NOTE ON A CLASS OF POLYNOMIALS OF APPROXIMATION*

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1. **Introduction.** In a recent paper† it was shown that if $f(x)$ is a given continuous function in the interval $a \leqq x \leqq b$, n a given positive integer, and m a given real number greater than 1, there exists one and just one polynomial $\phi(x)$, of degree n or lower, which reduces to a minimum the value of the integral

$$\int_a^b |f(x) - \phi(x)|^m \, dx.$$

The purpose of this note is to establish the truth of the same proposition in the case that $m = 1$, for which the proof is considerably less simple.

The function $\phi(x)$ in the earlier paper was, more generally, a linear combination, with constant coefficients, of an arbitrarily given set of linearly independent continuous functions. The existence of at least one minimizing function is proved with the same degree of generality here. The argument involves no new difficulty, and in fact is slightly simpler than for $m > 1$. For the proof of uniqueness, however, $\phi(x)$ is essentially a polynomial as indicated,‡ though a corresponding treatment could be given for other approximating functions, such as finite trigonometric sums, which are similarly determined by their roots.

2. **Existence of an approximating function.** Let $p_0(x), p_1(x), \cdots, p_n(x)$ be $n + 1$ functions of x, continuous throughout the interval $a \leqq x \leqq b$, and linearly independent in this interval. Let

$$\phi(x) = c_0 p_0(x) + c_1 p_1(x) + \cdots + c_n p_n(x)$$

be an arbitrary linear combination of these functions with constant coefficients, and let H be the maximum of $|\phi(x)|$ in (a, b).

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† D. Jackson, *On functions of closest approximation*, these Transactions, vol. 22 (1921), pp. 117–128. The main problem of the paper cited, going somewhat beyond the result quoted here, had previously been treated, for the polynomial case, by Pólya, in a note with which I was not acquainted at the time of writing: *Sur un algorithme*, etc., Comptes Rendus, vol. 157 (1913), pp. 840–843.

‡ The limitation is in the fact, not merely in the proof. For example, if $f(x) = 1$, $\phi(x) = cx$, in the interval $-1 \leqq x \leqq 1$, the integral of $|f - \phi|$ takes on its minimum value for any value of the coefficient c in the interval $-1 \leqq c \leqq 1$. Here ϕ is still a polynomial, to be sure, but not an arbitrary polynomial of its degree. The coefficients for the best polynomial of the form $c_0 + c_1 x$ are evidently uniquely determined, namely $c_0 = 1, c_1 = 0$.

Then there exists a constant P, completely determined by the system of functions $p_0(x), \cdots, p_n(x)$, such that

$$|c_k| \leqq PH \qquad (k = 0, 1, \cdots, n)$$

for all functions $\phi(x)$.

This lemma, due originally to Sibirani, is proved in the paper already cited,* and the proof need not be repeated here.

Let

$$G = \int_a^b |\phi(x)|\,dx;$$

then there exists a constant Q, completely determined by the system of functions† $p_0(x), \cdots, p_n(x)$, such that

$$|c_k| \leqq QG \qquad (k = 0, 1, \cdots, n),$$

for all functions $\phi(x)$. For‡

$$|c_0 \int_a^x p_0(x)\,dx + c_1 \int_a^x p_1(x)\,dx + \cdots + c_n \int_a^x p_n(x)\,dx|$$

$$= |\int_a^x \phi(x)\,dx| \leqq \int_a^x |\phi(x)|\,dx \leqq \int_a^b |\phi(x)|\,dx = G,$$

and it suffices to apply the lemma previously stated to the set of linearly independent functions

$$\int_a^x p_0(x)\,dx, \ \cdots, \int_a^x p_n(x)\,dx,$$

the constant in this case being denoted by Q.

Finally, let $f(x)$ be a function continuous for $a \leqq x \leqq b$, arbitrary at the outset, but to be kept unchanged throughout the remainder of the discussion; let M be the maximum of $|f(x)|$ in (a, b); let

$$g = \int_a^b |f(x) - \phi(x)|\,dx;$$

and let $\Delta = b - a$. Then, for all functions $\phi(x)$,

$$|c_k| \leqq Q(g + M\Delta),$$

* Loc. cit., Lemma I; the notation has been changed somewhat.

† It is understood that the definition of the functions includes a specification of the interval in which they are defined; when the p's are the successive powers of x, for example, the value of Q will depend on the interval in which they are considered.

‡ Cf. loc. cit., proof of Lemma II.

where Q has the same value as before. For

$$|\phi(x)| \leqq |f(x) - \phi(x)| + |f(x)|,$$

$$\int_a^b |\phi(x)| dx \leqq g + \int_a^b |f(x)| dx \leqq g + M\Delta,$$

and the preceding lemma is applicable.

From this third lemma it follows that if a function $\phi(x)$ is sought to make the value of g as small as possible, all the coefficients $c_0, \cdots, c_n$ that come into consideration belong to a bounded region in the corresponding space of $n+1$ dimensions, a region which may be regarded as closed. As g is a continuous function of the coefficients, there will surely be at least one determination of ϕ for which g actually attains its lower limit γ. Such a function ϕ will be called for brevity an *approximating function.*

3. **Necessary condition for the existence of two approximating functions.** Suppose there are two functions,* $\phi_0(x)$ and $\phi_1(x)$, linear combinations of the p's, for each of which g has the minimum value γ. Let

$$\phi_2(x) = \frac{1}{2}[\phi_0(x) + \phi_1(x)],$$

and let

$$r_i(x) = f(x) - \phi_i(x) \qquad (i = 0, 1, 2).$$

Then

$$r_2(x) = \frac{1}{2}[r_0(x) + r_1(x)].$$

Hence

$$|r_2(x)| \leqq \frac{1}{2}[|r_0(x)| + |r_1(x)|],$$

the sign of equality holding, for any particular value of x, if $r_0 r_1 \geqq 0$, while the relation is an inequality if r_0 and r_1 have opposite signs. If the inequality were to hold for any value of x, it would hold throughout an interval, because of the continuity of the functions, and it would follow that

$$\int_a^b |r_2(x)| dx < \gamma,$$

which is impossible. Consequently either of the remainder functions $r_0(x)$, $r_1(x)$, must be positive or zero at every point where the other is positive, and negative or zero at every point where the other is negative.

* In the earlier paper that has been cited, $\phi_m(x)$ denoted the function for which the integral of the mth power of the absolute value of the error is a minimum. Here the exponent is restricted to the value 1, and the subscript is used without any such special significance.

Suppose there are two points x_1, x_2, in the interval (a, b), such that

$$r_0(x_1) < 0, \qquad r_0(x_2) > 0.$$

Then

$$r_1(x_1) \leqq 0, \qquad r_1(x_2) \geqq 0.$$

There must be at least one point in the interval (x_1, x_2) at which r_0 vanishes, and there must be at least one point where r_1 vanishes. It is to be shown that there must be at least one point where both remainders vanish simultaneously. Let x_1' be the upper limit of the values of x in (x_1, x_2) for which $r_0(x) < 0$, and x_2' the lower limit of the values of x in (x_1', x_2) for which $r_0(x) > 0$. Then $r_0(x)$ is identically zero for $x_1' \leqq x \leqq x_2'$, is negative for points immediately at the left of x_1', and is positive for points immediately at the right of x_2'. For definiteness, and to cover the less obvious case, it will be assumed that $x_1' < x_2'$. If $x_1' = x_2'$, so that the interval reduces to a point, the discussion is essentially the same, though a little simpler. In the interval $x_1' \leqq x \leqq x_2'$, the function $r_1(x)$, being continuous, must either vanish somewhere, or be positive everywhere, or be negative everywhere. But if it were positive throughout the interval, it would be positive, in particular, at the point x_1', and so, by continuity, would be positive at points immediately at the left of x_1', where $r_0(x)$ is negative, and this is inadmissible. Similarly the hypothesis that $r_1(x) < 0$ everywhere has to be rejected. It is certain, then, that $r_1(x)$ must vanish at some point of the interval in question. *In every interval where one of the remainders changes sign, there is at least one point where both remainders vanish simultaneously. At such a point, ϕ_0 and ϕ_1, being both equal to f, are equal to each other.*

To draw further consequences from the relation $r_0 r_1 \geqq 0$, let

$$\phi_t(x) = \phi_0 + t(\phi_1 - \phi_0), \qquad r_t(x) = f(x) - \phi_t(x), \qquad 0 \leqq t \leqq 1.$$

The function previously denoted by $\phi_2(x)$, for example, now has the representation $\phi_{\frac{1}{2}}(x)$. It will always be true that

$$|r_t(x)| = (1 - t)|r_0(x)| + t|r_1(x)|,$$

and therefore

$$\int_a^b |r_t(x)|dx = \gamma.$$

Everything that has been said about the pair of functions ϕ_0, ϕ_1, can be said with equal force about the pair of functions ϕ_t corresponding to any two values of t in the interval $(0, 1)$.

4. **Uniqueness of the approximating polynomial.** From now on the generality of the discussion will be restricted, and it will be assumed that

$$p_k(x) = x^k \qquad (k = 0, 1, \cdots, n),$$

so that $\phi(x)$ is a polynomial of the nth degree or lower. In other respects the previous notation will be retained, and it is to be shown that the hypothesis of the existence of two distinct polynomials of approximation, ϕ_0 and ϕ_1, leads to a contradiction.

There are two cases to be distinguished.* Suppose first that $r_0(x)$ changes sign $n+1$ times or more in (a, b). By the italics of the preceding section there must then be at least $n+1$ points where r_0 and r_1 vanish simultaneously, and so at least $n+1$ points where ϕ_0 and ϕ_1 are equal to each other. The identity of the two polynomials follows immediately. If there is any value of t in $(0, 1)$ such that $r_t(x)$ changes sign $n+1$ times or more, the same conclusion can be drawn.

Otherwise, suppose that $r_t(x)$ never changes sign more than n times. For any particular value of t, let $r_t(x)$ change sign just q times, $q \leqq n$. It will be possible to select q points, $x_1, x_2, \cdots, x_q$, so that $r_t(x)$ will be of constant sign (wherever different from zero) in each of the intervals $a \leqq x \leqq x_1$, $x_1 \leqq x \leqq x_2, \cdots, x_q \leqq x \leqq b$, and will take on opposite signs in successive intervals. Let

$$\psi(x) = A(x-x_1)(x-x_2)\cdots(x-x_q),$$

the constant multiplier A being chosen so that the maximum of $|\psi(x)|$ in (a, b) is 1, and so that $\psi(x)$ has the same sign as $r_t(x)$ at some point where $r_t(x) \neq 0$. Then $\psi(x)$ will have the same sign as $r_t(x)$ at every point where $r_t(x)$ is different from zero. If r_t does not change sign at all, it will be understood that $\psi = \pm 1$.

Let ϵ be any positive quantity, and let R_1 be the set of points in (a, b) at which $|r_t(x)| \geqq \epsilon$, S_1 the complementary set, where $|r_t(x)| < \epsilon$. Then, in R_1,

$$|r_t(x) - \epsilon\psi(x)| = |r_t(x)| - \epsilon|\psi(x)|,$$

since r_t and ψ have the same sign, and $|r_t| \geqq \epsilon \geqq \epsilon|\psi|$. In S_1,

$$|r_t(x) - \epsilon\psi(x)| \leqq |r_t(x)| + \epsilon|\psi(x)|.$$

Hence

$$\int_a^b |r_t(x) - \epsilon\psi(x)|\,dx$$
$$= \int_{R_1} + \int_{S_1} \leqq \int_a^b |r_t(x)|\,dx - \epsilon\int_{R_1}|\psi(x)|\,dx + \epsilon\int_{S_1}|\psi(x)|\,dx,$$

* If it were a question of the integral of the mth power of the absolute value of the error, $m > 1$, the remainder for the approximating polynomial, if not identically zero, would necessarily change sign at least $n+1$ times; cf. loc. cit., end of § 7, noticing that the polynomial there is of degree $n-1$ at most. For $m = 1$, however, it is readily seen that there are other possibilities.

so that

$$\int_a^b |r_t(x) - \epsilon\psi(x)| dx < \int_a^b |r_t(x)| dx$$

unless

$$\int_{S_1} |\psi(x)| dx \geqq \int_{R_1} |\psi(x)| dx .$$

That is, the last relation must necessarily be satisfied, since otherwise the polynomial

$$\phi_t(x) + \epsilon\psi(x)$$

would give a better approximation, in the sense of the integral of the absolute value of the error, than $\phi_t(x)$, which is contrary to hypothesis.

As ϵ is arbitrarily small, it follows further that if R is the set of points where $|r_t(x)| > 0$, and S the complementary set where $r_t(x) = 0$,

$$\int_S |\psi(x)| dx \geqq \int_R |\psi(x)| dx ,$$

or, what amounts to the same thing,

$$\int_S |\psi(x)| dx \geqq \frac{1}{2}\int_a^b |\psi(x)| dx . \tag{1}$$

In detail, the argument is as follows: Let $\epsilon_1, \epsilon_2, \cdots$, be a diminishing sequence of positive quantities approaching zero, and let R_j and S_j be the sets corresponding to ϵ_j, after the manner of the preceding paragraph. Then* R is the sum of all the sets R_j, each of which is included in the following, and S is the product of all the sets S_j, each of which is included in the preceding, and therefore

$$\lim_{j=\infty} mR_j = mR, \qquad \lim_{j=\infty} mS_j = mS ,$$

the symbol m denoting *measure of*. As

$$\int_R = \int_{R_j} + \int_{R-R_j} \qquad \int_S = \int_{S_j} - \int_{S_j - S} ,$$

and $\lim_{j=\infty} m(R - R_j) = \lim_{j=\infty} m(S_j - S) = 0$, the desired relation follows immediately.

From (1) it can be inferred that

$$mS \geqq h ,$$

where h is a positive quantity depending only on n, a, and b, and independent

* It is convenient to use the language of the theory of Lebesgue integrals, though the problems of measure involved are very simple, since all the functions concerned are continuous, and each point-set that comes into consideration is made up of a finite number or an enumerable infinity of non-overlapping intervals and points.

of the coefficients in the particular polynomial ψ. For, let $\psi(x)$ be any polynomial of degree not higher than n, and let

$$\int_a^b |\psi(x)|dx = G.$$

Then, for each coefficient c_k in ψ,

$$|c_k| \leqq QG,$$

where Q is the constant of § 2. If W is the greatest value of any of the quantities $1, |x|, \cdots, |x^n|$ in (a, b),

$$|\psi(x)| \leqq (n+1)QGW,$$

and for any measurable set S in (a, b),

$$\int_S |\psi(x)|dx \leqq (n+1)QGWmS.$$

Hence (1) implies that*

$$(n+1)QGWmS \geqq \frac{1}{2}G,$$

$$mS \geqq \frac{1}{2(n+1)QW},$$

and the last quantity may be taken as h.

For every value of t in $(0, 1)$, then, it has been proved that $f(x)$ must be identical with $\phi_t(x)$ throughout a point-set of measure at least h. But this is impossible. For the sets S corresponding to two different values of t can not have more than n points in common, since a common point is a point where the two polynomials $\phi_t(x)$ are equal, and hence the measure of the sum of the sets is the sum of their measures; and the necessary condition cannot be satisfied for more than N values of t, where N is the greatest integer contained in $(b-a)/h$. So the desired contradiction has been obtained, and the uniqueness of the approximating polynomial is assured.

It may be emphasized once more that there is no assertion that the remainder for the polynomial of approximation must necessarily change sign $n+1$ times; it must *either* change sign $n+1$ times, *or* vanish identically throughout a finite number or an enumerable infinity of intervals, having at least a certain specified aggregate length.

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* It may be assumed that $g \neq 0$, since there is no occasion for considering a function ψ that vanishes identically.