A PROPERTY OF TWO $(n+1)$-GONS INSCRIBED IN A NORM-CURVE IN n-SPACE*

BY

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§ 1. Introduction

Two cubic equations fix two triangles inscribed in a conic, if the coördinates of the generating point are given as quadric functions of one parameter. So for a gauche cubic curve, where homogeneous coördinates are given by rational integral functions of degree 3 in a parameter, two inscribed tetrahedrons may have the parameters of their vertices determined by any two quartic equations.

Two triangles inscribed in one conic determine a second conic which touches their six sides; and there exists a third conic with respect to which the two are reciprocal polar curves.† On a twisted cubic curve the analogous theorem still bears the name of von Staudt‡ as its originator, while Hurwitz has given its most accessible proof. It states that two tetrahedra inscribed in a gauche cubic determine uniquely a symmetric polarity in which they are self-reciprocal, and hence that their eight faces are osculating planes of a second gauche cubic curve.

Geometric proof of either theorem is not difficult, but a formula can be constructed which renders either one immediately visible. It is of interest to observe that the proof of the theorem simply as stated is most obvious if the formula is allowed to retain a certain extraneous factor; but the removal of this factor and the resulting condensation of the formula discloses more clearly the further fact that in each case the two sets of points employed are not unique, but are random selections from an infinite linear system of triads

* Presented to the Society, April, 1919, under different title.

† Brianchon, *Mémoire sur les lignes du second ordre.* Paris (1817), p. 35.

Steiner, *Die geometrischen Constructionen ausgeführt mittelst der geraden Linie und eines festen Kreises.* Berlin (1833), p. 67.

‡ See von Staudt, *Beiträge zur Geometrie der Lage*, p. 378; and A. Hurwitz, *Beweis eines Satzes aus der Theorie der Raumcurven III. Ordnung,* Mathematische Annalen, vol. 20 (1882), pp. 135–137. The latter establishes the existence of infinitely many such tetrahedra.

See also the *Encyklopädie der math. Wissenschaften,* III C 2, p. 236, § 108.

or quartettes. Moreover this method has an obvious extension to the norm-curve in flat space of n dimensions.

For each curve both primitive and reduced formulas will be exhibited.

§ 2. The conics and two triangles

On a conic the theorem cited amounts to asserting the existence of a $(2, 2)$ correspondence, symmetrical, among values of one parameter, which will convert each of three points into both the others, in each of two sets of three (or triads). Denote those parameters, in the two sets, by

$$a, b, c \qquad \text{and} \qquad a', b', c'.$$

If u is the original parameter, v the transformed, the following is a $(2, 2)$ correspondence or transformation which satisfies the requirements:

$$\begin{aligned}\phi(u, v) \equiv\ & \\ & (c-b)\cdot(u-c)(u-b)\cdot(v-c)(v-b)\cdot(a-a')(a-b')(a-c') \\ & +(a-c)\cdot(u-a)(u-c)\cdot(v-a)(v-c)\cdot(b-a')(b-b')(b-c') \\ & +(b-a)\cdot(u-b)(u-a)\cdot(v-b)(v-a)\cdot(c-a')(c-b')(c-c')=0.\end{aligned}$$

For this relation is evidently satisfied identically by $u = a$, $v = b$, as each term contains either the factor $u - a$ or $v - b$, or both. Similarly for the pairs a, c and b, c. As for $u = a'$ and $v = b'$, insert those values in ϕ and remove the factor $(a' - a)(a' - b)(a' - c)\cdot(b' - a)(b' - b)(b' - c)$, whereupon the quotient remaining is the determinant

$$\begin{vmatrix} a - c' & b - c' & c - c' \\ 1 & 1 & 1 \\ a & b & c \end{vmatrix}$$

which vanishes.

This demonstrates the conic theorem, since when in $\phi(u, v)$ the quadric functions of u and v respectively are replaced by their equivalents in trilinear coördinates (x_1, x_2, x_3) and (y_1, y_2, y_3) of two points on the conic, $\phi(u, v)$ becomes symmetrically bilinear, and equated to zero gives a polar reciprocity

$$\phi(u, v) \equiv \Phi(x, y) = 0$$

with respect to a conic $\Phi(x, x) = 0$. Accordingly each vertex as a has its polar, as bc, touching the reciprocal of the conic on which the six vertices were taken to lie.

The extraneous factor in this formula $\phi(u, v)$ is

$$(a - b)(b - c)(c - a).$$

Remove that factor, and replace symmetric functions of either triad by the proper coefficient from one of these cubics:

$$f_1(u) = (u-a)(u-b)(u-c) = u^3 - Au^2 + Bu - C,$$
$$f_2(u) = (u-a')(u-b')(u-c') = u^3 - A'u^2 + B'u - C'.$$

We have then the reduced form

$$\begin{aligned}\phi_1(u,v) \equiv u^2v^2(A-A') - uv(u+v)(B-B') \\ + (u^2+uv+v^2)(C-C') + uv(AB'-A'B) \\ - (u+v)(AC'-A'C) - (BC'-B'C) = 0.\end{aligned}$$

Note that the constants are determinants from the array

$$\left\| \begin{matrix} 1 & A & B & C \\ 1 & A' & B' & C' \end{matrix} \right\|,$$

and that these are invariant save for the factor $(k_1 l_2 - k_2 l_1)$ when $f_1(u)$ and $f_2(u)$ are replaced by any two cubics

$$k_1 f_1(u) + k_2 f_2(u), \qquad l_1 f_1(u) + l_2 f_2(u)$$

of the linear system determined by the former two. Therefore all cubics of this linear system give polar triangles of the conic $\Phi(x,x) = 0$, and the sides of all such triangles touch one common curve of the second class.

§ 3. The twisted cubic and two tetrahedra

For the twisted cubic and the theorem of von Staudt and Hurwitz, the primitive formula is obviously the following,—summation covering cyclic permutations of a, b, c, d:

$$\begin{aligned}\sum \pm \begin{vmatrix} 1 & 1 & 1 \\ b & c & d \\ b^2 & c^2 & d^2 \end{vmatrix} \cdot (u-b)(u-c)(u-d)\cdot(v-b)(v-c)(v-d) \\ \cdot (a-a')(a-b')(a-c')(a-d') \equiv \phi(u,v) = 0.\end{aligned}$$

Point coördinates (x) and (y) upon the gauche cubic replace cubic expressions in u and v, giving a bilinear symmetric polarity

$$\phi(u,v) \equiv \Phi(x_1, x_2, x_3, x_4;\ y_1, y_2, y_3, y_4), \qquad \text{or} \qquad \Phi(x,y) = 0.$$

The quadric surface $\Phi(x,x) = 0$ is the one with respect to which the two tetrahedra are self-polar,—as the theorem asserts.

Here also, as in the preceding section, occurs a difference-product as an extraneous factor, and on its removal the polarity is seen to be covariant (or

combinant) in a linear system of tetrahedra. For if we denote by $f_1(u) = 0$ and $f_2(u) = 0$ the quartics whose roots are parameters of the vertices of the two tetrahedra,

$$f_1(u) \equiv (u-a)(u-b)(u-c)(u-d) \equiv u^4 - Au^3 + Bu^2 - Cu + D,$$

$$f_2(u) \equiv (u-a')\cdots(u-d') \equiv u^4 - A'u^3 + B'u^2 - C'u + D',$$

the function $\phi(u, v)$ can be represented as a determinant:

$$\phi(u, v) \equiv \begin{vmatrix} f_2(a) & f_2(b) & \cdots & f_2(d) \\ (u-a)(v-a) & (u-b)(v-b) & \cdots & (u-d)(v-d) \\ a(u-a)(v-a) & b(u-b)(v-b) & \cdots & d(u-d)(v-d) \\ a^2(u-a)(v-a) & b^2(u-b)(v-b) & \cdots & d^2(u-d)(v-d) \end{vmatrix} = 0.$$

By a well-known theorem on alternants* this is reduced and the difference-product of a, b, c, d may be removed. The result, the essential form of the $(3, 3)$ relation, is the determinantal equation

$$\begin{vmatrix} D & C & B & A & 1 \\ D' & C' & B' & A' & 1 \\ uv & -(u+v) & 1 & 0 & 0 \\ 0 & uv & -(u+v) & 1 & 0 \\ 0 & 0 & uv & -(u+v) & 1 \end{vmatrix} = 0.$$

Thus either with or without the desirable symmetry of $\phi(u, v)$ in parameters of the first and second sets, *we have obtained a type-formula extensible at once to norm-curves in any number of dimensions.*

VASSAR COLLEGE,
April, 1919.

* Muir, *A treatise on the theory of determinants*, § 127.