AN EXPANSION THEOREM FOR A SYSTEM OF LINEAR DIFFEREN-
TIAL EQUATIONS OF THE FIRST ORDER *

BY

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Introduction. The object of the present paper is to make a first step in the extension of the well-known Sturm-Liouville expansion theorem for a differential equation with one parameter to the case of systems of equations. A certain type of system of two equations of the first order in two unknowns will be considered. The essential facts deduced may be summarized in the following

THEOREM. *The values of λ for which the system*

$$u'(x) - [a(x) + \lambda] v(x) = 0,$$
$$v'(x) + [b(x) + \lambda] u(x) = 0,$$
$$\alpha_0 u(0) + \beta_0 v(0) = 0,$$
$$\alpha_1 u(1) + \beta_1 v(1) = 0,$$

(where $a(x)$, $b(x)$ are continuous functions, $0 \leqq x \leqq 1$, and $|\alpha_0| + |\beta_0| \neq 0$, $|\alpha_1| + |\beta_1| \neq 0$) possesses non-trivial solutions, are all real, and infinite in number, extending to infinity positively and negatively and having no finite limit-point. For each such value λ_n there is one and only one solution $[u_n, v_n]$; the set of solutions corresponding to all values of λ is orthogonal and may be normalized, these terms being used to characterize the relations:

$$\int_0^1 [u_m u_n + v_m v_n] \, dx = \begin{cases} 0, & m \neq n, \\ 1, & m = n. \end{cases}$$

If $f(x)$, $g(x)$ are any two functions having continuous second derivatives and satisfying the conditions

$$\alpha_0 f(0) + \beta_0 g(0) = 0, \qquad \alpha_1 f(1) + \beta_1 g(1) = 0,$$

then

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n u_n(x),$$

$$g(x) = \sum_{n=-\infty}^{+\infty} c_n v_n(x),$$

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* Presented to the Society, Sept. 5, 1917.

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where

$$c_n = \int_0^1 [fu_n + gv_n]\, dx,$$

the convergence being uniform.

The method of treatment follows closely that of Kneser[*] on a single equation of the second order. A further extension of the present results to a more complicated form of boundary condition has been made by Dr. C. C. Camp.

It may be noted that the several systems of integral equations used are in each case of the Volterra type; solutions could therefore be obtained directly in the form of uniformly convergent infinite series without any use of the notation for the resolvent system to the kernel system; this notation serves only to abbreviate the presentation.

We deal with the homogeneous system of differential equations

(1) $$u' - (a + \lambda) v = 0, \qquad v' + (b + \lambda) u = 0,$$

and the corresponding non-homogeneous system

(2) $$u' - (a + \lambda) v = \varphi, \qquad v' + (b + \lambda) u = \psi.$$

Here the independent variable x ranges over a finite real interval, which for convenience we take to be the interval $0 \leqq x \leqq 1$; λ is a parameter which may have any real or complex value. The functions a, b, φ, ψ are real and depend on x alone; they are continuous, and the first two have continuous first derivatives. The unknown functions u, v depend on x and λ; the notation u', v' denotes differentiation with respect to x. Throughout the paper the variable λ will be included in or omitted from the notation for function depending on it, as seems convenient.

A function-pair $[u, v]$, such that u, v have continuous first derivatives and satisfy (1) or (2), will be called a *solution;* the product of a solution $[u, v]$ by a constant c is to mean the solution $[cu, cv]$. The solution $[0, 0]$ of (1) is a *trivial solution.*

We consider also the boundary conditions

(B_0) $\alpha_0 u(0) + \beta_0 v(0) = 0,$

(B_1) $\alpha_1 u(1) + \beta_1 v(1) = 0,$

where α_0, β_0, α_1, β_1 are real constants such that

$$|\alpha_0| + |\beta_0| \neq 0, \qquad |\alpha_1| + |\beta_1| \neq 0.$$

In view of the latter conditions, there will be no loss of generality in writing, when this is convenient,

(3) $\alpha_0 = \cos\theta_0,$ $\beta_0 = \sin\theta_0;$ $\alpha_1 = \cos\theta_1,$ $\beta_1 = \sin\theta_1.$

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[*] M a t h e m a t i s c h e   A n n a l e n, vol. 58 (1904), p. 81; vol. 60 (1905), p. 402.

We shall use the notation $F = O(\varphi(\lambda))$, where F is a function of λ with or without x and other real variables, to denote that there exist positive real constants Λ, K such that $|F| \leqq K|\varphi|$ when $|\lambda| > \Lambda$; thus if F depends on other variables besides λ, the notation involves the idea of uniformity. A similar notation will be used with a real (positive or negative) integer n in place of λ.

The homogeneous system. THEOREM I. *The system* (1) *with initial conditions*

(4)
$$u(0) = \alpha, \qquad v(0) = \beta,$$

where α, β are any constants, possesses for each λ one and only one solution $[u, v]$; this solution is of the form

(5)
$$u(x) = \alpha \cos \xi + \beta \sin \xi + O\left(\frac{1}{\lambda}\right),$$

$$v(x) = \beta \cos \xi - \alpha \sin \xi + O\left(\frac{1}{\lambda}\right),$$

where

(6)
$$\xi = \lambda x + \tfrac{1}{2} \int_0^x [a(s) + b(s)]\, ds.$$

Write

(7)
$$u(x) = U(x) + \left(1 + \frac{a}{2\lambda}\right)(\alpha \cos \xi + \beta \sin \xi),$$

$$v(x) = V(x) + \left(1 + \frac{b}{2\lambda}\right)(\beta \cos \xi - \alpha \sin \xi).$$

By substitution in (1) we find that

(8)
$$U' - (a + \lambda)V = \frac{P}{\lambda},$$

$$V' + (b + \lambda)U = \frac{Q}{\lambda},$$

where

(9)
$$P(x, \lambda) = \frac{-\alpha a(b-a) - 2\beta a'}{4} \sin \xi + \frac{\beta a(b-a) - 2\alpha a'}{4} \cos \xi,$$

$$Q(x, \lambda) = \frac{\alpha b(b-a) - 2\beta b'}{4} \cos \xi + \frac{\beta b(b-a) + 2\alpha b'}{4} \sin \xi,$$

or

(10)
$$P(x, \lambda) = \frac{a(b-a)}{4}(\beta \cos \xi - \alpha \sin \xi) - \frac{a'}{2}(\alpha \cos \xi + \beta \sin \xi),$$

$$Q(x, \lambda) = \frac{b(b-a)}{4}(\alpha \cos \xi + \beta \sin \xi) - \frac{b'}{2}(\beta \cos \xi - \alpha \sin \xi),$$

and that

$$(11) \qquad U(0) = -\frac{\alpha a(0)}{2\lambda}, \qquad V(0) = -\frac{\beta b(0)}{2\lambda}.$$

If we multiply equations (8) respectively by $\cos \lambda x$, $-\sin \lambda x$, add, and integrate from 0 to x; and similarly multiply by $\sin \lambda x$, $\cos \lambda x$, add, and integrate, we obtain

$$(12) \quad U \cos \lambda x - V \sin \lambda x$$
$$= \int_0^x [a(s)V(s)\cos \lambda s + b(s)U(s)\sin \lambda s]\, ds + \frac{M}{\lambda},$$

$$U \sin \lambda x + V \cos \lambda x$$
$$= \int_0^x [a(s)V(s)\sin \lambda s - b(s)U(s)\cos \lambda s]\, ds + \frac{N}{\lambda},$$

where

$$(13) \quad \begin{aligned} M(x, \lambda) &= -\tfrac{1}{2}\alpha a(0) + \int_0^x [P(s)\cos \lambda s - Q(s)\sin \lambda s]\, ds, \\ N(x, \lambda) &= -\tfrac{1}{2}\beta b(0) + \int_0^x [P(s)\sin \lambda s + Q(s)\cos \lambda s]\, ds. \end{aligned}$$

Now solving (12) algebraically for the U, V appearing on the left, we have

$$(14) \quad \begin{aligned} U &= \frac{F}{\lambda} + \int_0^x [-b(s)U(s)\sin \lambda(x-s) \\ &\qquad\qquad\qquad + a(s)V(s)\cos \lambda(x-s)]\, ds, \\ V &= \frac{G}{\lambda} + \int_0^x [-b(s)U(s)\cos \lambda(x-s) \\ &\qquad\qquad\qquad - a(s)V(s)\sin \lambda(x-s)]\, ds, \end{aligned}$$

where

$$(15) \quad \begin{aligned} F(x, \lambda) &= M \cos \lambda x + N \sin \lambda x, \\ G(x, \lambda) &= -M \sin \lambda x + N \cos \lambda x. \end{aligned}$$

This is a system of integral equations of Volterra type for U, V:

$$(16) \quad \begin{aligned} U(x) &= \frac{F(x)}{\lambda} + \int_0^x [K_{11}(x, s)U(s) + K_{12}(x, s)V(s)]\, ds, \\ V(x) &= \frac{G(x)}{\lambda} + \int_0^x [K_{21}(x, s)U(s) + K_{22}(x, s)V(s)]\, ds, \end{aligned}$$

where

$$(17) \quad \begin{aligned} K_{11}(x, s, \lambda) &= -b(s)\sin \lambda(x-s), \\ K_{12}(x, s, \lambda) &= a(s)\cos \lambda(x-s), \\ K_{21}(x, s, \lambda) &= -b(s)\cos \lambda(x-s), \\ K_{22}(x, s, \lambda) &= -a(s)\sin \lambda(x-s). \end{aligned}$$

To the kernel system K_{ij} there exists a resolvent system Q_{ij}. The system (16) has the unique solution

(18)
$$U(x) = \frac{F(x)}{\lambda} + \frac{1}{\lambda} \int_0^x [\, Q_{11}(x, s) F(s) + Q_{12}(x, s) G(s)\,]\, ds,$$
$$V(x) = \frac{G(x)}{\lambda} + \frac{1}{\lambda} \int_0^x [\, Q_{21}(x, s) F(s) + Q_{22}(x, s) G(s)\,]\, ds.$$

By (9),
$$P = O(1), \qquad Q = O(1);$$
hence by (13),
$$M = O(1), \qquad N = O(1);$$
hence by (15),
$$F = O(1), \qquad G = O(1);$$
and obviously by (17),
$$K_{ij} = O(1).$$

If we write out the infinite series for the functions Q_{ij} of the resolvent system, analogous to the usual series of iterated kernels of a single kernel, it is clear that

(19)
$$Q_{ij} = O(1).$$

Thus, finally, by (18), we see that
$$U = O\left(\frac{1}{\lambda}\right), \qquad V = O\left(\frac{1}{\lambda}\right);$$

and this result substituted in (7) yields (5), and completes the proof of the theorem.

Furthermore, we have, from (9),
$$\partial P/\partial \lambda = O(1), \qquad \partial Q/\partial \lambda = O(1);$$
from (13),
$$\partial M/\partial \lambda = O(1), \qquad \partial N/\partial \lambda = O(1);$$
from (15),
$$\partial F/\partial \lambda = O(1), \qquad \partial G/\partial \lambda = O(1);$$
and from (17),
$$\partial K_{ij}/\partial \lambda = O(1).$$

Differentiating (16), we have
$$\frac{\partial U(x)}{\partial \lambda} = \frac{S(x)}{\lambda} + \int_0^x \left[K_{11}(x, s) \frac{\partial U(s)}{\partial \lambda} + K_{12}(x, s) \frac{\partial V(s)}{\partial \lambda} \right] ds,$$
$$\frac{\partial V(x)}{\partial \lambda} = \frac{T(x)}{\lambda} + \int_0^x \left[K_{21}(x, s) \frac{\partial U(s)}{\partial \lambda} + K_{22}(x, s) \frac{\partial V(s)}{\partial \lambda} \right] ds.$$

where
$$S(x) = \frac{\partial F(x)}{\partial \lambda} - \frac{F(x)}{\lambda} + \int_0^x \left[\frac{\partial K_{11}(x, s)}{\partial \lambda} U(s) + \frac{\partial K_{12}(x, s)}{\partial \lambda} V(s) \right] ds,$$
$$T(x) = \frac{\partial G(x)}{\partial \lambda} - \frac{G(x)}{\lambda} + \int_0^x \left[\frac{\partial K_{21}(x, s)}{\partial \lambda} U(s) + \frac{\partial K_{22}(x, s)}{\partial \lambda} V(s) \right] ds.$$

Evidently $S = O(1)$, $T = O(1)$. Solving the integral system for $\partial U/\partial\lambda$, $\partial V/\partial\lambda$, we have

$$\frac{\partial U(x)}{\partial\lambda} = \frac{S(x)}{\lambda} + \frac{1}{\lambda}\int_0^x [Q_{11}(x,s)S(s) + Q_{12}(x,s)T(s)]\,ds,$$

$$\frac{\partial V(x)}{\partial\lambda} = \frac{T(x)}{\lambda} + \frac{1}{\lambda}\int_0^x [Q_{21}(x,s)S(s) + Q_{22}(x,s)T(s)]\,ds.$$

Hence

$$\frac{\partial U}{\partial\lambda} = O\left(\frac{1}{\lambda}\right), \qquad \frac{\partial V}{\partial\lambda} = O\left(\frac{1}{\lambda}\right).$$

Combining this result with (7), we have

THEOREM II. *If $[u,v]$ represents the solution of the system* (1), (4), *then*

(20)
$$\frac{\partial u(x)}{\partial\lambda} = x(\beta\cos\xi - \alpha\sin\xi) + O\left(\frac{1}{\lambda}\right),$$

$$\frac{\partial v(x)}{\partial\lambda} = -x(\alpha\cos\xi + \beta\sin\xi) + O\left(\frac{1}{\lambda}\right).$$

Turning now to the system (1), (B_0), (B_1), we seek first a solution of (1), (B_0). Two such solutions for a single value of λ are clearly linearly dependent; it will therefore suffice to choose a single non-trivial solution. Such a solution is obtained by using with (1) the initial conditions

$$u(0) = -\beta_0 = -\sin\theta_0, \qquad v(0) = \alpha_0 = \cos\theta_0.$$

By Theorems I, II we have the

LEMMA I. *For every value of λ a solution of* (1), (B_0) *exists, and satisfies the conditions*:

(21)
$$u(x) = \sin(\xi - \theta_0) + O\left(\frac{1}{\lambda}\right),$$

$$v(x) = \cos(\xi - \theta_0) + O\left(\frac{1}{\lambda}\right),$$

(22)
$$\frac{\partial u(x)}{\partial\lambda} = x\cos(\xi - \theta_0) + O\left(\frac{1}{\lambda}\right),$$

$$\frac{\partial v(x)}{\partial\lambda} = -x\sin(\xi - \theta_0) + O\left(\frac{1}{\lambda}\right).$$

Any other solution of (1), (B_0) *for a fixed value of λ is a constant multiple of this one.*

It is now clear that a non-trivial solution of (1), (B_0), (B_1) will exist only in case the solution just found for (1), (B_0) satisfies also (B_1); this gives a

condition on λ. We define

$$D(\lambda) = \alpha_1 u(1) + \beta_1 v(1) = u(1)\cos\theta_1 + v(1)\sin\theta_1,$$

and note that

$$\frac{dD(\lambda)}{d\lambda} = \frac{\partial u(1)}{\partial\lambda}\cos\theta_1 + \frac{\partial v(1)}{\partial\lambda}\sin\theta_1.$$

Then the condition for a solution of (1), (B_0) to satisfy also (B_1) is that

(23) $$D(\lambda) = 0.$$

If we now evaluate $D(\lambda)$, $dD(\lambda)/d\lambda$ by use of (21), we find that

(24) $$D(\lambda) = \sin(\xi_1 + \theta_1 - \theta_0) + O\left(\frac{1}{\lambda}\right),$$

(25) $$\frac{dD(\lambda)}{d\lambda} = \cos(\xi_1 + \theta_1 - \theta_0) + O\left(\frac{1}{\lambda}\right),$$

where

(26) $$\xi_1 = \xi(1) = \lambda + \tfrac{1}{2}\int_0^1 [a(s) + b(s)]\, ds,$$

or, if we write $\tfrac{1}{2}\displaystyle\int_0^1 [a(s) + b(s)]\, ds + \theta_1 - \theta_0 = -\theta$,

(27) $$D(\lambda) = \sin(\lambda - \theta) + O\left(\frac{1}{\lambda}\right),$$

(28) $$\frac{dD(\lambda)}{d\lambda} = \cos(\lambda - \theta) + O\left(\frac{1}{\lambda}\right).$$

In order to investigate the roots of (23), we first prove

LEMMA II. *If* $[u_1, v_1]$, $[u_2, v_2]$ *represent solutions of* (1), (B_0), (B_1) *for two distinct values* λ_1, λ_2 *respectively of* λ, *then*

$$\int_0^1 [u_1 u_2 + v_1 v_2]\, dx = 0.$$

If we multiply the equations

$$u_1' - (a + \lambda_1) v_1 = 0,$$
$$v_1' + (b + \lambda_1) u_1 = 0,$$
$$u_2' - (a + \lambda_2) v_2 = 0,$$
$$v_2' + (b + \lambda_2) u_2 = 0,$$

respectively by v_2, $-u_2$, $-v_1$, u_1, and add, we have

$$\frac{d}{dx}[u_1 v_2 - u_2 v_1] = (\lambda_1 - \lambda_2)[u_1 u_2 + v_1 v_2];$$

hence

$$(\lambda_1 - \lambda_2) \int_0^1 [u_1 u_2 + v_1 v_2] \, dx = [u_1 v_2 - u_2 v_1]_0^1.$$

But since both solutions satisfy (B_0), (B_1), the expression on the right vanishes. Therefore if $\lambda_1 \neq \lambda_2$, the integral on the left must vanish.

From this result we deduce

LEMMA III. $D(\lambda)$ *has only real roots.*

If λ is a root, to which corresponds the non-trivial solution $[u, v]$, then we see by inspection of (1), (B_0), (B_1) that $\bar{\lambda}$ must also be a root with the solution $[\bar{u}, \bar{v}]$. If $\lambda, \bar{\lambda}$ were distinct, then by Lemma II,

$$\int_0^1 [u\bar{u} + v\bar{v}] \, dx = 0;$$

but this would imply that $u = v = 0$, and the solution would be trivial. Hence $\lambda = \bar{\lambda}$, and λ must be real.

We wish to show also that no root of $D(\lambda)$ can be double. For a double root we should have simultaneously

$$(29) \qquad \begin{aligned} D(\lambda) &= \alpha_1 u(1) + \beta_1 v(1) = 0, \\ \frac{dD(\lambda)}{d\lambda} &= \alpha_1 \frac{\partial u(1)}{\partial \lambda} + \beta_1 \frac{\partial v(1)}{\partial \lambda} = 0. \end{aligned}$$

Hence it would follow that

$$(30) \qquad u(1) \frac{\partial v(1)}{\partial \lambda} - v(1) \frac{\partial u(1)}{\partial \lambda} = 0.$$

If however we multiply the equations

$$(31) \qquad \begin{aligned} u' - (a + \lambda) v &= 0, \\ v' + (b + \lambda) u &= 0, \\ \left(\frac{\partial u}{\partial \lambda}\right)' - (a + \lambda) \frac{\partial v}{\partial \lambda} &= v, \\ \left(\frac{\partial v}{\partial \lambda}\right)' + (b + \lambda) \frac{\partial u}{\partial \lambda} &= -u, \end{aligned}$$

(of which the last pair are obvious consequences of the first pair) respectively by $\partial v/\partial \lambda$, $-\partial u/\partial \lambda$, $-v$, u, add, and integrate from 0 to 1, we find

$$\left[u \frac{\partial v}{\partial \lambda} - v \frac{\partial u}{\partial \lambda} \right]_0^1 = - \int_0^1 [u^2 + v^2] \, dx,$$

which, since $\partial u/\partial \lambda$ and $\partial v/\partial \lambda$ vanish at $x = 0$, reduces to

$$- \int_0^1 [u^2 + v^2] \, dx = u(1) \frac{\partial v(1)}{\partial \lambda} - v(1) \frac{\partial u(1)}{\partial \lambda} = 0,$$

by (30).   But from this it would follow that u and v are identically zero, which is untrue.   Hence

LEMMA IV. $D(\lambda)$ *has only simple roots.*

In order to determine the location of the real roots, we divide the whole range of real values of λ into two sets of intervals:

$$I_n : |\lambda - \theta - n\pi| \leqq \frac{\pi}{4},$$

$$J_n : |\lambda - \theta - (n + \tfrac{1}{2})\pi| \leqq \frac{\pi}{4},$$

where n is any integer.   At the beginning of any interval I_n,

$$\lambda = \theta + (n - \tfrac{1}{4})\pi,$$

and by (27),

$$D(\lambda) = \sin(n - \tfrac{1}{4})\pi + O\left(\frac{1}{\lambda}\right)$$

$$= \frac{(-1)^{n-1}}{\sqrt{2}} + O\left(\frac{1}{\lambda}\right).$$

Similarly at the end of an interval I_n, $\lambda = \theta + (n + \tfrac{1}{4})\pi$, and

$$D(\lambda) = \frac{(-1)^n}{\sqrt{2}} + O\left(\frac{1}{\lambda}\right).$$

For sufficiently large $|\lambda|$, therefore, $D(\lambda)$ has opposite signs at the beginning and end of I_n, and hence vanishes at least once inside I_n.

In any interval J_n. $\lambda = \theta + (n + \tfrac{1}{2})\pi + \delta$, $|\delta| \leqq \pi/4$, and

$$D(\lambda) = (-1)^n \cos \delta + O\left(\frac{1}{\lambda}\right).$$

But since $|\cos \delta| \geqq 1/\sqrt{2}$, it follows that for sufficiently large $|\lambda|$, $D(\lambda) \neq 0$ throughout J_n.

An argument analogous to the preceding shows that for sufficiently large $|\lambda|$, $dD(\lambda)/d\lambda \neq 0$ throughout I_n.   Therefore $D(\lambda)$ cannot have more than one root in each I_n.

Having now proved that for sufficiently large $|\lambda|$, $D(\lambda)$ has one and only one root in each I_n and none outside I_n, we may write for the large roots

(32) $\lambda_n = \theta + n\pi + \epsilon_n$, $|\epsilon_n| \leqq \frac{\pi}{4}$.

This expression, substituted in (27), gives

$$0 = (-1)^n \sin \epsilon_n + O\left(\frac{1}{n}\right),$$

or

$$\sin \epsilon_n = O\left(\frac{1}{n}\right).$$

Since the choice among the angles ϵ_n corresponding to a given $\sin \epsilon_n$ is here uniquely determined by the condition $|\epsilon_n| \leqq \pi/4$, we have

$$\epsilon_n = O\left(\frac{1}{n}\right),$$

and finally from (32),

(33)
$$\lambda_n = \theta + n\pi + O\left(\frac{1}{n}\right).$$

By using this formula for λ_n in (21), we find for the solution $[u_n, v_n]$ corresponding to the root λ_n of $D(\lambda)$,

$$u_n(x) = \sin(\xi_n - \theta_0) + O\left(\frac{1}{n}\right),$$

$$v_n(x) = \cos(\xi_n - \theta_0) + O\left(\frac{1}{n}\right),$$

where ξ_n is the value of ξ for $\lambda = \lambda_n$.

We have seen that two solutions $[u_m, v_m]$, $[u_n, v_n]$ are in a general sense *orthogonal;* that is,

$$\int_0^1 [u_m u_n + v_m v_n] \, dx = 0.$$

We now seek to have each solution *normalized:*

$$\int_0^1 [u_n^2 + v_n^2] \, dx = 1.$$

This can be accomplished by dividing u_n and v_n by the constant

$$\sqrt{\int_0^1 [u_n^2 + v_n^2] \, dx}.$$

But it is clear from the expressions just found for u_n, v_n, that

$$\int_0^1 [u_n^2 + v_n^2] \, dx = 1 + O\left(\frac{1}{n}\right);$$

hence normalization leaves the asymptotic forms for $u_n(x)$, $v_n(x)$ unchanged. We shall continue to use the same notation for the normalized solutions.

We collect the preceding results as follows:

THEOREM III. *The roots of $D(\lambda)$ are real and simple. For sufficiently large values of $|\lambda|$ they have the asymptotic expression*

(33)
$$\lambda_n = \theta + n\pi + O\left(\frac{1}{n}\right),$$

n being an integer.   To each root corresponds one and (except for multiplication by a constant) only one solution $[u_n, v_n]$ of (1), (B_0), (B_1).   These solutions may be taken to form a normal orthogonal set

$$(34) \qquad \int_0^1 [u_m u_n + v_m v_n] \, dx = \begin{cases} 0, \ m \neq n, \\ 1, \ m = n; \end{cases}$$

and have the asymptotic expressions

$$u_n(x) = \sin(\xi_n - \theta_0) + O\left(\frac{1}{n}\right),$$

$$v_n(x) = \cos(\xi_n - \theta_0) + O\left(\frac{1}{n}\right).$$

The non-homogeneous system. Consider now the system (2). Let $[u_0, v_0]$ denote a solution satisfying the initial conditions

$$(35) \qquad\qquad u_0(0) = 0, \qquad v_0(0) = 0.$$

We call $[u_1, v_1]$ the previously discussed solution of (1) satisfying the initial conditions

$$(36) \qquad\qquad u_1(0) = -\sin\theta_0, \qquad v_1(0) = \cos\theta_0,$$

and introduce a second solution $[u_2, v_2]$ of (1) satisfying the initial conditions

$$(37) \qquad\qquad u_2(0) = \cos\theta_0, \qquad v_2(0) = \sin\theta_0.$$

Each of the six functions $u_0, v_0, u_1, v_1, u_2, v_2$ is an integral function of the complex variable λ.   The solutions $[u_1, v_1]$, $[u_2, v_2]$ are linearly independent, since

$$\begin{vmatrix} u_1(0) & v_1(0) \\ u_2(0) & v_2(0) \end{vmatrix} = -1 \neq 0.$$

Recalling the notation

$$(38) \qquad\qquad D(\lambda) = u_1(1)\cos\theta_1 + v_1(1)\sin\theta_1,$$

we also write

$$(39) \qquad\qquad D_0(\lambda) = u_0(1)\cos\theta_1 + v_0(1)\sin\theta_1.$$

Both $D(\lambda)$ and $D_0(\lambda)$ are clearly integral functions of λ.

Any solution of (2) can be written in the form

$$(40) \qquad\qquad \begin{aligned} u &= u_0 + c_1 u_1 + c_2 u_2, \\ v &= v_0 + c_1 v_1 + c_2 v_2, \end{aligned}$$

by proper choice of the numbers c_1, c_2, which are independent of x, but vary with λ.   We seek to choose c_1, c_2 so as to satisfy (B_0), (B_1).   For (B_0) we need merely take $c_2 = 0$.   Placing this value in (40) and writing out

(B_2), we have

(41) $$D_0(\lambda) + c_1 D(\lambda) = 0.$$

In case $D(\lambda) \neq 0$, we see that

$$c_1 = -\frac{D_0(\lambda)}{D(\lambda)},$$

and hence find for (2), (B_0), (B_1) the unique solution

(42)
$$u = u_0 - \frac{D_0(\lambda)}{D_1(\lambda)} u_1,$$
$$v = v_0 - \frac{D_0(\lambda)}{D_1(\lambda)} v_1.$$

Thus we may state

LEMMA V. *If $D(\lambda) \neq 0$, the system (2), (B_0), (B_1) has one and only one solution $[u, v]$. Throughout any region in which $D(\lambda) \neq 0$, u and v are analytic functions of λ.*

If however λ has one of the values λ_n, so that $D(\lambda) = 0$, then (37) cannot be satisfied for any c_1 unless also $D_0(\lambda) = 0$, and is satisfied for every c_1 in case $D_0(\lambda) = 0$. Let us examine the meaning of the simultaneous vanishing of $D(\lambda)$, $D_0(\lambda)$. Writing out (2) for $[u_0, v_0]$ and (1) for $[u_1, v_1]$, multiplying the four equations in order by v_1, $-u_1$, $-v_0$, u_0, adding, and integrating from 0 to 1, we have

$$u_0(1)v_1(1) - v_0(1)u_1(1) = \int_0^1 [\varphi v_1 - \psi u_1]\, dx.$$

But if
$$D(\lambda) = u_1(1)\cos\theta_1 + v_1(1)\sin\theta_1 = 0,$$
$$D_0(\lambda) = u_0(1)\cos\theta_1 + v_0(1)\sin\theta_1 = 0,$$

then
$$u_0(1)v_1(1) - v_0(1)u_1(1) = 0,$$

and hence
$$\int_0^1 [\varphi v_1 - \psi u_1]\, dx = 0,$$

or, since $[u_1, v_1]$ here represents the solution $[u_n, v_n]$ corresponding to the root λ_n of $D(\lambda)$,

(43) $$\int_0^1 [\varphi v_n - \psi u_n]\, dx = 0.$$

Conversely, if this condition is satisfied and $D(\lambda) = 0$, then $D_0(\lambda) = 0$. Thus when $D(\lambda) = 0$, (43) is a necessary and sufficient condition that also $D_0(\lambda) = 0$.

If then (43) holds, the system (2), (B_0), (B_1) still has solutions for $\lambda = \lambda_n$,

which are of the form

$$u = u_0 + c_1 u_n,$$
$$v = v_0 + c_1 v_n,$$

for any c_1. We inquire whether a particular c_1 can be chosen in such a way that the solution $[u, v]$ of (2), (B_0), (B_1) for neighboring values of λ shall join analytically to the solution for λ_n. Since u_0, v_0, u_1, v_1 are analytic for all λ, we see from (42) that it is sufficient that the quotient $D_0(\lambda)/D(\lambda)$ approach a limit as λ approaches λ_n. This will always occur:

$$\lim_{\lambda \to \lambda_n} \frac{D_0(\lambda)}{D(\lambda)} = \frac{D_0'(\lambda_n)}{D'(\lambda_n)},$$

since $D'(\lambda_n) \neq 0$ by Lemma IV. The resulting $[u, v]$, being analytic in λ near λ_n and continuous at λ_n, will be analytic also at λ_n. Thus we have

LEMMA VI. *For $\lambda = \lambda_n$, the system (2), (B_0), (B_1) has solutions if and only if*

$$\int_0^1 [\varphi v_n - \psi u_n] \, dx = 0.$$

If this condition is satisfied, a solution $[u, v]$ can be selected which joins with the unique solution for neighboring values of λ in such a way that u and v are analytic throughout a region surrounding λ_n.

From this result we have further

LEMMA VII. *If for every λ_n,*

$$\int_0^1 [\varphi v_n - \psi u_n] \, dx = 0,$$

then the system (2), (B_0), (B_1) possesses a solution $[u, v]$ such that u and v are integral functions of λ.

Let us now interpret further the meaning of the condition just obtained. Suppose that (43) holds for each λ_n. Then the solution $[u, v]$ of (2), (B_0), (B_1), being analytic for all λ by Lemma VII, can be written

$$u = \sum_{n=0}^{\infty} \lambda^n y_n(x),$$

(44)

$$v = \sum_{n=0}^{\infty} \lambda^n z_n(x),$$

where the series converge, uniformly in x, for all λ in the complex plane and for $0 \leqq x \leqq 1$. It is easily seen that y_n, z_n are in all cases real. Substituting in (2) and equating coefficients, we have

(45)
$$y_n' - az_n = \quad\; z_{n-1},$$
$$z_n' + by_n = -\, y_{n-1}$$

for all values of n, provided we agree to define

$$(46) \qquad\qquad y_{-1} = -\psi, \qquad z_{-1} = \varphi.$$

Write (45) and a similar pair of equations obtained by putting m for n; multiply these four equations respectively by z_m, $-y_m$, $-z_n$, y_n, add, and integrate from 0 to 1; we have

$$\int_0^1 [y_{m-1}\, y_n + z_{m-1}\, z_n]\, dx = \int_0^1 (y_m\, y_{n-1} + z_m\, z_{n-1})\, dx.$$

By repeated application of this result it is seen that if we define

$$(47) \qquad\qquad W_r = \int_0^1 [y_0\, y_r + z_0\, z_r]\, dx,$$

then

$$(48) \qquad\qquad \int_0^1 [y_m\, y_n + z_m\, z_n]\, dx = W_{m+n}.$$

Integrating the inequality

$$\left| \begin{matrix} y_{m+1}(x) & y_{m+1}(\xi) \\ y_{m-1}(x) & y_{m-1}(\xi) \end{matrix} \right|^2 + \left| \begin{matrix} y_{m+1}(x) & z_{m+1}(\xi) \\ y_{m-1}(x) & z_{m-1}(\xi) \end{matrix} \right|^2$$

$$+ \left| \begin{matrix} z_{m+1}(x) & y_{m+1}(\xi) \\ z_{m-1}(x) & y_{m-1}(\xi) \end{matrix} \right|^2 + \left| \begin{matrix} z_{m+1}(x) & z_{m+1}(\xi) \\ z_{m-1}(x) & z_{m-1}(\xi) \end{matrix} \right|^2 \geqq 0$$

with respect to both x and ξ between the limits 0 and 1, we have

$$(49) \qquad\qquad W_{2m-2}\, W_{2m+2} - W_{2m}^2 \geqq 0.$$

If we multiply (44) respectively by y_0, z_0 and integrate from 0 to 1, we have the series

$$\sum_{n=0}^{\infty} W_n \lambda^n,$$

convergent for all λ. Omitting alternate terms, we have the series

$$(50) \qquad\qquad W_0 + \lambda^2\, W_2 + \lambda^4\, W_4 + \cdots,$$

also (since the convergence of a power series is absolute) convergent for all λ. By (48),

$$W_{2m} = \int_0^1 [y_m^2 + z_m^2]\, dx \geqq 0.$$

We shall show that some $W_{2m} = 0$. If $W_{2m} \neq 0$ for all m, then by (48), $W_{2m} > 0$, and by (49),

$$\frac{W_{2m-2}}{W_{2m}} \geqq \frac{W_{2m}}{W_{2m+2}}.$$

Hence

$$\frac{W_0}{W_2} \geqq \frac{W_2}{W_4} \geqq \frac{W_4}{W_6} \geqq \cdots.$$

Therefore the series (50) could not converge for $\lambda = \sqrt{W_0/W_2}$, since for that value the terms of (50) would be respectively greater numerically than the terms of the series

$$W_0 + W_0 + W_0 + \cdots.$$

Thus, for some m, $W_{2m} = 0$; that is, by (48),

$$\int_0^1 [y_m^2 + z_m^2] \, dx = 0,$$

whence

$$y_m = z_m = 0;$$

and, by repeated applications of (45),

$$y_{m-1} = z_{m-1} = 0,$$
$$y_{m-2} = z_{m-2} = 0,$$
$$\vdots \qquad \cdots \qquad \cdots$$
$$y_0 \quad = z_0 \quad = 0,$$
$$\varphi \quad = \psi \quad = 0.$$

Thus we have

THEOREM IV. *If $\varphi(x)$, $\psi(x)$ are continuous functions such that for every solution $[u_n, v_n]$ of (1), (B_0), (B_1),*

$$\int_0^1 [\varphi v_n - \psi u_n] \, dx = 0,$$

then

$$\varphi = \psi = 0.$$

Expansion of an arbitrary function-pair. Let $f(x)$, $g(x)$ be a pair of continuous functions, and let there exist the simultaneous expansions

(51)
$$f(x) = \sum_{n=-\infty}^{+\infty} c_n u_n(x),$$
$$g(x) = \sum_{n=-\infty}^{+\infty} c_n v_n(x).$$

If both series converge uniformly, then the coefficients can be determined by multiplying respectively by $u_n(x)$, $v_n(x)$, adding, and integrating from 0 to 1. In view of the normal orthogonality of the set of $[u_n, v_n]$, this process gives

(52)
$$c_n = \int_0^1 [f u_n + g v_n] \, dx.$$

If term-wise differentiation of (51) is permissible, it is clear that f, g must satisfy also the boundary conditions

(C_0) $\qquad\qquad\qquad\qquad \alpha_0 f(0) + \beta_0 g(0) = 0,$

(C_1) $\qquad\qquad\qquad\qquad \alpha_1 f(1) + \beta_1 g(1) = 0.$

We now proceed to investigate the convergence of the two series (51), in which the coefficients are determined by (52) in terms of two given functions $f(x)$, $g(x)$. It is assumed that f and g have continuous second derivatives. Using (1), and integrating by parts twice, we have

$$\int_0^1 f u_n \, dx = - \int_0^1 \frac{f v_n'}{\lambda_n + b} \, dx$$

$$= \left[-\frac{f v_n}{\lambda_n + b} \right]_0^1 + \int_0^1 v_n \frac{d}{dx} \frac{f}{\lambda_n + b} \, dx$$

$$= \left[-\frac{f v_n}{\lambda_n + b} \right]_0^1 + \int_0^1 \frac{u_n'}{\lambda_n + a} \frac{d}{dx} \frac{f}{\lambda_n + b} \, dx$$

$$= \left[-\frac{f v_n}{\lambda_n + b} + \frac{u_n}{\lambda_n + a} \frac{d}{dx} \frac{f}{\lambda_n + b} \right]_0^1$$

$$\qquad\qquad\qquad - \int_0^1 u_n \frac{d}{dx} \left\{ \frac{1}{\lambda_n + a} \frac{d}{dx} \frac{f}{\lambda_n + b} \right\} dx .$$

Similarly,

$$\int_0^1 g v_n \, dx = \left[\frac{g u_n}{\lambda_n + a} + \frac{v_n}{\lambda_n + b} \frac{d}{dx} \frac{g}{\lambda_n + a} \right]_0^1$$

$$\qquad\qquad\qquad - \int_0^1 v_n \frac{d}{dx} \left\{ \frac{1}{\lambda_n + b} \frac{d}{dx} \frac{g}{\lambda_n + a} \right\} dx .$$

By direct computation, we find that

$$\frac{1}{\lambda_n + a} \frac{d}{dx} \frac{f}{\lambda_n + b} = O\left(\frac{1}{\lambda_n^2} \right) = O\left(\frac{1}{n^2} \right),$$

$$\frac{1}{\lambda_n + b} \frac{d}{dx} \frac{g}{\lambda_n + a} = O\left(\frac{1}{\lambda_n^2} \right) = O\left(\frac{1}{n^2} \right),$$

$$\frac{d}{dx} \left\{ \frac{1}{\lambda_n + a} \frac{d}{dx} \frac{f}{\lambda_n + b} \right\} = O\left(\frac{1}{\lambda_n^2} \right) = O\left(\frac{1}{n^2} \right),$$

$$\frac{d}{dx} \left\{ \frac{1}{\lambda_n + b} \frac{d}{dx} \frac{g}{\lambda_n + a} \right\} = O\left(\frac{1}{\lambda_n^2} \right) = O\left(\frac{1}{n^2} \right).$$

Hence

$$\int_0^1 f u_n \, dx = \left[-\frac{f v_n}{\lambda_n + b} \right]_0^1 + O\left(\frac{1}{n^2} \right),$$

$$\int_0^1 g v_n \, dx = \left[\frac{g u_n}{\lambda_n + a} \right]_0^1 + O\left(\frac{1}{n^2} \right),$$

and

$$c_n = \left[-\frac{f v_n}{\lambda_n + b} + \frac{g u_n}{\lambda_n + a} \right]_0^1 + O\left(\frac{1}{n^2}\right).$$

But

$$-\frac{f v_n}{\lambda_n + b} + \frac{g u_n}{\lambda_n + a} = \frac{-f v_n + g u_n}{\lambda_n} + O\left(\frac{1}{n^2}\right),$$

and

$$c_n = \left[-\frac{f v_n + g u_n}{\lambda_n} \right]_0^1 + O\left(\frac{1}{n^2}\right).$$

In case f, g satisfy (C_0), (C_1),

$$[-f v_n + g u_n]_0^1 = 0,$$

and

$$c_n = O\left(\frac{1}{n^2}\right).$$

As the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges, and as

$$u_n = O(1), \qquad v_n = O(1),$$

both the series (51) converge uniformly. It remains to be seen whether they represent f and g. Write

(53)
$$F(x) = \sum_{n=-\infty}^{+\infty} c_n u_n(x),$$

$$G(x) = \sum_{n=-\infty}^{+\infty} c_n v_n(x).$$

From (53),

$$c_n = \int_0^1 [F u_n + G v_n] \, dx,$$

while also by (52),

$$c_n = \int_0^1 [f u_n + g v_n] \, dx.$$

Hence, subtracting,

$$\int_0^1 [(F - f) u_n + (G - g) v_n] \, dx = 0,$$

and by Theorem IV,

$$F = f, \qquad G = g.$$

Two details which have been neglected in the above proof require further consideration. In the first place, the use of $\lambda_n + a$, $\lambda_n + b$ in denominators is permissible, since for sufficiently large λ_n, they do not vanish. Secondly, u_n, v_n have been used above with the understanding that the subscript denotes

the serial order of the solution in question, while in the earlier part of the paper, u_n, v_n have indicated the solution corresponding to the root λ_n of $D(\lambda)$ lying in I_n. Since for sufficiently large λ one and only one root of $D(\lambda)$ lies in each I_n, the two interpretations of the subscript differ only by a constant, so that the evaluation $\lambda_n = O(n)$ is still correct. We have thus finally proved

THEOREM V. *If* $f(x)$, $g(x)$ *are any two functions possessing continuous second derivatives,* $0 \leqq x \leqq 1$, *and satisfying the boundary conditions* (C_0), (C_1), *then they may be simultaneously expanded in the uniformly convergent series:*

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n u_n(x),$$

$$g(x) = \sum_{n=-\infty}^{+\infty} c_n v_n(x),$$

where

$$c_n = \int_0^1 [fu_n + gv_n]\,dx.$$

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AUTHOR'S CORRECTION

O. E. Glenn. *A memoir upon formal invariancy with regard to binary modular transformations. Invariants of relativity* (T r a n s a c t i o n s, vol. 21, pp. 285–312).

(1) My attention was directed by Dr. W. L. G. Williams to the existence of a certain seminvariant of f_4 modulo 3, of degree four, which is not reducible in terms of the nine seminvariants given on page 295 and stated there, in a theorem, to form a complete system. The seminvariant is of a type for which the starred assumption on page 296 is not valid. The error makes my result for the seminvariants of the quartic modulo 3 much less general than is indicated in the theorem. However, when the system is completed by discovery of the requisite new forms my starred assumption can perhaps be proved for it, and thus my theory would be completed.

(2) The following quadratic covariant was inadvertently omitted from the lists (35), (40), (47):

$$g = sx_1^2 + (s + b_0 b_1 + b_0^2)x_1 x_2 + b_0(b_0 + b_1 + b_2)x_2^2.$$