suppose that $O$ belongs to $M - H$. Neither of the sets $H$ and $M - (H + O)$ contains a limit point of the other one. But $K$ is connected and each sect from $O$ is connected. It follows that $K$ is one of the sects of $M$ from $O$. Thus the supposition that $O$ belongs to $M - H$ leads to a contradiction. Hence $O$ belongs to $H$. Thus for every point $P$ of $M$ such that one sect from $P$ contains neither $A$ nor $B$ there exists a point $O_P$ such that (1) the sect $O_P P$ contains neither $A$ nor $B$, (2) the sect $O_P P$ is not a subset of any other sect that contains neither $A$ nor $B$. Let $\overline{H}$ denote the set of all points $P$ of the segment $AB$ such that one sect from $P$ contains neither $A$ nor $B$. Let $\overline{N}_1$ denote the set of all points $O_P$ for all points $P$ of $\overline{H}$. For each point $X$ of $\overline{H}$ let $M_X$ denote that sect from $X$ which contains neither $A$ nor $B$. Let $N_2$ denote the set of all those points of $M$ that belong neither to $N_1$ nor to any $M_X$ for any point $X$ of $N_1$. Let $N$ denote the set $N_1 + N_2$. Every point of $N_1$ will be called an improper point and every point of $N_2$ will be called a proper point. Either $N_2$ contains a limit point of $\overline{H}$ or $H$ contains a limit point of $N_2$. Suppose first that $N_2$ contains a point $O$ which is a limit point of $\overline{H}$. Then there exists a sequence of distinct points $X_1, X_2, X_3, \ldots$ belonging to $N_1$ and a sequence of points $P_1, P_2, P_3, \ldots$ such that $O$ is the sequential limit point of the sequence $P_1, P_2, P_3, \ldots$ and such that, for every $n$, $P_n$ either coincides with $X_n$ or belongs to $M_{X_n}$. There exist about $O$ two regions $\overline{R}_1$ and $\overline{R}_2$ such that $\overline{R}_2$ is a subset of $\overline{R}_1$. There exists a positive integer $n$ such that the points $P_n, P_{n+1}, P_{n+2}, \ldots$ are all in $\overline{R}_2$. But each of the point-sets $P_n + M_{P_n}, P_{n+1} + M_{P_{n+1}}, P_{n+2} + M_{P_{n+2}}, \ldots$ is closed, connected, and unbounded. It follows that for each $m$, $M_{P_{\overline{t}_m}}$ contains a closed, connected subset $\overline{t}_m$ which contains at least one point on the boundary of $\overline{R}_1$ and at least one point on the boundary of $\overline{R}_2$ and every point of which is either on the boundary of $\overline{R}_1$ or of $\overline{R}_2$ or in the domain $\overline{R}_1 - \overline{R}_2$. No two of the point-sets $\overline{t}_1, \overline{t}_2, \overline{t}_3, \ldots$ have a point in common. That this leads to a contradiction follows by an argument analogous to (or identical with) that used in a similar connection above.

Suppose secondly that $\overline{H}$ contains a point $O$ which is a limit point of $N_2$. Clearly $O$ must belong to $N_1$. If $X$ is a point of $N_2$ distinct from $A$ and from $B$ and $Y$ is a point belonging either to $N_1$ or to $N_2$, $X$ will be said to precede $Y$ or to follow $Y$ according as $Y$ belongs to the sect $XB$ or to the sect $XA$. It may be easily proved that there exists in $N_2$ a sequence of distinct points $X_1, X_2, X_3, \ldots$, all distinct from $A$ and from $B$, such that $O$ is a sequential limit point of this sequence and such that either (1) for every $n$, $X_n$ precedes $O$ and $X_{n+1}$ or (2) for every $n$, $X_n$ follows $O$ and $X_{n+1}$. Suppose that, for every $n$, $X_n$ precedes $O$ and $X_{n+1}$. Each of the intervals $X_1 X_2, X_2 X_3, \ldots$ is closed and connected, no two of them contain in common any point other than a common endpoint and no one of them contains the point $O$. That $O$ is
the only limit point of \( X_1 X_2 + X_2 X_3 + \cdots \) that does not belong to \( X_1 X_2 + X_2 X_3 + \cdots \) may be proved by an argument similar to that used above in the proof that \( M - H \) does not contain more than one limit point of \( H \).

Now sect

\[ OA' = M - M_0 - O = (AX_1 + X_1 X_2 + X_2 X_3 + \cdots) + (B + V) \]

where \( V \) is the point-set \( M - (B + M_0 + O + AX_1 + X_1 X_2 + X_2 X_3 + X_3 X_4 + \cdots) \). Neither of the two sets \((AX_1 + X_1 X_2 + X_2 X_3 + \cdots)\) and \((B + V)\) contains a limit point of the other one. Hence the sect \( OA' \) is not connected. This involves a contradiction. Hence it is not true that, for every \( n \), \( X_n \) precedes \( O \) and \( X_{n+1} \). In an entirely similar way it may be proved that \( X_n \) cannot follow \( O \) and \( X_{n+1} \) for every \( n \). Thus the supposition that \( H \) contains a limit point of \( N_2 \) has led to a contradiction.

It follows that the set \( H \) does not exist. Hence if \( P \) is any point of \( M - (A + B) \), \( M - P \) is the sum of two sects of which one contains \( A \) and the other contains \( B \). It follows that the two sects \( PA \) and \( PB \) have no point in common and neither of them contains a limit point of the other one. Suppose now that \( M \) is a proper subset of \( M \) that contains both \( A \) and \( B \). Then \( M \) contains a point \( P \) that does not belong to \( M \). Let \( M_1 \) denote the set of all points common to \( M \) and the sect \( PA \) and let \( M_2 \) denote the set of all points common to \( M \) and the sect \( PB \). Neither of the sets \( M_1 \) and \( M_2 \) contains a limit point of the other one. But \( M = M_1 + M_2 \). Hence \( M \) is not connected. Thus \( M \) contains no proper connected subset that contains both \( A \) and \( B \). It follows that \( M \) satisfies all the requirements of Lennes' definition of an arc with the exception of the requirement that it should be bounded. That it also satisfies the latter requirement follows from the theorem of Hallett referred to above. The truth of Theorem 1 is therefore established.

**Definition 2.**† If \( A \) and \( B \) are two distinct points a simple continuous arc from \( A \) to \( B \) is a closed, connected, and bounded point-set containing \( A \) and \( B \) which is disconnected by the omission of any one of its points which is distinct from \( A \) and from \( B \).

**Theorem 2.** In a space satisfying Axioms\(^*\) 1' and 4 of F.A. every point-
set $M$ that satisfies the requirements of Definition 2 satisfies also those of Lennes' definition.

Proof. Let $M$ be a set of points satisfying the requirements of Definition 2. Certain portions of the proof of Theorem 1 clearly apply here. In particular the same argument that was used there applies here to show that if $P$ is any point of $M$ distinct from $A$ and from $B$ and $M - P$ is the sum of two subsets neither of which contains a limit point of the other one then if one of these subsets contains neither $A$ nor $B$ it must be unbounded. But here $M$ itself is bounded. Therefore, for every point $P$ of $M - (A + B)$, $M - P$ is* the sum of two mutually separated† point-sets $M_A$ and $M_B$ such that $M_A$ contains $A$ and $M_B$ contains $B$. It follows‡ that $M$ contains no proper connected subset containing both $A$ and $B$. The truth of Theorem 2 is therefore established.

Definition 3. An open curve is a closed and connected point-set which is separated into two connected subsets by the omission of any one of its points.

If $P$ is a point of an open curve $M$ the point-set obtained by adding $P$ to either of the two sects into which $M$ is separated by the omission of $P$ is called a ray. The two rays of $M$ so determined by a point $P$ are said to start from $P$. If $A$ is a point of $M$ distinct from $P$ that ray of $M$ which starts from $P$ and contains $A$ will be called the ray $PA$.

Theorem 3. In euclidean space of two dimensions, if $A$ and $B$ are two points of the open curve $M$ the interval $AB$ of $M$ is a simple continuous arc from $A$ to $B$.§

Proof. It is clear that the interval $AB$ is closed. It is connected. For suppose it is not. Then it is the sum of two mutually separated point-sets. Let $M_A$ denote that one of these sets which contains $A$ and let $M_A$ denote the other one. Let $AC$ denote that ray from $A$ which does not contain $B$ and let $BD$ denote that ray from $B$ which does not contain $A$. If $M_A$ contains $B$ then neither of the complementary sets $AC + M_A$ and $M_A + BD$ contains a limit point of the other one. If $M_A$ does not contain $B$ then neither of the complementary sets $AC + BD + M_B$ and $M_B$ contains a limit point of the other one. Thus in either case $M$ is not connected. Thus the supposition that $AB$ is not connected has led to a contradiction.

* Cf. an argument given by Sierpinski, loc. cit. pp. 137–140. His argument assumes separability while the proof given here holds good in every space satisfying Axioms 1' and 4. Such spaces are of course not necessarily separable.

† Two point-sets are said to be mutually separated if neither contains a point or a limit point of the other one.

‡ See latter part of proof of Theorem 1.

§ The definition of an open curve given on page 159 of F.A. is (aside from phraseology) equivalent to Definition 3. Theorem 49 is however not fully proved in F.A. In particular it is there assumed without proof that $t$ contains no proper connected subset that contains both $A$ and $B$. 

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Suppose that $O$ is any point of the interval $AB$ distinct from $A$ and from $B$. It will be shown that $AB - O$ is the sum of two connected point-sets neither of which contains a limit point of the other one. Suppose that one of the sects of $M$ from $O$ contains both $A$ and $B$. Let $OX$ denote the other sect of $M$ from $O$. Let $N$ denote the set of all those points which are common to the sect $OA$ and the interval $AB$. The point-set $N$ contains $A$ and $B$. Suppose it is not connected. Then it is the sum of two sets neither of which contains a limit point of the other one. Let $N_a$ denote one of these sets which contains $A$ and let $N_b$ denote the other one. If $N_a$ contains $B$ then the sect $OA$ is the sum of the two mutually separated sets $AC + BD + N_a$ and $N_b$. If $N_a$ does not contain $B$ then the sect $OA$ is the sum of the two mutually separated sets $AC + N_a$ and $BD + N_b$. In either case the sect $OA$ is not connected. Thus the supposition that one sect of $M$ from $O$ contains both $A$ and $B$ leads to a contradiction. It follows that the sects $OA$ and $OB$ have no point in common. Hence neither of the point-sets $OA - O$ and $OB - O$ contains a point or a limit point of the other one. It is easy to see that these point-sets are connected and that their sum is $AB - 0$. Thus if $O$ is any point of the interval $AB$ distinct from $A$ and from $B$ then $AB - O$ is the sum of two connected point-sets neither of which contains a point or a limit point of the other one. Thus the interval $AB$ satisfies all the requirements for an arc as given in Definition 1.

For further results concerning open curves see F.A., loc. cit., and a paper by J. R. Kline.*

**Definition 4.** A simple closed curve is a closed, connected, and bounded point-set which is disconnected by the omission of any two of its points.

**Theorem 4.** In a space satisfying Axioms 1' and 4 of F.A., a set of points $M$ satisfying the requirements of Definition 4 is the sum of two simple continuous arcs that have only their endpoints in common.

**Proof.** Suppose the set of points $M$ satisfies the requirements of Definition 4. Let $A$ and $B$ denote two distinct points of $M$. By hypothesis the set $M - (A + B)$ is the sum of two mutually separated point-sets $M_1$ and $M_2$. I will show that $M_1 + A + B$ and $M_2 + A + B$ are simple continuous arcs from $A$ to $B$. That $M_1 + A + B$ is closed is evident.

It is also connected. For suppose it is not. Then it is the sum of two closed, mutually exclusive point-sets $N$ and $K$. If $N$ should contain both $A$ and $B$ then $M$ would be the sum of the two separated sets $N + M_2$ and $K$ which is contrary to hypothesis. Similarly $K$ cannot contain both $A$ and $B$. It follows that one of the sets $N$ and $K$ contains $A$ and the other contains $B$. Suppose $K$ contains $A$. The set $M_2 + A + B$ must be connected. For if

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*J. R. Kline, The converse of the theorem concerning the division of a plane by an open curve, these Transactions, vol. 18 (1917), pp. 177–184.
it were the sum of two mutually separated sets $\bar{N}$ and $\bar{K}$ where $\bar{K}$ contains both $A$ and $B$ then $M$ would be the sum of the two mutually separated sets $\bar{K} + M_1$ and $\bar{N}$, while if it were the sum of two mutually separated sets $\bar{N}$ and $\bar{K}$ where $\bar{N}$ contains $B$ and $\bar{K}$ contains $A$ then $M$ would be the sum of the two mutually separated sets $K + \bar{K}$ and $N + \bar{N}$. The set $K$ is connected. For otherwise it would be the sum of two mutually separated sets $K_1$ and $K_2$ (where $K_1$ contains $A$) and $M$ would be the sum of two separated sets $M_2 + K_1 + N$ and $K_2$. Likewise $N$ is connected. Hence $N + M_2 + A$ is connected. Either $K$ or $N$ contains more than one point.

Case I. Suppose that $K$ contains more than one point but $N$ contains only the point $B$ and that the set $M_2 + A$ is not connected. Then $M_2 + A$ is the sum of two mutually separated point-sets $L_1$ and $L_2$ where $L_2$ contains $A$, $B$ is a limit point both of $L_1$ and of $L_2$, the set $M - B$ is the sum of the two mutually separated sets $L_1$ and $L_2 + K$, and the set $M - (A + B)$ is the sum of the two separated sets $L_2 - A$ and $K - A + L_1$. That this leads to a contradiction may be proved by an argument entirely analogous to that employed in the sub-case of Case II below in which $N$ contains more than one point.*

Case II. Suppose that $K$ contains more than one point and that either $M_2 + A$ is connected or $N$ contains more than one point. In the first case let $Y_0$ denote the point $B$. Otherwise let $Y_0$ denote some definite point of $N$ other than $B$. In either case, if $X$ is any point of $K$ other than $A$, $M - X - Y_0$ is the sum of two mutually separated sets $M_{xy_0}$ and $\bar{M}_{xy_0}$ where $M_{xy_0}$ contains $M_2 + A$. The set $\bar{M}_{xy_0} + X + Y_0$ is the sum of two sets $K_X$ and $N_{y_0}$, where $K_X$ is a subset of $K$ and $N_{y_0}$ is a subset of $N$. For every point $X$ of $K$ distinct from $A$, the set $K_X$ is connected. For otherwise it would be the sum of two separated sets $K_{x_1}$ and $K_{x_2}$ where $K_{x_1}$ contains $X$ and in this case the set $M$ would be the sum of two mutually separated sets, $K_{x_2}$ and $N + M_{xy_0}$, which is contrary to hypothesis. The set $M_{xy_0} + X + Y_0 + N_{y_0}$ is connected. For if it were the sum of two mutually separated sets $L$ and $T$ where $L$ contains $X$ then $M$ would be the sum of two mutually separated sets, $L + K_X$ and $T$, which is contrary to hypothesis. Suppose that $X_1$ and $X_2$ are two points of $K$ and that $X_2$ is in $K_{x_1}$. If $X_1$ were in $K_{x_1}$ then the connected point-set $M_{xy_0} + X_2 + Y_0 + N_{y_0}$ would contain one point $X_2$ of $K_{x_1}$ but would not contain the point $X_1$ and therefore would necessarily be a subset of $K_{x_1}$ and therefore of $K$ which is not the case. Hence $X_1$ is not in $K_{x_2}$. But $K_{x_2}$ is connected and, by hypothesis, contains a point $X_3$ in $K_{x_1}$. Hence $K_{x_3}$ is a subset of $K_{x_1}$. Suppose now that $X_1$ and $X_2$ are two points in $K$ and that $K_{x_1}$ and $K_{x_2}$ have a point in common. Then, unless $X_2$ is in $K_{x_3}$, $K_{x_3}$ must contain the whole of $K_{x_1}$. But if $X_2$ is in $K_{x_2}$ and $X_2$ is not in $K_{x_3}$ then $X_2$ is in $K_{x_3}$. But $K_{x_3}$ is connected and, by hypothesis, contains a point $X_2$ in $K_{x_1}$. Hence $K_{x_3}$ is a subset of $K_{x_1}$.

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*In the proof in question below merely replace $N$ by $L_1$ and $M_2$ by $L_2 - A$. 

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Then, as has been shown above, \( K_x \) contains the whole of \( K_{x_4} \). We thus have the result that if \( K_x \) and \( K_{x_4} \) have a point in common then one of them contains the other one. The points of \( K \) can be arranged in a well-ordered sequence \( \beta \). Let \( P_1 \) be the first point in the sequence \( \beta \). Let \( P_2 \) be the first point of \( \beta \) which lies in \( K_{P_1} \). Let \( P_3 \) be the first point of \( \beta \) which is common to \( K_{P_2} \) and \( K_{P_1} \). Let \( P_4 \) be the first point of \( \beta \) which is common to \( K_{P_3} \), \( K_{P_2} \), and \( K_{P_1} \). This process may be continued. It follows that the sequence \( \beta \) contains a subsequence \( \alpha \) such that if \( T \) is a subset of the elements of \( \alpha \) there is an element of \( \alpha \) which follows, in \( \alpha \), all the elements of \( T \) if, and only if, there exists a point which belongs to \( K_P \) for every point \( P \) of \( T \) and, if there does exist such a point, then the first element of \( \alpha \) that follows all the elements of \( T \) is the first point of \( \beta \) which belongs to \( K_P \) for every point \( P \) of \( T \). For every two distinct points \( X \) and \( Y \) of the sequence \( \alpha \) either \( K_X \) contains \( K_Y \) or \( K_Y \) contains \( K_X \). Moreover the set \( K \) is bounded. It follows* that there exists a point \( P \) which belongs to every \( K_x \) for every point \( X \) of \( \alpha \). It is clear that there is only one such point \( P \) and that \( K_P \) consists of the single point \( P \). For each point \( X \) of \( \alpha \) the set \( M_{XY_0} + X + Y_0 \) is connected. But every point of \( M - P \) belongs to \( M_{XY_0} + X + Y_0 + N_0 \) for some point \( X \) of \( \alpha \). It follows that both \( M_{P_0} + Y_0 \) and \( M - P \) are connected. If \( Y \) is any point of \( N_0 \) then \( M_{P_0} + P + Y = P + N_Y \). By an argument similar to that used to establish the existence of \( P \) it can be shown that there exists in \( N \) a point \( Q \) such that \( N_Q = Q \). The set \( M - (P + Q) \) is connected. Thus the supposition that \( M_1 + A + B \) is not connected has led to a contradiction. Similarly \( M_2 + A + B \) is connected.

Suppose now that \( P_1 \) is any point of \( M_1 \). Let \( P_2 \) denote any point of \( M_2 \). The set \( M - (P_1 + P_2) \) is the sum of two mutually separated sets \( \overline{M} \) and \( \overline{M}_0 \). Since \( \overline{M}_1 + P_1 + P_2 \) and \( \overline{M}_2 + P_1 + P_2 \) are connected, \( P_1 \) is a limit point both of \( \overline{M}_1 \) and of \( \overline{M}_2 \). It follows that both of these sets contain points of \( M_1 \). Let \( \overline{N}_1 \) denote the set of points common to \( \overline{M}_1 \) and \( M_1 + A + B \) and let \( \overline{N}_2 \) denote the set of points common to \( \overline{M}_2 \) and \( M_1 + A + B \). The set \( (M_1 + A + B) - P_1 \) is the sum of the two mutually separated sets \( \overline{N}_1 \) and \( \overline{N}_2 \).

It has now been shown that the point-set \( M_1 + A + B \) satisfies all the requirements of a simple continuous arc from \( A \) to \( B \) as given in Definition 2. Similarly \( M_2 + A + B \) is a simple continuous arc from \( A \) to \( B \). Clearly these two arcs have only \( A \) and \( B \) in common. The truth of Theorem 4 is therefore established.

*See my paper, *On the most general class L of Fréchet in which the Heine-Borel-Lebesgue theorem holds true*, loc. cit.
Definition. If the point-set $M$ is a proper subset of the point-set $N$, the boundary of $M$ with respect to $N$ is the set of all points $[X]$ such that $X$ is either a point or a limit point of $M$ and also either a point or a limit point of $N - M$.

Theorem 5. If the continuous* point-set $M$ contains no continuous set of condensation† then every two points of $M$ are the extremities of a simple continuous arc that lies wholly in $M$.

Indication of proof. By an argument largely similar to (but not entirely identical with) one used in my paper, Concerning continuous sets that have no continuous sets of condensation,‡ it may be proved that $M$ is “connected in kleinem.” By an argument similar to that used in my paper, A theorem concerning continuous curves,§ it may be proved that every two points of $M$ are the extremities of a simple continuous arc lying wholly in $M$.

Theorem 6. In euclidean space of two dimensions if no continuous subset of the continuous point-set $M$ has more than two boundary points with respect to $M$ then $M$ is a simple continuous arc, a simple closed curve, a simple open curve, or a ray of a simple open curve.

Proof. The set $M$ contains no continuous set of condensation. Hence by Theorem 5 every two points of $M$ are the extremities of at least one simple continuous arc that lies wholly in $M$. Let $G$ denote the family of all arcs which lie in $M$. If $AB$ is an arc of $G$ no point of $AB$ distinct from $A$ and from $B$ is a limit point of $M - AB$. For if there should exist such a point $P$ there would exist on $AB$ two points $\tilde{A}$ and $\tilde{B}$ in the order $A\tilde{A}P\tilde{B}B$ on $AB$ and the interval $AB$ would be a continuous subset of $M$ that has three distinct boundary points with respect to $M$, these boundary points being the points $\tilde{A}$, $P$, and $\tilde{B}$.

If $g_1$ and $g_2$ are two arcs of $G$ with a common point then the point-set $g_1 + g_2$ is either a simple continuous arc or a simple closed curve. For otherwise one of the arcs $g_1$ and $g_2$ (call it $g$) would contain a point $P$ which is not an endpoint of $g$ but which is a limit point of $M - g$.

Let $A$ and $B$ denote two definite distinct points of $M$ and let $AB$ denote a definite arc of the set $G$ having $A$ and $B$ as endpoints. Let $K_C$ ($C = A, B$) denote the set of all points $X$ of $M$ such that $C$ and $X$ are the extremities of a simple continuous arc which is a subset of $M$ but which contains no point except $C$ in common with $AB$. An arc from $C$ to $X$ satisfying these conditions

* A set of points is said to be continuous if it is closed and connected and contains more than one point.
† The continuous set of points $N$ is said to be a continuous set of condensation of the set $M$ if $N$ is a proper subset of $M$ and every point of $N$ is a limit point of $N - M$.
§ Ibid., vol. 23 (1917), pp. 233–236.
will be called the arc $CX$. If neither $K_A$ nor $K_B$ exists then $M$ is the arc $AB$. Suppose that $K_A$ exists. If $X_1$ and $X_2$ are points of $K_A$, $X_1$ is said to precede $X_2$ if $AX_1$ is a subset of $AX_2$. It is clear that of any two distinct points of $K_A$ one of them precedes the other one. The set $K_A + A + B$ is closed. For suppose there exist two points $O$ and $\bar{O}$, distinct from $A$, which are limit points of $K_A$ but which do not belong to $K_A$. It can easily be proved with the help of Theorem 5 of F.A. that there exist two sets of points $A_1, A_2, A_3, \cdots$ and $A_1', A_2', A_3', \cdots$ belonging to $K_A$ such that, for every $n$, $A_n$ precedes $A_n'$ and $A_n'$ precedes $A_{n+1}$ and such that $O$ is the sequential limit point of $A_1, A_2, A_3, \cdots$ while $\bar{O}$ is the sequential limit point of $A_1', A_2', A_3', \cdots$. It follows, by an argument similar to that used in establishing the existence of the set $t$ in the proof of Theorem 1, that there exists a continuous set of points $\bar{K}$ containing $O$ such that if $Y$ is a point of $\bar{K}$ then every region about $Y$ contains points of infinitely many of the arcs $A_1 A_2, A_2 A_3, \cdots$. For no value of $n$ does the arc $A_n A_{n+1}$ contain a limit point of the point-set composed of the arcs $A_{n+2}, A_{n+3}, A_{n+4}, \cdots$. Hence $\bar{K}$ contains no point of any of the arcs $A_1 A_2, A_2 A_3, \cdots$. It follows that $\bar{K}$ is a continuous set of condensation of $M$. But it was shown above that $M$ contains no continuous set of condensation. Thus the supposition that there exist two limit points of $K_A + A$ that do not belong to $K_A + A$ has led to a contradiction. It follows that either $K_A + A + B$ is closed or there exists one and only one point $P$, distinct from $B$, which does not belong to $K_A$ but is a limit point of $K_A$. In the latter case $P$ is a sequential limit point of a sequence of points $P_1, P_2, P_3, \cdots$ belonging to $K_A$ such that, for every $n$, $P_n$ precedes $P_{n+1}$ and the point-set $P + A_1 P + P_1 P_2 + P_2 P_3 + \cdots$ is a simple continuous arc from $A$ to $P$. It follows that $P$ either coincides with $B$ or belongs to $K_A$. But this is contrary to supposition. Hence the set $K_A + A + B$ is closed.

There are several cases to be considered.

Case I. Suppose that $K_A + A$ is unbounded and that $K_B$ does not exist. The set $K_A + A$ contains a countably infinite set of distinct points $A_1, A_2, A_3, \cdots$ such that $A_1 + A_2 + A_3 + \cdots$ has no limit point. Let $\bar{A}_2$ be the first point of this sequence that follows $A_1$, $\bar{A}_3$ the first one that follows $\bar{A}_2$, etc. There results an infinite sequence $\bar{A}_1, \bar{A}_2, \bar{A}_3, \cdots$ of points belonging to the set $A_1, A_2, A_3, \cdots$ such that, for every $n$, $\bar{A}_n$ precedes $\bar{A}_{n+1}$. For each $n$ there exists a region $R_{n+1}$ containing $\bar{A}_{n+1}$ but containing no point of $M$ that does not belong to the interval $A_n A_{n+2}$ of $M$. It can be shown that there exists a countable set of distinct points $\bar{B}_1, \bar{B}_2, \bar{B}_3, \cdots$ (where $\bar{B}_1 = B$), with no limit point, and a set of arcs $\bar{B}_1, \bar{B}_2, \bar{B}_3, \bar{B}_4, \cdots$ such that

* Here, for every $n$, $A_n A_{n+1}$ denotes the interval of the arc $AA_n$ whose endpoints are $A_n$ and $A_{n+1}$.

† Here, for every $n$, $P_n P_{n+1}$ denotes the interval of $AP_{n+1}$ whose endpoints are $P_n$ and $P_{n+1}$.
(1) for every $n$, $B_{n+1}$ is in $B_{n+1}$, (2) the set $M + B_1 B_2 + B_2 B_3 + \cdots$ is an open continuous curve. The set $M$ is a ray of this curve.

**Case II.** Suppose that $K_A$ is unbounded and that $K_B$ exists but is bounded. Then $K_A$ and $K_B$ can have no point in common and the set $K_B$ cannot contain more than one last point. If $P$ is a point of $K_B$ which is not a last point then $K_B + B - P$ is not connected. Thus the set $K_B + B$ is a simple continuous arc (see Definition 2). That $(K_B + B) + K_A$ is a ray follows by an argument similar to that employed in Case I.

**Case III.** Suppose $K_A$ and $K_B$ are both unbounded. In this case the closed sets $K_A$ and $K_B$ have no point in common and the set $K_B + AB + K_A$ is evidently an open curve.

**Case IV.** Suppose $K_A$ is bounded and that $K_B$ either is bounded or does not exist and that $K_A$ and $K_B$ have no point in common. In this case $M$ is clearly a simple continuous arc.

**Case V.** Suppose that $K_A$ is bounded and that $K_B$ and $K_A$ have at least one point in common. It is clear that in this case $K_A + B$ is a simple continuous arc from $A$ to $B$ and that $M$ is a simple closed curve.

If the term simple continuous curve is applied only to point-sets which are either arcs, closed curves, open curves, or rays then it is clear in view of the above results that these point-sets may be defined as follows.

**Definition 5.** A simple continuous curve* is a continuous point-set $M$ no continuous subset of which has more than two boundary points with respect to $M$.

**Definition 6.** A simple closed curve is a continuous point-set $M$ every proper continuous subset of which has just two boundary points with respect to $M$.

**Definition 7.** A simple continuous open curve is a continuous point-set $M$ every bounded continuous subset of which has just two boundary points with respect to $M$.

**Definition 8.** A simple continuous arc is a continuous curve $M$ containing a point $A$ such that every continuous proper subset of $M$ that contains $A$ has just one boundary point with respect to $M$.

**Definition 9.** A ray is a continuous curve $M$ containing a point $A$ such that every bounded continuous subset of $M$ that contains $A$ has just one boundary point with respect to $M$.

* The term continuous curve is ordinarily not applied to sets that are not bounded. Thus, according to the ordinary terminology, a straight line is not a continuous curve. I suggest that the term continuous curve be applied to every closed point-set which is connected "inkleinem" (in the sense of H. Hahn) whether it be bounded or unbounded. If this terminology is adopted what is now ordinarily called a continuous curve may be characterized as a bounded continuous curve.

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