A PROOF AND EXTENSION OF THE JORDAN-BROUWER SEPARATION THEOREM*

BY

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1. The theorem on the separation of n-space by an $(n - 1)$-dimensional manifold† suggests the following more general problem of analysis situs.

Given a figure C of known connectivity immersed in an n-space H, what can be said about the connectivity of the domain H–C residual to C?

It will be shown that a certain duality exists between the topological invariants of C and H–C, and that when C is an $(n - 1)$-dimensional manifold the separation theorem is merely one aspect of this duality. The paper also touches upon a number of well known related questions,—among them, the invariance of dimensionality and regionality, the approachability of points of C from H–C, the invariance of the topological constants of C and H–C, and so on.

Of course, the main difficulties in such problems as the above are of a point-theoretic order. They all yield, however, to simple pinching processes, except for the use of which the following treatment will be purely combinatorial. The earlier sections, §§ 2–8, are expository and give a rapid though essentially complete survey of the terminology and combinatorial machinery needed in the sequel. The fundamental part of the discussion, with illustrations and applications, is really all contained in §§ 9–12.

The theory of connectivity may be approached from two different angles depending on whether or not the notion of *sense* is developed and taken into consideration. We have adopted the second and somewhat simpler point of view in this discussion in order to condense the necessary preliminaries as much as possible. A treatment involving the idea of sense would be somewhat more complicated but would follow along much the same lines.

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† First completely proved for the case $n = 1$ (Jordan's theorem) by O. Veblen, these Transactions, vol. 6 (1905), p. 83, cf. addendum, vol. 14 (1913), p. 65 (second footnote). Proved for a general n by L. E. J. Brouwer, Mathematische Annalen, vol. 71 (1912), p. 314. A proof of Jordan's theorem along lines similar to those used in this paper was given by the author in the Annals of Mathematics, vol. 21 (1920), p. 180.

CHAINS AND THEIR COMBINATORIAL PROPERTIES

2. Certain advantages of symmetry are gained by setting the problem in the space of n-dimensional spherical geometry rather than in euclidean n-space. We shall therefore take as our fundamental domain the n-sphere

$$H^n: \qquad x_0^2 + x_1^2 + \cdots + x_n^2 = 1$$

in the space of $n + 1$ real variables. The geodesics (great circles) determined on the n-sphere H^n by its intersection with $n - 1$ linearly independent n-planes through the origin play the rôle in this geometry of the straight lines in ordinary euclidean space. A region will be said to be *convex* if any two of its points may be joined by one and only one geodesic arc made up of points of the region.

An n-plane through the origin subdivides the n-sphere H^n into a pair of n-regions bounded and separated by an $(n - 1)$-sphere. The latter may in turn be subdivided in the same way into a pair of $(n - 1)$-regions separated by an $(n - 2)$-sphere, and so on down to a pair of 0-regions, or points. The resulting partition of the n-sphere, consisting of two regions of every dimensionality from 0 to n, will be called an *elementary subdivision* of H^n. It is evident that any k-region of an elementary subdivision may be cut up by an n-plane through the origin into a pair of convex k-regions separated by a convex $(k - 1)$-region and that by repeating this process of repartitioning, the n-sphere may be cut up into arbitrarily small convex regions of dimensionalities 0 to n. These regions will be called k-cells, where k denotes dimensionality.

If the repartitioning is done in a perfectly random fashion, there is nothing to prevent the boundary of a k-cell from containing a part but not all of a cell of lower dimensionality. It is then possible to carry the repartitioning still further, beginning with the boundaries of the n-cells and working down to the boundaries of the 1-cells, until, finally, the boundary of each cell consists of complete cells only. The resulting collection of cells will be called a *subdivision* of the n-sphere H^n.

A subdivision S' will be said to be *derived* from a subdivision S if it can be obtained from S without the use of any other operations than the repartitioning of cells in the manner just described. Thus, by definition, every subdivision is derived from an elementary one. It should be observed that the operations of repartitioning may always be performed in such an order that all intermediate figures will themselves be subdivisions. This would be the case, for example, if the cells were repartitioned in such an order that no one of them was ever touched until the cutting up of its boundary had been completed.

3. There is, of course, no difficulty in writing out explicitly the analytical expressions that determine a cell. They consist merely of the equation of H^n taken in conjunction with certain linear equalities and inequalities depending on

the n-planes that cut out the cell. From the form of these expressions, it follows at once that every cell is convex. Consequently, any two i-cells are *homeomorphic* with one another (i.e., in point-for-point continuous correspondence), and the boundary of any i-cell is homeomorphic with an $(i - 1)$-sphere, since the latter bounds each of two i-cells on an i-sphere.

4. Any set of cells from the same subdivision of H^n will be called a *chain* provided the set never contains a cell E without also containing all cells on the boundary of E. A chain may therefore be a very mixed agglomeration of cells. If, however, it consists only of i-cells and the cells of their boundaries, it will be called an *i-chain*.

The simplest i-chain is one containing a single i-cell. It will be called a *cellular i-chain*. Any i-chain will be said to be the *sum modulo* 2 of the cellular i-chains determined by its individual i-cells; in symbols,

$$K^i = K_1^i + K_2^i + \cdots + K_s^i \qquad \text{(mod. 2)} \tag{1}$$

We shall also speak of the sum of two or more arbitrary i-chains of the subdivision, and it is here that the *modulo* 2 feature of the operation first comes into evidence. To form the sum, we express each i-chain in terms of its cellular components, as in (1), add components, and reduce coefficients modulo 2. In other words, the sum-chain contains the i-cells that belong to an odd number of chains of the sum, but no others.

5. An i-chain K^i $(i > 0)$ will be said to be *closed* if each of its $(i - 1)$-cells belongs to the boundary of an even number of its i-cells; otherwise, it will be called *open*, or *bounded*, and its boundary K^{i-1} will be the $(i - 1)$-chain determined by such $(i - 1)$-cells as belong to the boundary of an odd number of i-cells of K^i. It will be convenient to express the relation of K^{i-1} to K^i symbolically by the notation (adapted from the congruences of Poincaré)

$$K^i \equiv K^{i-1} \qquad \text{(mod 2)} \tag{2}$$

which may be read "K^i is bounded by K^{i-1}". The expression

$$K^i \equiv 0 \qquad \text{(mod 2)}$$

signifies that K^i has no boundary and is therefore closed. We shall frequently condense the notations (1) and (2) by omitting to write mod 2.

A 0-chain will be open or closed according as it consists of an odd or an even number of points.

The boundary K^{i-1} of an i-cell is a simple illustration of a closed $(i - 1)$-chain, since every $(i - 2)$-cell of K^{i-1} belongs to precisely two $(i - 1)$-cells of K^{i-1}

Now, the relations

$$K_s^i \equiv K_s^{i-1} \qquad (s = 1, 2, \ldots, t) \qquad \pmod 2$$

evidently imply

$$\sum_s K_s^i \equiv \sum_s K_s^{i-1} \qquad \pmod 2$$

or, in words, the boundary of a sum of i-chains is the sum of the boundaries of the i-chains themselves. Thus, in particular, the sum of two or more closed i-chains is itself closed, when it does not vanish. It also follows that the boundary of an open $(i+1)$-chain is a closed i-chain, since it is the sum of the boundaries of the individual $(i+1)$-cells of the $(i+1)$-chain.

6. We proceed to define the connectivity numbers of a chain C. Let K^i be an i-chain of C, that is to say, a chain composed of cells of C. Then, by a second adaptation from Poincaré, we shall write

$$K^i \sim 0 \qquad (\mathrm{mod}\ 2,\ C) \tag{3}$$

(K^i is *homologous to zero*, or *bounds* on C) provided K^i is the boundary of some open $(i+1)$-chain of C. The relations $K_1^i \sim 0$ and $K_2^i \sim 0$ evidently imply $K_1^i + K_2^i \sim 0$; therefore, it will be legitimate to operate with homologies as though they were linear equations modulo 2. The expression $K_1^i \sim K_2^i$ will, of course, be just another way of writing $K_1^i + K_2^i \sim 0$. We denote by $R^i - 1$ the maximum number of closed non-bounding i-chains $K_1^i, K_2^i, \ldots,$ of C that are independent with respect to homologies; that is, such that there exists no relation between the chains K_s^i of the form

$$\sum_{s=1}^{R^i-1} \epsilon_s K_s^i \sim 0 \qquad (\mathrm{mod}\ 2,\ C)$$

unless the coefficients ϵ_s are all zero, and such that every other closed i-chain K^i of C is related to the chains K_s^i by an homology

$$K^i \sim \sum \epsilon_s K_s^i \qquad (\mathrm{mod}\ 2,\ C).$$

The number R^i is called the *ith connectivity number* of C.* It will be seen later on that R^i is not only an invariant of C but also of the set of points determined by C. For the moment, it will be sufficient to observe that R^0 denotes the number of separate connected parts of C. It is sometimes advantageous to

* The numbers R^i are the modulo 2 analogues of the Betti numbers of Poincaré. They were first introduced in a paper by Professor Veblen and the author, Annals of Mathematics, vol. 14 (1913), p. 163.

consider connectivity numbers of higher dimensionalities those that of any cell of C. Such numbers are automatically unity, from their definition.

The connectivity number R^i of C which is of the same dimensionality i as the cells of highest dimensionality appearing in C satisfies a relation that we shall now recall for future reference. Let there be α i-cells

$$A_1^i, A_2^i, \ldots, A_\alpha^i$$

in C, and let the symbols associated with these i-cells be regarded as variables free to take on either of the values 0 or 1. Then, to every choice of a set of values for these variables such that at least one variable is not zero, there may be associated an i-chain of C determined by the i-cells with symbols unity. Conversely, by reversing the process, to every i-chain of C there may be associated a set of values of the variables. Now, if $i > 0$, let us write a modular equation

$$E_q: \qquad \sum_p \epsilon_{qp} A_p^i \equiv 0 \pmod{2}$$

corresponding to each $(i-1)$-cell A_q^{i-1} of C, where the coefficient ϵ_{qp} has the value unity or zero according as the cell A_q^{i-1} is or is not on the boundary of the cell A_p. Then, to any set of values of the variables A_p^i satisfying the simultaneous equations E_q, there will be associated a *closed* i-chain of C, and conversely. For, among the variables with non-vanishing coefficients appearing in each equation, there must be an even number or zero that have the value unity, which means that each $(i-1)$-cell of C is on the boundary of an even number or none of the i-cells of the i-chain determined by the solution of the equation E_q. Thus, if ρ be the number of linearly independent equations E_q, the maximum number of independent solutions must be

$$R^i - 1 = \alpha - \rho,$$

which is the relation we set out to find.

If $i = 0$, there are no equations E_q. For this special case, we evidently have

$$R^0 - 1 = \alpha - 1.$$

We say that a closed n-chain is *irreducible* if its invariant R^n has the value 2, that is, if the chain is not the sum of two or more closed sub-chains. The points of an irreducible closed n-chain form an *n-dimensional manifold*.

7. Just as in the case of subdivisions, §2, a chain C' will be said to be *derived* from a chain C if it is one of a sequence of chains beginning with C and such that each member of the sequence is transformable into the next one by partitioning a single k-cell E^k into a pair of k-cells F_1^k and F_2^k separated by a $(k-1)$-cell F^{k-1}. The invariants of the chain C' are the same as those of C, for, when

the cell E^k is cut, the only i-chains to appear in the figure that are not mere subdivisions of old ones are the $(k - 1)$-chains containing F^{k-1} and the k-chains containing one but not both of F_1^k and F_2^k. The latter chains are open and therefore do not increase the connectivity numbers; nor do they decrease them by setting up new relations of bounding among old $(k - 1)$-chains, for their boundaries all contain the new cell F^{k-1}. The $(k - 1)$-chains containing F^{k-1}, whether open or closed, are transformed into old chains which no longer contain F^{k-1} by the addition of the boundary of F^k; therefore, they can have no effect on the connectivity numbers, one way or the other. Thus, we pass from C to C' by a series of operations which do not alter the connectivity numbers.

THEOREM S^n. *The connectivity numbers of any subdivision S^n of an n-sphere are all unity except the nth one which has the value* 2.

For, by §2, the subdivision S^n is derivable from an elementary subdivision and therefore has the same connectivity numbers as the latter. But in an elementary subdivision, every closed chain bounds a cell, with the exception of the n-chain determined by the two n-cells.

COROLLARY. *Any closed $(n - 1)$-chain K^{n-1} of the subdivision S^n of the n-sphere bounds exactly two open n-chains. Moreover, these two n-chains have only the points of K^{n-1} in common.**

For, since the number R^{n-1} is unity, there must exist an open n-chain K^n such that

$$K^n \equiv K^{n-1} \qquad (\text{mod } 2,\ C),$$

and consequently a second n-chain $K^n + S^n$, determined by the n-cells of S^n which do not belong to K^n, such that

$$K^n + S^n \equiv K^{n-1} \qquad (\text{mod } 2,\ C)$$

and such that it has in common with K^n only the points of K^{n-1}.

Now, if there were a third n-chain L^n bounded by K^{n-1}, there would be two independent closed n-chains

$$L^n + K^n \equiv 0 \text{ and } L^n + K^n + S^n \equiv 0,$$

and the number R^n would be at least 3, contrary to the theorem.

8. Let $\bar{H}^m$ and H^n be an m- and an n-sphere respectively, and let $\bar{C}$ be any chain of a subdivision of $\bar{H}^m$. Then, by an extension of terminology, we shall

* This is a somewhat weakened form of the theorem on the separation of n-space by a generalized polyhedron. For a proof of the latter theorem making use of modulo 2 equations, see O. Veblen, these Transactions, vol. 14 (1913), p. 65, and vol. 15 (1914), p. 506.

speak of any set of points C of H^n in reciprocal one-one continuous correspondence with $\bar{C}$ as a *chain immersed* in H^n. For example, if $\bar{C}$ is the boundary of a 2-cell, C may be any simple closed curve of H^n. The *cells* of the chain C will be the images in H^n of the cells of $\bar{C}$, so that C and $\bar{C}$ will both have the same cellular structure and, consequently, the same connectivity numbers. We shall frequently make use of the fact that there exists a derived chain of C made up of arbitrarily small cells. That such a chain does exist follows at once from the uniform continuity of the correspondence between the closed sets of points determined by C and $\bar{C}$, for we know that the cells of $\bar{C}$ may be redivided to any degree of smallness.

If the chain C does not fill up the entire space H^n, the residual part of H^n will form a certain domain $H^n - C$ made up of inner points. We proceed to define the connectivity numbers of this domain. Any chain of any subdivision of H^n will be called a chain of $H^n - C$ provided it is wholly contained in $H^n - C$. Among the chains of $H^n - C$ will be set up the following homologies: (1) Each closed i-chain will be said to be homologous to its derived chains; (2) each closed i-chain which bounds an open $(i + 1)$-chain of $H^n - C$ will be said to be homologous to zero. We combine homologies (1) and (2) like linear equations modulo 2 and denote by $(\bar{R}^i - 1)$ the maximum number of linearly independent closed i-chains of $H^n - C$. *A priori*, there is no reason why the number $\bar{R}^i$ should be finite in this case, since we are now dealing with equations in an infinite number of variables. It will be proved further on, however, that the numbers $\bar{R}^i$ are all finite and also pure topological invariants of the domain $H^n - C$, in spite of the fact that a metric on H^n has been used in defining them. The number $\bar{R}^0$ is of particular importance and evidently denotes the number of separate connected regions in $H^n - C$.

Since we shall only be concerned with the relations between chains under homologies, it will be legitimate to do away with the distinction between a chain of $H^n - C$ and its derived chains. We shall therefore regard any two chains with a common derived chain as equivalent chains, to be denoted by the same symbol K^i. A closed i-chain will then be said to bound if it bounds in any of its derived forms, so that the terms *bounding* and *homologous to zero* will henceforth be synonymous.

ON THE DUAL CONNECTIVITIES OF C AND $H^n - C$

9. We now come to the body of the discussion.

THEOREM T^i. *Let C^i be a cellular i-chain* (§4) *immersed in the n-sphere H^n. Then, the connectivity numbers of the domain $H^n - C^i$ residual to C^i are all unity. In other words, every closed chain L^k of $H^n - C^i$ bounds.*

The theorem is trivial if i is zero, in which case, C^i reduces to a point C^0.

For there are no closed chains in $H^n - C^i$ of dimensionality greater than or equal to n. Moreover, by the corollary to Theorem S^n, every closed $(n-1)$-chain of $H^n - C^0$ bounds twice in H^n and therefore once in $H^n - C^c$. Every closed chain of lower dimensionality bounds as often as we please in $H^n - C^0$.

The general case will be handled by induction with respect to i. We shall assume the validity of Theorem T^{i-1} and first prove a lemma.

Lemma U^i. *Let the cellular i-chain C^i be subdivided into two cellular i-chains A and B, respectively, meeting in a cellular $(i-1)$-chain C^{i-1}. Then every k-chain L^k of $H^n - C^i$ which bounds both in $H^n - A$ and $H^n - B$ must also bound in $H^n - C^i$.*

If $k = n - 1$, the chain L^k bounds exactly two open n-chains of H^n meeting in L^k, by the corollary of § 7. The connected set of points $C^i = A + B$ must therefore lie wholly within one of these two open n-chains, since it does not meet L^k. Consequently, the other n-chain lies in $H^n - C^i$, and L^k bounds in this region.

Now, suppose that $k < n - 1$, so that there exist chains in H^n of dimensionalities as high as $k + 2$. By hypothesis, there exist two open chains L^{k+1}_A and L^{k+1}_B such that

$$(4) \qquad \begin{aligned} L^{k+1}_A &\equiv L^k \qquad \pmod{2,\ H^n - A} \\ L^{k+1}_B &\equiv L^k \qquad \pmod{2,\ H^n - B} \end{aligned}$$

and these combine to form the closed chain

$$L^{k+1}_A + L^{k+1}_B \equiv 0 \qquad \pmod{2,\ H^n - C^{i-1}},$$

as illustrated in the figure which is purely schematic. We shall assume that L^{k+1}_A and L^{k+1}_B meet B and A, respectively, otherwise the lemma would be true without further argument.

Now, by Theorem T^{i-1}, which we are assuming in the induction, there exists an open $(k+2)$-chain M^{k+2} such that

$$M^{k+2} \equiv L^{k+1}_A + L^{k+1}_B \qquad \pmod{2,\ H^n - C^{i-1}}.$$

This chain cuts the chains A and B in mutually exclusive closed sets of points and may therefore be broken up into cellular $(k+2)$-chains so small that no one of them meets both A and B, since there is a definite interval of separation between two non-overlapping closed sets of points. Now, let $\overline{M}^{k+2}$ be the sum of the cellular $(k+2)$-chains of M^{k+2} that meet A, and therefore not B, and let $\overline{L}^{k+1}$ be the boundary of $\overline{M}^{k+2}$ Then we have

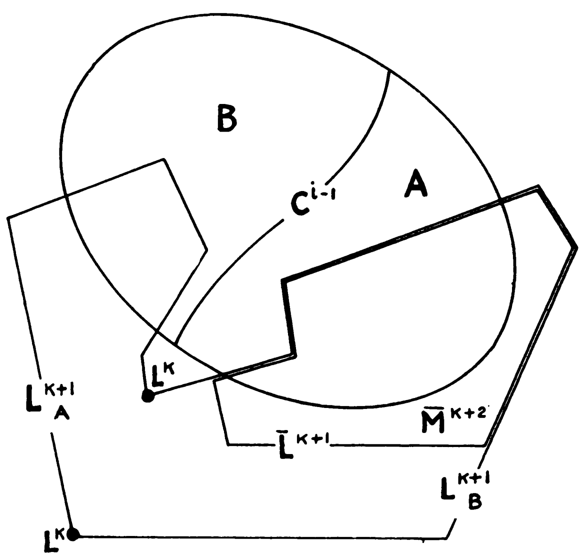
$$(5) \qquad M^{k+2} + \overline{M}^{k+2} \equiv (L^{k+1}_B + \overline{L}^{k+1}) + L^{k+1}_A \equiv 0 \pmod{2,\ H^n - C^{i-1}}.$$

But $L^{k+1}_B + \bar{L}^{k+1}$ meets neither A nor B and therefore lies in $H^n - C^i$. Hence, by (5) and the first relation in (4),

$$L^{k+1}_B + \bar{L}^{k+1} \equiv L^k \qquad (\text{mod } 2,\ H^n - C^i),$$

which establishes the lemma.

Theorem T^i now follows at once by the ordinary pinching process. If the theorem were false for the chain C^i, it would be false for one of two cellular sub-chains, by the lemma, and by repeating the argument it would be possible to find a sequence of sub-chains C^i_s of C^i closing down upon a single point C^0 and for each of which the theorem would be false. But, by Theorem T^0, every closed k-chain L^k of $H^n - C^i$, and therefore of $H^n - C^0$, would bound a chain



L^{k+1} of H^n which did not meet C^0 and which therefore could not meet all of the chains C^i_s converging on that point. Therefore, the theorem must be true, since the assumption that it is false leads to a contradiction.

COROLLARY V^i. *A cellular i-chain immersed in an n-sphere H^n cannot fill H^n.*

For let C^i be broken up into two cellular parts, A and B, as in Lemma U^i, and let P_A and P_B be points of A, and B, respectively but not of the chain C^{i-1} common to A and B. Then, by Theorem T^{i-1}, the 0-chain $P_A + P_B$ bounds a 1-chain in $H^n - C^{i-1}$ which must contain a broken line of geodesics connecting P_A with P_B. But this broken line meets A and B in mutually exclusive closed sets of points and must therefore contain points that belong to neither of these sets. Such points must be points of $H^n - C^i$.

COROLLARY W^i. *Let C be the sum and C^{i-1} the intersection of two closed sets of points A and B. Then every closed k-chain L^k $(k<n-1)$ of $H^n - C$ which bounds a chain L_A^{k+1} of $H^n - A$ and a chain L_B^{k+1} of $H^n - B$ must also bound in $H^n - C$ provided the chains L_A^{k+1} and L_B^{k+1} may be so chosen that $L_A^{k+1} + L_B^{k+1}$ bounds in $H^n - C^{i-1}$. Moreover, the corollary is valid even if $k = n-1$ unless C^{i-1} is the null set.*

For the proof of Lemma U^i is applicable here with scarcely a change.

10. We are now in a position to prove the duality theorem mentioned in the introduction. In order to separate out the difficulties, however, let us first consider an important special case which admits of a simpler proof than the general one.

THEOREM X^i. *Let C^i be an i-sphere immersed in an n-sphere H^n. Then the connectivity numbers R^s of C^i are related to the connectivity numbers R^s of the residual space $H^n - C^i$ by the equations*

$$R^i = \bar{R}^{n-i-1} = 2, \quad R^s = \bar{R}^{n-s-1} = 1 \qquad (s \neq i).$$

The theorem states, in other words, that there exists but one independent closed non-bounding chain in $H^n - C^i$. This chain will be of dimensionality $(n - i - 1)$. It will be said to *link* the i-sphere C^i.

If $i = 0$, the i-sphere C^i is a pair of points, so that the theorem is both trivial and obvious. The $(n-1)$-chain linking C^i is any closed $(n - 1)$-chain of L^{n-1} such that one of the points of C^i lies in each of the two open n-chains bounded by L^{n-1} in H^n, (§7). All closed chains of lower dimensionalities bound as often as we please in $H^n - C^i$.

The case $i>0$ will be solved by induction with respect to i. Let us subdivide the i-sphere C^i into a pair of cellular i-chains A and B meeting in an $(i-1)$-sphere C^{i-1}. Then by Theorem T^i, every closed chain L^k of $H^n - C^i$ must bound two open chains L_A^{k+1} and L_B^{k+1} in $H^n - A$ and $H^n - B$, respectively. Therefore, by Corollary W^i, L^k must also bound in $H^n - C^i$ unless the closed chain $L_A^{k+1} + L_B^{k+1}$ fails to bound in $H^n - C^{i-1}$. But, by Theorem X^{i-1}, which we have a right to assume in the induction, this can only occur if $L_A^{k+1} + L_B^{k+1}$ is the $(n - i)$-chain linking C^{i-1}, in which case $k = n - i - 1$. Consequently,

$$R^s = \bar{R}^{n-s-1} = 1 \qquad (s \neq i).$$

On the other hand, the chain L^{n-i} of $H^n - C^{i-1}$ which does link C^{i-1} necessarily meets both A and B in mutually exclusive closed sets of points, for if it failed to meet A, for example, it would bound in $H^n - A$, by Theorem T^i, and hence *a fortiori* in $H^n - C^{i-1}$, contrary to hypothesis. The chain L^{n-i} may thus be written as the sum of two open chains (cf. Lemma U^i),

$$L^{n-i} = L_A^{n-i} + L_B^{n-i} \tag{6}$$

lying in $H^n - A$ and $H^n - B$, respectively, and having a common boundary L^{n-i-1} in $H^n - C^i$. The chain L^{n-i-1} is then the required chain linking C^i. For, otherwise, there would be an open chain $\bar{L}^{n-i}$, such that

$$\bar{L}^{n-i} \equiv L^{n-i-1} \qquad (\text{mod } 2, H^n - C^i),$$

and, consequently, one or the other of the closed chains $\bar{L}^{n-i} + L_A^{n-i}$ or $\bar{L}^{n-i} + L_B^{n-i}$ would have to link C^{i-1}, since their sum $L_A^{n-i} + L_B^{n-i}$ would. But suppose, for example, that $\bar{L}^{n-i} + L_A^{n-i}$ linked C^{i-1}. Then, $\bar{L}^{n-i} + L_A^{n-i}$ would have to meet A, which would be impossible since, from their definitions, neither of its parts $\bar{L}^{n-i}$ nor L_A^{n-i} could. A similar contradiction would arise if we assumed that $\bar{L}^{n-i} + L_B^{n-i}$ linked C^{i-1}.

Finally, L^{n-i-1} is the only independent chain linking C^i. For, if M^{n-i-1} denote any $(n-i-1)$-chain of $H^n - C^i$ linking C^i, there is associated with M^{n-i-1} a closed chain $M_A^{n-i} + M_B^{n-i}$ linking C^i, defined after the manner of the chain $L_A^{n-i} + L_B^{n-i}$ in (6) associated with L^{n-i-1}. Consequently, with the chain $L^{n-i-1} + M^{n-i-1}$, there is associated the chain

$$(L_A^{n-i} + M_A^{n-i}) + (L_B^{n-i} + M_B^{n-i})$$

which cannot link C^{i-1}. Therefore, by Corollary W^i, $L^{n-i-1} + M^{n-i-1}$ bounds in $H^n - C^i$, and M^{n-i-1} is dependent on L^{n-i-1}. Thus, finally

$$R^i = \bar{R}^{n-i-1} = 2.$$

The chain of $H^n - C^i$ which links C^i may evidently be chosen to be irreducible, for if it consisted of several irreducible parts, one at least of these parts would have to link C^i.

11. This brings us to the central theorem:

THEOREM Y. *Let C be any chain immersed in an n-sphere H^n. Then, between the invariants R^i of C and the invariants $\bar{R}^i$ of $H^n - C$ there exists the following duality relation*:

$$R^i = \bar{R}^{n-i-1} \qquad (0 \leqq i \leqq n-1).$$

To lay the foundations for a proof by induction, let us first examine the trivial case where the chain C consists of 0-cells only. Obviously, a closed chain of $H^n - C$ of dimensionality less than $n-1$ bounds as often as we please in

$H^n - C$, so that $R^{n-i-1} = R^i = 1$ $(i>0)$. To determine the remaining connectivity number $\overline{R}^{n-1}$, let us make a subdivision of H^n such that each point A_s^0 of C appears as an interior point of some cellular n-chain M_s^n of the subdivision and such that no two of the points A_s^0 belong to the same cellular n-chain M^n. The boundary of each cellular n-chain M_s^n will be a closed $(n-1)$-chain which we shall denote by L_s^{n-1}:

$$M_s^n \equiv L_s^{n-1} \qquad (\text{mod } 2,\ H^n). \tag{7}$$

Now, every closed $(n-1)$-chain L^{n-1} of $H^n - C$ is homologous to some combination of the chains L_s^{n-1}; for the chain L^{n-1} surely bounds in H^n,

$$M^n \equiv L^{n-1} \qquad (\text{mod } 2,\ H^n), \tag{8}$$

and if the bounded chain M^n contains points of C, we have merely to add to (8) such relations of the set (7) as correspond to the points in question to obtain an open n-chain free from points of C and bounded by L^{n-1} together with some linear combination of the chains L_s^{n-1}.

Finally, there is one and only one homology between the chains L_s^{n-1}. For by §7, any linear combination of the chains L_s^{n-1} bounds exactly two n-chains in H^n, one of which must be free of points of C if the combination is to be bound in $H^n - C$. Evidently, this can only occur if the linear combination includes all the chains L_s^{n-1}. Thus, if there be α points to C, we have

$$\overline{R}^{n-1} - 1 = \alpha - 1 = R^0 - 1,$$

by §6, which establishes the equality of $\overline{R}^{n-1}$ and R^0.

As usual, we treat by induction the case where the chain C contains at least one cell of dimensionality greater than zero.

Let B be the chain obtained by leaving off an i-cell of C of the highest dimensionality possible, and let A be the cellular i-chain determined by this i-cell. We shall assume that the theorem holds for B and shall prove that when A is restored, every change of connectivity on B is balanced by a dual change in the residual space so that the theorem continues to hold for C.

Evidently, such changes of connectivity as are produced on B by the addition of A are caused by the appearance of new independent i-chains containing A or the disappearance of independent $(i-1)$-chains by bounding open i-chains containing A. We denote the boundary of A by C^{i-1} and distinguish two cases according as

(I) The chain C^{i-1} does not bound on B;

(II) The chain C^{i-1} bounds some open i-chain A^i of B.

In the first case, no new closed i-chain can be created, since such a chain would be of the form $A + A^i$ which would imply

$$A^i \equiv C^{i-1} \qquad (\text{mod } 2,\ B),$$

contrary to hypothesis. On the other hand, we have

$$A \equiv C^{i-1} \qquad (\text{mod } 2,\ C),$$

so that the independent chain C^{i-1} is lost. No other independent chain is lost, for a second relation

$$A + D^i \equiv D^{i-1} \qquad (\text{mod } 2,\ C)$$

would imply

$$D^i \equiv C^{i-1} + D^{i-1} \qquad (\text{mod } 2,\ B)$$

showing that D^{i-1} was dependent on C^{i-1}.

The second case is treated with equal facility. Without going into details, we find that a single independent i-chain $A + A^i$ is gained and that no independent $(i - 1)$-chain is lost.

To calculate the compensating changes of connectivity produced in the residual space, we define a closed $(n - i - 1)$-chain of the residual space which will be said to be *dual* to the chain A. Let an irreducible $(n - i)$-chain L^{n-i} of $H^n - C^{i-1}$ be chosen linking the boundary C^{i-1} of A and therefore meeting A in a closed set of points. Whenever we can, we shall choose the chain L^{n-i} in such a way that it contains at least one point not of A. It will then be possible, by the process already so frequently employed, to break the chain L^{n-i} up into a pair of open chains bounded by a closed chain L^{n-i-1} of $H^n - C^i$ and such that one of the open chains which we shall call M^{n-i} contains all the points of intersection of L^{n-i} with A but no point of B:

$$M^{n-i} \equiv L^{n-i-1} \qquad (\text{mod } 2,\ H^n - \text{B}). \tag{9}$$

The chain L^{n-i-1} will be said to be dual to A. It evidently links any i-sphere contained in C and containing A, for it has been obtained by exactly the construction given in the proof of Theorem X^i for finding the chain linking such an i-sphere. Moreover, as we note for future reference, if ϵ be any positive constant, the chain M^{n-i} bounded by L^{n-i-1} may evidently be so chosen that each of its points is within a distance ϵ of some point of intersection of M^{n-i} with A.

We derived the dual L^{n-i-1} of A on the assumption that a chain L^{n-i} linking C^{i-1} could be found which contained a point not of A. Now, as a matter of fact, it is easy to prove that any chain linking C^{i-1} must contain such a point. Rather than digress to prove this, however, let us merely say that if no chain

L^{n-i} contains a point not of A, then $M^{n-i} = L^{n-i}$ and the boundary of M^{n-i} is the "null" $(n-i-1)$-chain.

We now prove two lemmas.

(α) *If a closed chain L^{k+1} of $H^n - B$ does not link the boundary C^{i-1} of A, there is always some chain of $H^n - C$ (that is to say, some chain not meeting A) which is homologous to L^{k+1} in $H^n - B$. If the chain L^{k+1} does link C^{i-1}, however, there is no chain of $H^n - C$ homologous to L^{k+1}.*

For if L^{k+1} links C^{i-1} it must cut the cellular i-chain A, otherwise it would bound in $H^{n+1} - A$, by Theorem T^i, and therefore *a fortiori* in $H^{n+1} - C^{i-1}$. The same is true of all chains homologous to L^{k+1}.

On the other hand, if L^{k+1} does not link C^{i-1}, there exists an open chain M^{k+2} such that

$$M^{k+2} \equiv L^{k+1} \qquad (\text{mod } 2, H^n - C^{i-1}).$$

We may again use the figure going with Lemma U^i to represent the situation schematically, provided we let $L^{k+1}_A + L^{k+1}_B$ represent the chain L^{k+1} and imagine that L^{k+1}_A is nearer to the eye than B and does not intersect B, though M^{k+2} may. Now M^{k+2} meets A and B, if at all, in mutually exclusive closed sets of points, therefore we may repeat the argument of Lemma U^i and find a chain $L^{k+1} + \bar{L}^{k+1}$ which does not meet A (nor B in this case). But, with the same notation as before,

$$\bar{M}^{k+2} \equiv \bar{L}^{k+1} \sim 0 \qquad (\text{mod } 2, H^n - B).$$

Therefore,

$$L^{k+1} + \bar{L}^{k+1} \sim L^{k+1},$$

showing that there remains in $H^n - C$ a representative of the family of chains homologous to L^{k+1} in $H^n - B$.

Thus, in view of Theorem X^i, the addition of A to B cannot reduce the number of independent non-bounding chains of the residual space aside from those of dimensionality $n - i$. Moreover, not more than one independent $(n - i)$-chain can be destroyed, since the sum of two chains linking C^{i-1} cannot link C^{i-1}.

(β) *Let L^k be a closed chain of $H^n - C$ which bounds in $H^n - B$;*

$$L_B^{k+1} \equiv L^k \qquad (\text{mod } 2, H^n - B)$$

and, necessarily, of course, in $H^n - A$

$$L_A^{k+1} \equiv L^k \qquad (\text{mod } 2, H^n - A), \tag{10}$$

by Theorem T^i. Then, L^k fails to bound in $H^n - C$ if and only if the closed chain

$$L_A^{k+1} + L_B^{k+1} \equiv 0 \qquad (\text{mod } 2,\ H^n - C^{i-1}) \tag{11}$$

links C^{i-1} for every posible choice of the chain L_B^{k+1}.

For, as we saw in (α), $L^{k+1} = L_A^{k+1} + L_B^{k+1}$ must cut A if it links C^{i-1}. Consequently, L_B^{k+1} must cut A, since L_A^{k+1} does not (see figure). But if this occurs for every choice of L_B^{k+1}, L cannot bound in $H^n - C$.

On the other hand, if L_B^{k+1} can be so chosen that $L_A^{k+1} + L_B^{k+1}$ does not link C^{i-1}, then, by Corollary W^i, L^k must bound in $H^n - C$. As an immediate consequence of (β), no new independent non-bounding closed chains of dimensionalities other than $n - i - 1$ can be created in the residual space when A is added to B.

Let us now combine the two results obtained.

(I′) If the space $H^n - B$ contains an $(n - i)$-chain L linking C^{i-1}, no new independent $(n - i - 1)$-chain can be created in the residual space. For, in place of relation (10) in Lemma (β), we may equally well write

$$L + L_A^{k+1} \equiv L^k \qquad (\text{mod } 2,\ H^n - \text{A}), \tag{10′}$$

(where $k = n - i - 1$), and so obtain in place of (11) the expression

$$(L + L_A^{k+1}) + L_B^{k+1} \equiv 0. \tag{11′}$$

But not both of the chains (11) and (11′) can link C^{i-1}; therefore L^k must bound in $H^n - C$, by Corollary W^i. The net result of adding A to B is therefore to diminish by unity the number $\bar{R}^{n-i}$ but to leave invariant the remaining connectivity numbers of the residual space.

(II′) If the space $H^n - B$ contains no $(n - i)$-chain linking C^{i-1}, none of the connectivity numbers $\bar{R}^s$ can be diminished. However, in this case, a chain of $H^n - C^{i-1}$ linking C^{i-1} must meet both A and B. Consequently by a literal transcription of the proof of Theorem X^i, we find that a single new independent $(n - i - 1)$-chain is created which is nothing more than the dual of A. The number $\bar{R}^{n-i-1}$ is therefore increased by unity. It also follows that every independent non-bounding $(n - i - 1)$-chain of $H^n - C$ is homologous in that region to some linear combination of the duals of the i-cells of C.

Now, it will be observed that the changes of connectivity (I′) and (II′) of the residual space are exactly the ones wanted to compensate for the changes of connectivity (I) and (II), respectively, of the immersed figure. Furthermore, whether or not similarly numbered changes occur together when A is added to B, the differences $R^i - R^{i-1}$ and $\bar{R}^{n-i-1} - \bar{R}^{n-i}$ both increase by unity in every case and therefore remain equal to one another. Thus to complete the proof we have only to show that the number R^i of C is equal to the number $\bar{R}^{n-i-1}$ of $H^n - C$. This we proceed to do.

Let C' be the chain obtained by eliminating all the cells of C of highest dimensionality i. Then, to each cellular $(i-1)$-chain A' of C' may be found a dual $(n-i)$-chain L^{n-i} of $H^n - C'$ so close to A' that it meets only such i-cells of C as contain A' on their boundaries. The latter cells it must meet, however, since it links their boundaries. We may therefore break up the chain L^{n-i} into a set of open i-chains each containing the points of intersection of L^{n-i} with one alone of the i-cells of C and each bounded by the dual of the chain determined by that i-cell. Thus, if L_p^{n-i-1} denote the dual of A_p^i, we shall have a set of homologies

$$\bar{E}_q \qquad \sum_p \epsilon_{qp} L_p^{n-i-1} \sim 0 \qquad (\text{mod } 2,\ H^n - C)$$

where the coefficients ϵ_{qp} have precisely the same significance as in the relations E_q of (§ 6). Furthermore, every homology among the duals L_p^{n-i-1} of chains A_p^i is expressible as a linear combination of the fundamental homologies $\bar{E}_q$. For suppose

$$(12) \qquad M^{n-i} \equiv \sum_u \alpha_u L_u^{n-i-1} \sim 0 \qquad (\text{mod } 2,\ H^n - C).$$

Then, by combining with (9), we obtain the closed chain

$$M^{n-i} + \sum_u \alpha_u M_u^{n-i} \equiv 0 \qquad (\text{mod } 2,\ H^n - C').$$

But this chain is expressible linearly in terms of the duals of the cellular $(i-1)$-chains of C'so that the homology (12) is expressible in terms of the homologies $\bar{E}_q$. We therefore have

$$R^i - 1 = \alpha - \rho = \bar{R}^{n-i-1} - 1$$

(cf. § 6), which establishes the equality of R^i and $\bar{R}^{n-i-1}$.

12. In closing we give a few corollaries of the fundamental theorem.

JORDAN-BROUWER THEOREM. *If M^{n-1} be an $(n-1)$-dimensional manifold (§ 6) immersed in an n-sphere H^n, the residual domain $H^n - M^{n-1}$ consists of exactly two connected regions.* For $\bar{R}^0 = \bar{R}^{n-1} = 2$.

Let us distinguish between these regions by arbitrarily calling one the *interior* and the other the *exterior* of M^{n-1}. Then,

If the manifold M^{n-1} be homeomorphic with an $(n-1)$-sphere, the connectivity numbers of both interior and exterior are unity. For $\bar{R}^i = R^{n-i-1} = 1$ $(i > 0)$.

ACCESSIBILITY THEOREM. *In every neighborhood of every point of* M^{n-1}, *there is a point of* M^{n-1} *which is accessible from any point P of* $H^n - M^{n-1}$ *by a broken line of geodesic arcs made up of points of* $H^n - M^{n-1}$.

For consider any subdivision of M^{n-1} into cells. Then, it is always possible to join the point P to a point Q on the other side of M^{n-1} by a broken line which meets M^{n-1} in one $(n-1)$-cell of the subdivision only. Because, if that one $(n-1)$-cell were omitted, we should have $R^{n-1} = \bar{R}^0 = 1$ and the space residual to M^{n-1} would become connected. But this $(n-1)$-cell may be chosen in an arbitrary neighborhood of an arbitrary point of M^{n-1}, which proves the theorem and also the following corollary:

There are interior and exterior points in every neighborhood of every point of M^{n-1}; for example, points of the broken line PQ.

By the same device of omitting an arbitrarily small $(n-1)$-cell, the theorem on the invariance of dimensionality may be proved.

Let C^k *be any cellular k-chain* $(k \leqq n-1)$ *immersed in an n-sphere* H^n. *Then, there are points of* $H^n - C^k$ *in every neighborhood of every point of* C^k; namely, points of a 1-chain linking the boundary of the omitted $(n-1)$-cell.

The theorem on the invariance of regionality is, of course, an immediate consequence of the separation theorem:

Let C^n *be a cellular n-chain immersed in an n-sphere* H^n. *Then no interior point of* C^n *is a limit point of points of* $H^n - C^n$. For the boundary of C^n is an $(n-1)$-dimensional manifold separating H^n into $H^n - C^n$ and the interior of C^n.

Finally, we note that Theorem Y establishes the purely topological character of the invariants R^i and $\bar{R}^{n-i-1}$. For $\bar{R}^{n-i-1}$ does not depend at all upon the particular cellular structure of C, but only on the set of points determined by C; therefore, the same must be true of R^i. Conversely, R^i is not affected by the choice of the metric on H^n; therefore, neither is $\bar{R}^{n-i-1}$.

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