ON TRANSFORMATIONS WITH INVARIANT POINTS*

BY

J. W. ALEXANDER

1. There is a well known theorem by Brouwer† to the effect that every sense-preserving transformation of a spherical surface into itself such that both the direct transformation and its inverse are one-valued leaves invariant at least one point of the surface. It is proposed to prove a somewhat more general theorem by a simplified process and to outline one or two applications of the theorem in its extended form. The theorem and its proof both generalize automatically to any number of dimensions.

The results contained in this paper were obtained by the author a number of years ago but never published. Later, a more concise method of presentation suggested itself and made it possible to condense the argument into the form given below. Analogous theorems have recently been worked out independently by Professors Birkhoff and Kellogg as a first step towards the solution of some important problems on functional transformations.‡

2. **The transformation $S_1 : S_2$.** Let S_1 be the sphere

$$x^2 + y^2 + z^2 = 1$$

in real 3-space. It will be convenient to express the coördinates of a point P_1 of S_1 by means of two parameters u and v, such, for example, as the angles of latitude and longitude in ordinary polar coördinates. By an image S_2 of the sphere S_1 will be meant the locus in 3-space of a point $P_2(u, v)$ which varies continuously with the point $P_1(u, v)$ of S_1. The image S_2 will thus be defined by certain parametric equations

$$x = X(u, v), \quad y = Y(u, v), \quad z = Z(u, v). \tag{1}$$

We shall not exclude the case where the transformation determined by (1) carries two or more points of S_1 into the same point (x, y, z); in fact, in an ex-

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† L. E. J. Brouwer. *Ueber eineindeutige, stetige Transformationen von Flächen in sich*, Mathematische Annalen, vol. 69 (1910). Also, Proceedings of the Section of Sciences Royal Academy at Amsterdam, vol. 13 (1911).

‡ G. D. Birkhoff and O. D. Kellogg, *Invariant points in function space*, these Transactions, vol. 23 (1922).

treme case, it may carry the entire surface S_1 into a single point. However, we shall make a convention similar to the one adopted in the theory of Riemann surfaces to the effect that a point of S_2 is not determined by position alone but also by association with a definite point of S_1, as determined by the relations (1). With this extension of language, there will be exactly one point of S_2 corresponding to each point of S_1.

3. **The index of the transformation $S_1 : S_2$.** Let us assume provisionally that the image S_2 of S_1 consists of a finite number of analytic pieces, so that the Gaussian integral

$$(2)\qquad \iint \begin{vmatrix} x & y & z \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} \frac{dudv}{r^3}, \qquad r = +\sqrt{x^2+y^2+z^2},$$

evaluated over S_2 has a meaning.* It is not difficult to see that this integral measures the solid angle subtended at the origin by the surface S_2 when the customary conventions are adopted as to the sign of the angle subtended by each surface element. For our present purposes, we merely notice that the conditions of integrability

$$(3)\qquad \frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r^3}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r^3}\right) = 0$$

are satisfied identically except at the origin and, consequently, that the value of the integral (2) remains unaltered when the surface S_2 is deformed continuously without meeting the origin, provided, of course, that the surface remains sufficiently regular so that the integral does not become meaningless. If S_2 crosses the origin, however, the integrand undergoes a discontinuity, and the integral changes abruptly by $\pm 4\pi$, the value obtained by integrating (2) over a small sphere with center at the origin. Evidently, therefore, the value of the integral over S_2 is $4k\pi$, where k is some integer denoting the difference between the number of times that a general ray issuing from the origin cuts the surface from the negative to the positive side and the number of times that it cuts the surface from the positive to the negative side.

In particular, the value of (2) integrated over S_1 itself is $\pm 4\pi$. We shall select the parameters u and v in such a way as to make this last number positive. The quantity k determined by the integration of (2) over S_2 for the same choice

* Cf. the note by Hadamard appended to Tannery's *Introduction à la Théorie des Fonctions d'une Variable.*

of parameters is then a function of the transformation $S_1 : S_2$ which remains invariant under a continuous deformation of S_2 which does not carry S_2 into contact with the origin. We shall call k the *index* of the transformation.

Since the conditions of integrability (3) are satisfied, we may do away with the restriction that the surface S_2 be composed of a finite number of analytic pieces and give the integral (2) a meaning, by extension, over any image S_2 of S_1 which does not meet the origin. For we shall prove that S_2 may always be approximated to any degree of accuracy by a second image S_2' of the analytic type and that the values of (2) for any two sufficiently good approximations of S_2 will always be the same. We may therefore say, by definition, that the value of the integral (2) over a general image S_2 is the same as its value over a sufficiently good approximating surface S_2'.

One obvious way of setting up the approximating surfaces S_2' is the following. Let α be any positive number less than the distance from the origin to the nearest point of S_2 to the origin. Then, owing to the uniform continuity of the transformation $S_1 : S_2$, it will be possible to cut up the sphere S_1 into a finite system of non-overlapping spherical triangles such that the images in S_2 of any two points from the interior and boundary of the same triangle are always within a distance of $\alpha/8$ from one another. Suppose, in the first instance, that the images of no three vertices of a spherical triangle are collinear. We may then approximate the image S_2 of S_1 by a second image S_2' consisting of plane triangular pieces, one corresponding to each spherical triangle of S_1 and such that the vertices of each plane triangle coincide with the images of the vertices of the corresponding spherical traingle. We thereby obtain an approximating surface S_2' composed of triangular regions (which may, of course, cut into one another) and such that each point of S_2' is within a distance of $\alpha/4$ from the corresponding point of S_2. When the images of the three vertices of a spherical triangle of S_1 are collinear, we may replace each of the images by a nearby point within a distance $\alpha/16$ of the original and so locate the new points that no three are collinear. We then proceed as before and build up an approximating surface S_2' out of triangles with vertices at the new points. This time, each point of S_2' will be within a distance of $\alpha/2$ from the corresponding point of S_2. It is therefore possible in every case to approximate the image S_2 to any degree of accuracy by a second image S_2' made up of analytic pieces (triangular regions, in fact).

Finally, the value of the integral (2) over two surfaces T and U which approximate the surface S_2 to within the accuracy of the surface S_2' defined above must be the same. For each point (x', y', z') of T is within a distance α of the corresponding point (x'', y'', z'') of U; consequently, the surface T may be deformed into the surface U without crossing the origin. Such a deformation could be defined, for instance, by the equations

$$x = \lambda x'' + (1-\lambda)x', \quad y = \lambda y'' + (1-\lambda)y', \quad z = \lambda z'' + (1-\lambda)z',$$

where λ is allowed to vary from 0 to 1.

It follows at once from the above that we may define the index of any transformation $S_1 : S_2$ provided S_2 does not meet the origin, and that the index remains invariant under the most general continuous deformation of S_2 which does not bring this surface into contact with the origin.

4. The extended form of Brouwer's theorem. We shall be concerned more particularly with the case where every point of the image S_2 coincides with a point of S_1, so that the transformation $S_1 : S_2$ carries the sphere S_1 into itself or a part of itself. Such a transformation is extremely general: it may have any index, and its inverse need neither be one-to-one or even everywhere defined on the sphere. It will be shown that

The transformation $S_1 : S_2$ must have an invariant point if its index is not equal to —1.

For let the point (x_1, y_1, z_1) of S_1 be paired with the point (x_2, y_2, z_2) of S_2 under the transformation $S_1 : S_2$. Then, if we put

$$x = x_1 - x_2, \quad y = y_1 - y_2, \quad z = z_1 - z_2, \quad r = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2},$$

the integral (2) assumes the form

$$\iint \begin{vmatrix} x_1 - x_2 & y_1 - y_2 & z_1 - z_2 \\ \dfrac{\partial}{\partial u}(x_1 - x_2) & \dfrac{\partial}{\partial u}(y_1 - y_2) & \dfrac{\partial}{\partial u}(z_1 - z_2) \\ \dfrac{\partial}{\partial v}(x_1 - x_2) & \dfrac{\partial}{\partial v}(y_1 - y_2) & \dfrac{\partial}{\partial v}(z_1 - z_2) \end{vmatrix} \frac{du\,dv}{r^3}, \tag{4}$$

which has a perfectly definite meaning provided r does not vanish, that is, provided the transformation $S_1 : S_2$ has no fixed point. Furthermore, (4) remains invariant under any continuous deformation of either S_1 or S_2 which does not bring a point of one surface into coincidence with its image on the other.

Now, consider the following two deformations:

(i) S_1 is held fixed but S_2 is shrunk into coincidence with the origin by a radial contraction. Then, in the limit, the integral (4) reduces to

$$\iint \begin{vmatrix} x_1 & y_1 & z_1 \\ \dfrac{\partial x_1}{\partial u} & \dfrac{\partial y_1}{\partial u} & \dfrac{\partial z_1}{\partial u} \\ \dfrac{\partial x_1}{\partial v} & \dfrac{\partial y_1}{\partial v} & \dfrac{\partial z_1}{\partial v} \end{vmatrix} \frac{du\,dv}{r^3} = 4\pi. \tag{5}$$

(*ii*) S_2 is held fixed but S_1 is shrunk into coincidence with the origin. The integral (4) then becomes

$$
(6) \qquad -\int\!\!\int \begin{vmatrix} x_2 & y_2 & z_2 \\ \frac{\partial x_2}{\partial u} & \frac{\partial y_2}{\partial u} & \frac{\partial z_2}{\partial u} \\ \frac{\partial x_2}{\partial v} & \frac{\partial y_2}{\partial v} & \frac{\partial z_2}{\partial v} \end{vmatrix} \frac{du\,dv}{r^3} = -\,4k\pi,
$$

where k is the index of the transformation $S_1 : S_2$. Equating (5) and (6), we have at once $k = -1$ which proves that if there is no fixed point the index of the transformation must be -1.

5. Generalization to n dimensions. Evidently, both theorem and proof generalize at once to any number of dimensions by the use of the integral

$$
(7) \qquad \underbrace{\int \cdots \int}_{(n)} \begin{vmatrix} x_0 & \cdots\cdots & x_n \\ \frac{\partial x_0}{\partial u_1} & \cdots\cdots & \frac{\partial x_n}{\partial u_1} \\ \cdot & \cdot\ \cdot\ \cdot\ \cdot & \cdot \\ \frac{\partial x_0}{\partial u_n} & \cdots\cdots & \frac{\partial x_n}{\partial u_n} \end{vmatrix} \frac{du_1 \ldots du_n}{r^{n+1}} = 2^n k\pi.
$$

It will be noticed, however, that the negative sign before the integral (6) will reappear in the generalizations to even values of n only, since it depends on the fact that the determinant in the integrand has an odd number of rows and columns. The form of the generalized theorem is, therefore, as follows:

Any one-valued continuous transformation $S_1 : S_2$ of an n-sphere

$$
x_0^2 + x_1^2 + \cdots + x_n^2 = 1
$$

into itself has an invariant point provided the index of the transformation is not $(-1)^{n+1}$.

6. Applications. Certain other theorems by Brouwer may now be proved in somewhat extended form.

Every one-valued continuous transformation of real projective n-space of even dimensions into itself or a part of itself has at least one invariant point. (In this and in the following theorem, the inverse transformation need not be single-valued.)

This theorem follows at once from the homeomorphism existing between the points of projective n-space and the point pairs of an n-sphere forming the extremities of the diameters of the n-sphere. (To set up the homeomorphism, we need only project the n-sphere from its center upon a projective n-plane.)

To every transformation of projective n-space into itself, there evidently correspond two transformations of the n-sphere into itself differing from one another by a reflection in the center of the sphere. But, if the number n is even, such a reflection changes the sign of the index, so that one of the two transformations surely has an index different from $(-1)^{n+1}$. This transformation therefore has a fixed point belonging to a fixed point pair, which means that the corresponding transformation of projective n-space also has a fixed point.

Every one-valued continuous transformation of the interior and boundary of an $(n-1)$*-sphere into itself or a part of itself has an invariant point.*

For we may assume that the $(n-1)$-sphere is situated in n-space of inversion which is homeomorphic with an n-sphere. It is then possible to extend the definition of the transformation to the entire n-space by specifying that every point exterior to the $(n-1)$-sphere shall be carried into the same point as its reflection within the sphere. Now, the transformation thus defined carries the entire n-space into a set of points bounded by the $(n-1)$-sphere. It is therefore of index 0 and consequently has an invariant point. Obviously, the fixed point is on or interior to the $(n-1)$-sphere, since all exterior points are carried into interior points or points on the $(n-1)$-sphere.

On an n*-sphere of even dimensions, there cannot exist a one-valued vector distribution which is everywhere continuous.*

For if there existed such a distribution, the points P of the n-sphere could all be displaced by a constant amount $\delta < 2\pi$ along great circles tangent to the vectors at the points, and the n-sphere would therefore undergo a transformation into itself without fixed point. But the index of this transformation would clearly be $+1$, since we could pass continuously from S_1 to S_2 by allowing all the points P to travel at a uniform rate along their respective great circles. We would thus be led to a contradiction with the theorem of §5.

7. The extended form of Brouwer's theorem also lends itself to a number of new applications. We indicate one of them by way of illustration.

Let R *be the interior and boundary of a finite region of the plane bounded by* p *non-intersecting circles, one enclosing all the others. We consider a transformation of* R *into itself which is continuous on* R*, which has a one-valued inverse on the boundary of* R*, and which carries each of the bounding circles into itself with preservation of sense. Then the transformation always has an invariant point if* p *is not equal to* 2.

For, here again, we may extend the transformation over the entire plane in such a way that the exterior of the outer circle is carried into its interior and the interiors of the $p-1$ inner circles into their exteriors. We then obtain a transformation of the plane of inversion which is of index $1-p$. Consequently, the transformation has a fixed point if p is not 2. Clearly, also, the fixed point is a point of R.

It would be easy to generalize the last theorem to n dimensions or to examine the case where the bounding circles are permuted, transformed into themselves with given indices, and so on. However, it seems hardly necessary to dwell on such obvious extensions.

PRINCETON UNIVERSITY,
PRINCETON, N. J.