PERMUTABLE RATIONAL FUNCTIONS*

By

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INTRODUCTION

We investigate, in this paper, the circumstances under which two rational functions, \( \Phi(z) \) and \( \Psi(z) \), each of degree greater than unity, are such that

\[
\Phi[\Psi(z)] = \Psi[\Phi(z)].
\]

A pair of functions of this type will be called permutable.

A memoir devoted to this problem has recently been published by Julia.† When \( \Phi(z) \) and \( \Psi(z) \) are polynomials, and are such that no iterate of one is identical with any iterate of the other, Julia shows how \( \Phi(z) \) and \( \Psi(z) \) can be obtained from the formulas for the multiplication of the argument in the functions \( e^z \) and \( \cos z \). His other results are mainly of a qualitative nature, and deal with the manner in which \( \Phi(z) \) and \( \Psi(z) \) behave when iterated.

Certain of Julia's results have been announced independently by Fatou.‡ Fatou's method is identical with that of Julia.

The method used in the present paper differs radically from that of Julia and Fatou, and leads to results of much greater precision. Its chief yield is the

**THEOREM.** If the rational functions \( \Phi(z) \) and \( \Psi(z) \), each of degree greater than unity, are permutable, and if no iterate of \( \Phi(z) \) is identical with any iterate of \( \Psi(z) \), there exist a periodic meromorphic function \( f(z) \), and four numbers \( a, b, c \) and \( d \), such that

\[
f(az + b) = \Phi[f(z)], \quad f(cz + d) = \Psi[f(z)].
\]

The possibilities for \( f(z) \) are: any linear function of \( e^z, \cos z, \varphi z \); in the lemniscatic case \( (g_8 = 0) \), \( \varphi^2 z \); in the equianharmonic case \( (g_2 = 0) \), \( \varphi' z \)

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* Presented to the Society, February 24, 1923.
† The case in which one of the functions is linear will be met incidentally in § X.
∥ The condition that no common iterate exists is certainly satisfied if the degrees of the two functions are relatively prime.
and $\varphi^3 z$. These are, essentially, the only periodic meromorphic functions which have rational multiplication theorems.\footnote{These Transactions, vol. 23 (1922), p. 16.}

The multipliers $a$ and $c$ must be such that if $\omega$ is any period of $f(z)$, $a\omega$ and $c\omega$ are also periods of $f(z)$.

If $p$ represents the order of $f(z)$, that is, the number of times $f(z)$ assumes any given value in a primitive period strip or in a primitive period parallelogram, the products

$$b \left( 1 - e^{2\pi i/p} \right), \quad d \left( 1 - e^{2\pi i/p} \right)$$

must be periods of $f(z)$.

Finally,

$$(a - 1) d - (c - 1) b$$

must be a period of $f(z)$.

The condition that $\Phi(z)$ and $\Psi(z)$ have no iterate in common, can be replaced by one which is certainly not stronger, and which is satisfied, for instance, if there does not exist a rational function $\sigma(z)$, of degree greater than unity, such that

$$\Phi(z) = \varphi[\sigma(z)], \quad \Psi(z) = \psi[\sigma(z)],$$

where $\varphi(z)$ and $\psi(z)$ are rational.

The existence of the periodic function $f(z)$ is demonstrated by a method which is almost entirely algebraic. It would be interesting to know whether a proof can also be effected by the use of the Poincaré functions employed by Julia.

Of the periodic functions listed above, the linear integral functions of $e^z$ and of $\cos z$ are the only ones whose multiplication theorems will produce a pair of permutable polynomials. In all other cases, at least one of the functions $\Phi(z)$ and $\Psi(z)$ will be fractional. In §X we settle completely the case in which $\Phi(z)$ and $\Psi(z)$ are both polynomials, obtaining the

**Theorem.** If $\Phi(z)$ and $\Psi(z)$ are a pair of permutable polynomials (non-linear), which do not come from the multiplication theorems of $e^z$ and $\cos z$, there exist a linear integral function $\lambda(z)$ and a polynomial

$$G(z) = \varepsilon R(z^\varepsilon),$$
where $R(z)$ is a polynomial, such that

$$\Phi(z) = \lambda^{-1} \{ \epsilon_1 G^{(\omega)}[\lambda(z)] \}, \quad \Psi(z) = \lambda^{-1} \{ \epsilon_2 G^{(\omega)}[\lambda(z)] \},$$

where $G^{(\omega)}(z)$ represents the $i$th iterate of $G(z)$, and where $\epsilon_1$ and $\epsilon_2$ are $r$th roots of unity.

Thus, neglecting a linear transformation, $\Phi(z)$ and $\Psi(z)$ are iterates of the same polynomial, multiplied sometimes by roots of unity.

As to the permutable fractional functions which do not come from the multiplication theorems (and which therefore have an iterate in common), we give a method for constructing them, which, while it is not everything to be desired, still throws considerable light upon the functions under consideration. This method, which involves two types of operations, applies to all rational functions, integral or fractional.

Let $\Phi(z)$ and $\Psi(z)$ be two permutable rational functions. If there exist three rational functions, $\varphi(z)$, $\psi(z)$ and $\sigma(z)$, each of degree greater than unity, such that

$$G(z) = \sigma[\varphi(z)], \quad \Psi(z) = \sigma[\psi(z)],$$

and that $\varphi[\sigma(z)]$ is permutable with $\psi[\sigma(z)]$, we shall call the act of passing from $\Phi(z)$ and $\Psi(z)$ to $\varphi[\sigma(z)]$ and $\psi[\sigma(z)]$ an operation of the first type.

If $\Phi(z)$ and $\Psi(z)$ are permutable, it is evident that $\Phi[\Psi(z)]$ will be permutable both with $\Phi(z)$ and with $\Psi(z)$. We shall call the act of passing from $\Phi(z)$ and $\Psi(z)$ to $\Phi[\Psi(z)]$ and $\Phi(z)$, or to $\Phi[\Psi(z)]$ and $\Psi(z)$, an operation of the second type.

We show in § X that if $\Phi(z)$ and $\Psi(z)$ do not come from the multiplication theorems of the periodic functions, there exists a linear function $\lambda(z)$ such that $\lambda^{-1} \Phi \lambda(z)$ and $\lambda^{-1} \Psi \lambda(z)$ can be obtained by repeated operations of the above two types, starting from a pair of functions

$$\epsilon R(z^r), \quad \epsilon z R(z^r),$$

where $R(z)$ is a rational function, and where $\epsilon$ is an $r$th root of unity (sometimes unity itself).

For polynomials, only operations of the second type are necessary, and we obtain the explicit formulas given above. In the case of the fractional functions, however, operations of the first type are sometimes necessary, so that there exist permutable pairs of fractional functions which come neither from
the multiplication theorems of the periodic functions, nor from the iteration of a function.

We have not succeeded thus far in determining all cases in which operations of the first type are possible. In fact, an illustration of the operations of that type, given in § X, will probably weaken any a priori conviction one might have to the effect that formulas as explicit as those stated above for polynomials can be found for the permutable fractional functions which have an iterate in common. It will not be inconceivable that too little order may prevail among the functions of that class for a complete enumeration of them to be possible.

The present paper is the outcome of efforts to solve, for fractional functions, the problem settled for polynomials in our paper *Prime and composite polynomials.*

1. Preliminaries

What we do principally in this section is to recall certain results proved in the above mentioned paper on prime and composite polynomials, on which the work in the present paper will be based.

Let \( \psi(z) \) and \( \varphi(z) \) be two rational functions of the respective degrees \( r > 1 \) and \( s > 1 \). Let \( F(z) = \psi[\varphi(z)] \). We put

\[ w = F(z) = \varphi(u), \quad u = \psi(z). \]

It is easy to see the Riemann surface for \( F^{-1}(w) \) is related to those for \( \varphi^{-1}(w) \) and \( \psi^{-1}(u) \). Suppose that, for \( u = c \), \( \psi^{-1}(u) \) has a critical point with a certain number of cycles. As \( \varphi^{-1}(w) \) assumes no value more than once on its Riemann surface, \( F^{-1}(w) \) will surely have a critical point for \( w = \varphi(c) \). If the value \( c \) is assumed by a branch \( u \) of \( \varphi^{-1}(w) \) which is uniform in the neighborhood of \( \varphi(c) \), those branches of \( F^{-1}(w) \) for which \( \psi(z) = u \) will be ramified at \( \varphi(c) \) as the branches of \( \psi^{-1}(u) \) are at \( c \). If the value \( c \) is assumed by a cycle of \( p \) branches of \( \varphi^{-1}(w) \), each cycle of \( \psi^{-1}(u) \) at \( u = c \) will lead to a cycle of \( F^{-1}(w) \) at \( \varphi(c) \) with \( p \) times as many sheets. If \( \varphi^{-1}(w) \) has a critical point for \( w = d \), and if none of the points \( u = \varphi^{-1}(d) \) is a critical point of \( \psi^{-1}(u) \), then each cycle of \( \varphi^{-1}(w) \) at \( d \) leads to \( s \) cycles of the same number of sheets for \( F^{-1}(w) \) at \( d \).

We call the sum of the orders of the branch points of an algebraic function which are superimposed on each other at a given critical point the index of the function at the point. The sum of the indices of the inverse of a rational function of degree \( r \), for all of its critical points, is \( 2r - 2 \).

* These Transactions, vol. 23 (1922), p. 51.
It is easy to see that the index of \( F^{-1}(w) \), at any critical point \( w_0 \) of \( \varphi^{-1}(w) \), is at least \( s \) times the index of \( \varphi^{-1}(w) \) at \( w_0 \). Also, if the index of \( F^{-1}(w) \) at \( w_0 \) is \( q \), and if \( w_0 \) is not a critical point of \( \varphi^{-1}(w) \), then \( \varphi^{-1}(u) \) must have critical points whose affixes are values of \( \varphi^{-1}(w) \) at \( w_0 \), and the sum of whose indices is \( q \).

With respect to the group of monodromy of \( F^{-1}(w) \), the \( rs \) branches of \( F(w) \) break up into \( r \) systems of imprimitivity, such that if the branches

\[
\begin{align*}
   \zeta_1, \zeta_2, \ldots, \zeta_r
\end{align*}
\]

constitute one of these systems, we have

\[
\begin{align*}
   \psi(\zeta_1) = \psi(\zeta_2) = \cdots = \psi(\zeta_r).^*
\end{align*}
\]

These \( r \) systems are said to be determined by \( \psi(z) \).

Conversely, let \( F(z) \) be any rational function of degree \( rs \) whose inverse has an imprimitive group. If (1) is a system of imprimitivity of the group of \( F^{-1}(w) \), there exists a rational function \( \psi(z) \), of degree \( s \), for which (2) holds, and we have

\[
\begin{align*}
   F(z) = \varphi[\psi(z)],
\end{align*}
\]

where \( \varphi(z) \) is a rational function of degree \( r \). If another rational function determines the same systems of imprimitivity as \( \psi(z) \), it is a linear function of \( \psi(z) \).

Suppose that (1) can be broken up into smaller systems of imprimitivity, each containing \( t \) letters, and let \( \sigma(z) \) be the rational function of degree \( t \) which determines these systems. Then \( \psi(z) \) is a rational function of \( \sigma(z) \).†

We shall deal next with five rational functions of degrees greater than unity; \( g_1(z) \) and \( g_2(z) \), each of degree \( r \), \( \psi_1(z) \) and \( \psi_2(z) \), each of degree \( s \), and \( F(z) \). We suppose that

\[
\begin{align*}
   F(z) = g_1[\psi_1(z)] = \psi_1[g_2(z)].
\end{align*}
\]

We put \( w = F(z) \), \( u = \psi_2(z) \) and \( v = g_2(z) \), so that

\[
\begin{align*}
   w = g_1(u) = \psi_1(v).
\end{align*}
\]

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* Loc. cit., p. 54.
† Loc. cit., p. 55.
The function $\psi_s(z)$ determines $r$ systems of imprimitivity of the group of $F^{-1}(w)$,

$$U_1, U_2, \ldots, U_r,$$

each containing $s$ letters, while $\varphi_s(z)$ determines the $s$ systems

$$V_1, V_2, \ldots, V_s,$$

each containing $r$ letters. If $w$ describes a closed path, the sets $U$ are permuted like the branches of $\varphi_1^{-1}(w)$, and the sets $V$ like the branches of $\psi_1^{-1}(w)$.

In what follows, we shall assume that each system $U$ has exactly one letter in common with each system $V$. We see directly that if some substitution of the group of $F^{-1}(w)$ interchanges the letters of some set $V_i$ among themselves, it interchanges the sets $U$ with a substitution similar to that which it effects on the letters of $V_i$.

A point $(w, u)$ on the Riemann surface of $\varphi_1^{-1}(w)$ for which $w$ is a critical point of $\varphi_1^{-1}(w)$ or of $\psi_1^{-1}(w)$ (that is of $F^{-1}(w)$) will be called a point of $\varphi_1^{-1}(w)$. A point of $\psi_1^{-1}(w)$ is defined similarly. If $p$ branches coalesce at a point, we shall say that the point is of order $p$. Thus a point of order $p$ is a branch point of order $p - 1$.

A point of order unity will be called a simple point. If $\varphi_1^{-1}(w)$ has a point of order $p$ for $w = w_0$, and if $\psi_1^{-1}(w)$ has a critical point at $w_0$ where its branches undergo a substitution whose order is not a factor of $p$, the point of order $p$ will be called an $A$-point of $\varphi_1^{-1}(w)$. An $A$-point of $\psi_1^{-1}(w)$ is defined similarly. If $w_0$ is a critical point of $\psi_1^{-1}(w)$, every simple point which $\varphi_1^{-1}(w)$ may have at $w_0$ is an $A$-point of $\varphi_1^{-1}(w)$.

Suppose that $\psi_1^{-1}(w)$ has a point of order $p$ for $w = w_0$, and let the value of $\psi_1^{-1}(w)$ at the point be $v_0$. Suppose that as $w$ makes a turn about $w_0$, the branches of $\varphi_1^{-1}(w)$ undergo a substitution $S$; $S$ will be identity if $w_0$ is not a critical point of $\varphi_1^{-1}(w)$. Let $w$ execute $p$ turns about $w_0$, so that the branches of $\varphi_1^{-1}(w)$, and therefore the systems $U$, undergo the substitution $S^p$. Let $v_i$ be one of those branches of $\psi_1^{-1}(w)$ which coalesce at the point of order $p$ now under consideration. As $w$ makes $p$ turns about $w_0$, the value of $v_i$ makes a single turn about $v_0$. Thus, by the $p$ turns, the letters of $V_i$ are interchanged among themselves; hence the substitution which these letters undergo is similar to $S^p$. This means that when $v$ makes a turn about $v_0$, the branches of $\varphi_2^{-1}(v)$ undergo a substitution similar to $S^p$. 
Thus, a necessary and sufficient condition that \( \psi^{-1}(v) \) have a critical point at \( v_0 \), is that the value \( v_0 \) be assumed by \( \psi^{-1}(w) \) at an \( A \)-point. A similar result holds for \( \psi^{-1}(u) \).

II. THE THREE SEQUENCES

We deal with the two permutable functions \( \Phi(z) \) and \( \Psi(z) \), of the respective degrees \( m > 1 \) and \( n > 1 \), and write

\[
\omega = F(z) = \Phi(\Psi(z)) = \Psi(\Phi(z)),
\]

or, more briefly,

\[
F = \Phi \Psi = \Psi \Phi.
\]

The greatest single source of work in this paper is the possibility of the existence of a rational function \( \sigma_0(z) \), of degree greater than unity, such that

\[
\Phi = \sigma_0 \Phi, \quad \Psi = \sigma_0 \Psi,
\]

where \( \sigma_0(z) \) and \( \Psi_0(z) \) are rational (even linear). Wherever the contrary is not stated, we shall assume that such a \( \sigma_0(z) \) exists.

With a view towards securing later a sharp separation of permutable pairs of functions into two classes, we establish now a definite method for selecting \( \sigma_0(z) \). We proceed as follows. As \( F = \Phi \Psi \), the function \( \Psi(z) \) determines \( m \) systems of imprimitivity of the group of \( F^{-1}(w) \), each containing \( n \) letters. Also, if \( \Phi = \Psi \sigma_0 \), the function \( \sigma_0(z) \) determines systems of imprimitivity of the group of \( F^{-1}(w) \). Two branches, \( \varepsilon_1 \) and \( \varepsilon_2 \), will be in the same one of the systems determined by \( \sigma_0(z) \) if \( \sigma_0(\varepsilon_1) = \sigma_0(\varepsilon_2) \). But as \( \Psi(\varepsilon_1) = \Psi(\varepsilon_2) \) in this case, \( \varepsilon_1 \) and \( \varepsilon_2 \) are both in one system determined by \( \Psi(z) \). Hence every system determined by \( \sigma_0(z) \) is contained in a system determined by \( \Psi(z) \), so that each system determined by \( \Psi(z) \) is composed of one or more systems determined by \( \sigma_0(z) \). A similar fact is true of \( \Phi(z) \).

Thus, given any system determined by \( \Phi(z) \), there is at least one system determined by \( \Psi(z) \) with which it has more than one letter in common. Now, it is a simple consequence of the elementary notions on imprimitivity that if a group has two sets of systems of imprimitivity, there exists a number \( t_0 \), such that if a system of the first set has at least one letter in common with a system of the second set, it has precisely \( t_0 \) letters in common with it. These systems of \( t_0 \) letters are themselves systems of imprimitivity.

We shall suppose, in what follows, that \( \sigma_0(z) \) is so taken that it determines the systems of imprimitivity of \( t_0 \) letters just shown to exist. This determines
σ₀(ζ) to within a linear function, and the particular disposition made in regard to this linear function is of no importance for what follows.

We see now that there exists no rational function β(ζ), of degree greater than unity, such that

\[ \varphi_0 = \zeta \beta, \quad \psi_0 = \xi \beta, \]

where ζ(ζ) and ξ(ζ) are rational. If there did, each system of the group of \( F^{-1}(\omega) \) determined by \( \Phi(\omega) \) would have more than \( t_0 \) letters in common with some system determined by \( \Psi(\omega) \).

We have

\[ \varphi_0 \sigma_0 \psi_0 \sigma_0 = \psi_0 \sigma_0 \varphi_0 \sigma_0, \]

so that

(3) \[ \varphi_0 \sigma_0 \psi_0 = \psi_0 \sigma_0 \varphi_0. \]

Let the degrees of \( \varphi_0(\omega), \psi_0(\omega), \sigma_0(\omega) \) be \( r_0, s_0, t_0 \), respectively. We represent each member of (3) by \( G \). The function \( \sigma_0 \psi_0 \) determines \( r_0 \) systems of imprimitivity of the group of \( G^{-1}, \) each containing \( s_0 t_0 \) letters. Also, \( \sigma_0 \varphi_0 \) determines \( s_0 \) systems of imprimitivity of the group of \( G^{-1}, \) each containing \( r_0 t_0 \) letters. The \( s_0 t_0 \) letters in any system determined by \( \sigma_0 \psi_0 \) are distributed among the \( s_0 \) systems determined by \( \sigma_0 \varphi_0 \). Hence given any system determined by \( \sigma_0 \psi_0 \), there is some system determined by \( \sigma_0 \varphi_0 \) with which it has at least \( t_0 \) letters in common. Thus, by what goes before, there exists a number \( t_1 \geq t_0 \), such that if a system determined by \( \sigma_0 \psi_0 \) has at least one letter in common with a system determined by \( \sigma_0 \varphi_0 \), the two systems have precisely \( t_1 \) letters in common. The systems determined by \( \sigma_0 \psi_0 \) and by \( \sigma_0 \varphi_0 \) break up into a third set of systems of imprimitivity, which are determined by a function \( \sigma_1(\omega) \), of degree \( t_1 \). Also, we have

\[ \sigma_0 \varphi_0 = \varphi_1 \sigma_1, \quad \sigma_0 \psi_0 = \psi_1 \sigma_1, \]

where \( \varphi_1(\omega) \) and \( \psi_1(\omega) \) are rational functions of the respective degrees \( r_1 \leq r_0 \) and \( s_1 \leq s_0 \). Furthermore, there exists no rational \( \beta(\omega) \) of degree greater than unity such that

\[ \varphi_1 = \zeta \beta, \quad \psi_1 = \xi \beta, \]

where ζ(ζ) and ξ(ζ) are rational. Finally, by (3),

\[ \varphi_0 \psi_1 \sigma_0 = \psi_0 \varphi_1 \sigma_0, \]

* If \( \varphi_0 \) is linear, we have to consider all of the branches of \( G^{-1} \) as forming a single system of imprimitivity.
so that

\[ \varphi_0 \psi_1 = \psi_0 \varphi_1. \]

We now subject the permutable functions \( \varphi_1 \sigma_i \) and \( \psi_i \sigma_i \) to the treatment given above to \( \Phi(z) \) and \( \Psi(z) \). What follows is plain. There exist three sequences

(A) \( \varphi_0, \varphi_1, \varphi_2, \ldots, \psi_i, \ldots \)

(B) \( \psi_0, \psi_1, \psi_2, \ldots, \psi_i, \ldots \)

(C) \( \sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_i, \ldots \)

the degrees of the functions in the first two sequences being non-increasing, those of the functions in the third sequence non-decreasing, and the following relations holding for every \( i \):

\[ \varphi_i \sigma_i \psi_i \sigma_i = \psi_i \sigma_i \varphi_i \sigma_i, \]

\[ \sigma_i \varphi_i = \varphi_{i+1} \sigma_{i+1}, \quad \sigma_i \psi_i = \psi_{i+1} \sigma_{i+1}, \]

\[ \varphi_i \psi_{i+1} = \psi_i \varphi_{i+1}; \]

furthermore, for no \( i \) does a function \( \beta(z) \) of degree greater than unity exist such that

\[ \varphi_i = z\beta, \quad \psi_i = \xi \beta, \]

where \( \zeta(z) \) and \( \xi(z) \) are rational.

For the case in which no \( \sigma_0(z) \) exists, we define the sequences (A) and (B) by the equations

\[ \varphi_i = \Phi, \quad \psi_i = \Psi. \]

and all facts proved for (A) and (B) when (C) exists will hold also in this case. From the monotonic character of the degrees of the functions in the three sequences, and from the fact that the degree of every \( \sigma_i(z) \) is not greater than the smaller of \( m \) and \( n \), we derive the

CONCLUSION. There exist a subscript \( i_0 \), and three integers \( r, s \) and \( t \), such that, for \( i \geq i_0 \), \( \varphi_i(z), \psi_i(z) \) and \( \sigma_i(z) \) are of the respective degrees \( r, s \) and \( t \).

* We have from (3), \( \sigma_0 \varphi_0 \sigma_0 \varphi_0 = \sigma_0 \psi_0 \sigma_0 \psi_0. \)
From this point on, until we come to the last section of our paper, we shall work under the assumption that $r$ and $s$ each exceed unity. We assume, that is, that for no $i$ is one of the permutable functions $\varphi_i$ or $\psi_i$ a rational function of the other. This understood, we prove the important

**Lemma.** There exist a subscript $i_0$, and two integers $h \leq 4$ and $k \leq 4$, such that, for $i \geq i_0$, every $\varphi_i(z)$ is of degree $r$, while its inverse has precisely $h$ critical points; and every $\psi_i(z)$ is of degree $s$, while its inverse has precisely $k$ critical points.

We write

$$w = F(z) = \varphi_i \left[ \psi_i \begin{smallmatrix} x \end{smallmatrix} \right] = \psi_i \left[ \varphi_i \begin{smallmatrix} x \end{smallmatrix} \right],$$

and assume that $i \geq i_0$, so that $\varphi_i(z)$ and $\varphi_{i+1}(z)$ are each of degree $r$, and $\psi_i(z)$ and $\psi_{i+1}(z)$ each of degree $s$. As there exists no non-linear rational $\beta(z)$ such that $\varphi_{i+1} = \zeta \beta$, $\psi_{i+1} = \xi \beta$, where $\zeta(z)$ and $\xi(z)$ are rational, each of the systems of imprimitivity of the group of $F^{-1}(w)$ determined by $\varphi_{i+1}(z)$ has precisely one letter in common with each system determined by $\psi_{i+1}(z)$. Hence we can employ the notion of the $A$-point.

Suppose that $\varphi^{-1}_i$ has $g$ critical points, $w_1, \ldots, w_g$. We seek a lower bound for the sum of the indices of $\varphi^{-1}_i$ at these $g$ points. That sum equals $gs - j$, where $j$ is the number of points which $\varphi^{-1}_i$ has at $w_1, \ldots, w_g$. If $p$ of these points are simple points, the other $j - p$ are at least of order 2, and we have

$$2(j - p) + p \leq gs,$$

so that $j \leq (gs + p)/2$, and the sum of the indices of $\varphi^{-1}_i$ at $w_1, \ldots, w_g$ is at least $(gs - p)/2$. We observe that if one of the $j$ points is of order greater than 2, or if there are fewer than $p$ simple points, (4) must be an inequality, and the sum of the indices of $\varphi^{-1}_i$ at $w_1, \ldots, w_g$ will exceed $(gs - p)/2$.

Suppose now that $\varphi^{-1}_{i+1}$ has fewer than $g$ critical points. Then $\varphi^{-1}_i$ must have fewer than $g$ simple points at $w_1, \ldots, w_g$, for every such simple point of $\varphi^{-1}_i$ is an $A$-point, and yields a critical point of $\varphi^{-1}_{i+1}$. Hence the sum of the indices of $\varphi^{-1}_i$ at $w_1, \ldots, w_g$ exceeds $(gs - g)/2$ and we have

$$\frac{gs - g}{2} < gs - 2.$$

so that $g < 4$. Thus $\varphi^{-1}_i$ has three critical points, and $\varphi^{-1}_{i+1}$ has two.

At each of the two critical points of $\varphi^{-1}_{i+1}$, its $r$ branches must be permuted in a single cycle. From the manner in which the critical points of $\varphi^{-1}_{i+2}$ depend
on those of \( g_{i+1}^{-1} \), we see that, at every critical point of \( g_{i+2}^{-1} \), its branches undergo a substitution which is a power of a cyclic substitution in \( r \) letters. Such a substitution must be regular, that is, it displaces every letter, and the order of its cycles are all equal. It follows that, for \( j > i \), the critical points of every \( g_j^{-1} \) have regular substitutions. Hence at every critical point of \( g_j^{-1} (j > i) \), all of the branches of \( g_j^{-1} \) are permuted, so that the index of \( g_j^{-1} \), at each of its critical points, is at least \( r/2 \). As the sum of the indices of every \( g_j^{-1} \) is \( 2r - 2 \), every \( g_j^{-1} \) has either two critical points or three.

If a \( g_j^{-1} (j > i) \) has three critical points, it cannot have one at which its branches are permuted in a single cycle; in that case the sum of its indices would exceed \( 2r - 2 \). Hence \( g_{j+1}^{-1} \) must also have three critical points, for \( g_j^{-1} \) cannot transmit to \( g_{j+1}^{-1} \) a critical point with a single cycle.

We see now that if \( g_{i+1}^{-1} \) has fewer critical points than \( g_i^{-1} \), then, either each \( g_j^{-1} \) has two critical points for \( j > i \), or else, for \( j \) sufficiently large, each \( g_j^{-1} \) has three critical points. It remains only to settle the case in which, for every \( i > i_0 \), \( g_{i+1}^{-1} \) has at least as many critical points as \( g_i^{-1} \). In this case, since \( g_i^{-1} \) cannot have more than \( 2r - 2 \) critical points, it is evident that an \( \bar{h} \) exists, such that, for \( i \) sufficiently large, each \( g_i^{-1} \) has precisely \( \bar{h} \) critical points. When \( g_i^{-1} \) and \( g_{i+1}^{-1} \) have an equal number of critical points, we find that the \( g \) of the preceding page does not exceed 4. Hence \( \bar{h} \leq 4 \).

The argument for \( \varphi_i(x) \) holds also for \( \psi_i(x) \), and the lemma is proved.

### III. The Critical Points of \( g_i^{-1} \) and \( \psi_i^{-1} \)

From this point on, every subscript \( i \) will be understood to be not less than the \( i_1 \) of the preceding lemma. We assume also that \( h \geq k \); if this is not so at the start, we need only interchange the designations of \( \Phi(x) \) and \( \Psi(x) \). Let \( h = 4 \), and let the critical points of \( g_i^{-1} \) be \( w_1, \ldots, w_s \). If \( \psi_i^{-1} \) has \( p \leq 4 \) simple points at \( w_1, \ldots, w_s \), the sum of the indices of \( \psi_i^{-1} \) at \( w_1, \ldots, w_s \) is at least \( (4s - p)/2 \); if one of the points of \( \psi_i^{-1} \) at \( w_1, \ldots, w_s \) is of order greater than 2, this lower bound must be increased. We must thus have

\[
\frac{4s - p}{2} \leq 2s - 2.
\]

If \( p \) were less than 4, or if \( \psi_i^{-1} \) had a point of order greater than 2 at \( w_1, \ldots, w_s \) (5) could not hold. Hence \( p = 4 \), and those points of \( \psi_i^{-1} \) at \( w_1, \ldots, w_s \) which are not simple are all of order 2. Furthermore, it is clear that \( \psi_i^{-1} \) can have no critical points other than \( w_1, \ldots, w_s \).
We shall see below that the points of \( \varphi_{t-1} \) are also all of order 2, except four which are simple.

Let \( h = 3 \), and let the critical points of \( \varphi_{t-1} \) be \( a, b, \) and \( c \). We show first that \( \varphi_{t-1} \) has no critical points other than \( a, b, c \).

Suppose first that \( s > 3 \). If \( a \) were not a critical point of \( \varphi_{t-1} \), \( \varphi_{t-1} \) would have at least four simple points at \( a \), so that \( \varphi_{t+1} \) would have at least four critical points. Hence \( a \), and similarly \( b \) and \( c \), are critical points of \( \varphi_{t-1}' \). As \( k \leq 3 \), \( \varphi_{t+1}^{-1} \) can have no other critical points.

Let \( s = 3 \). If \( a \) is not a critical point of \( \varphi_{t-1}^{-1} \), \( \varphi_{t-1} \) must have no simple point at \( b \) or at \( c \). This means that the branches of \( \varphi_{t-1}^{-1} \) are permuted in a single cycle at \( b \) and at \( c \), so that \( b \) and \( c \) are the only critical points of \( \varphi_{t-1}^{-1} \).

Let \( s = 2 \); \( \varphi_{t-1}^{-1} \) has just two branch points, which are both simple. If one or both of these were not placed at \( a, b, c \), at least two of these latter points would not be critical points of \( \varphi_{t-1}^{-1} \), and \( \varphi_{t-1}^{-1} \) would have at least four \( A \)-points.

Thus, when \( h = 3 \), \( \varphi_{t-1}^{-1} \) has no critical points other than \( a, b, c \). When \( h = 3 \), we must have \( r > 3 \). But we have seen above that when \( h = 3 \) and \( k = 2 \), we have \( s < 3 \). Hence, in the case of \( h = 3 \), it is legitimate to assume that \( r \geq s \). In §§ V, VII, VIII, IX, which deal with the case of \( h = 3 \), it will be understood, unless otherwise stated, that \( r \geq s \).

Suppose that \( h = 3 \), and that at the critical points \( a, b, c \), of \( \varphi_{t-1}^{-1} \), the branches of \( \varphi_{t-1}^{-1} \) undergo substitutions of orders \( \alpha_i, \beta_i, \gamma_i \), respectively. The function \( \varphi_{t-1}^{-1} \) must have precisely three \( A \)-points. Let \( x \) be the sum of the orders of the \( A \)-points of \( \varphi_{t-1}^{-1} \) at \( a, y \) at \( b \) and \( z \) at \( c \). The orders of those points of \( \varphi_{t-1}^{-1} \) at \( a \) which are not \( A \)-points are divisible by \( \alpha_i \). Hence their number is at most \( (s - x)/\alpha_i \). Thus, the total number of points of \( \varphi_{t-1}^{-1} \) at \( a, b, c \) is at most

\[
(6) \quad \frac{s - x}{\alpha_i} + \frac{s - y}{\beta_i} + \frac{s - z}{\gamma_i} + 3.
\]

As \( \varphi_{t-1}^{-1} \) has no critical points other than \( a, b, c \), the sum of its indices at \( a, b, c \) is \( 2s - 2 \). This means that \( \varphi_{t-1}^{-1} \) has \( s + 2 \) points at \( a, b, c \). Thus the expression (6) is at least \( s + 2 \), and we find, directly.

\[
(7) \quad \frac{1}{\alpha_i} + \frac{1}{\beta_i} + \frac{1}{\gamma_i} \geq 1 - \frac{1}{s} + \frac{1}{s} \left( \frac{x}{\alpha_i} + \frac{y}{\beta_i} + \frac{z}{\gamma_i} \right).
\]

In particular, the first member of (7) exceeds \((1-1/s)\). Suppose that \( \varphi_{t-1}^{-1} \) has at \( a \) an \( A \)-point of order \( g \). This \( A \)-point gives rise to a critical point of \( \varphi_{t+1}^{-1} \) at which the branches of \( \varphi_{t+1}^{-1} \) undergo a substitution similar to the \( g \)th power of the substitution which the branches of \( \varphi_{t-1}^{-1} \) undergo at \( a \). The order, call it \( \alpha_{i+1} \), of the substitution at this critical point of \( \varphi_{t+1}^{-1} \) is \( \alpha_i \), divided by
the greatest common divisor of \( \alpha_i \) and \( g \). Certainly, then, \( 1/\alpha_{i+1} \) is not greater than \( g/\alpha_i \). It is easy now to see that

\[
\frac{x}{\alpha_i} + \frac{y}{\beta_i} + \frac{z}{\gamma_i} \geq \frac{1}{\alpha_{i+1}} + \frac{1}{\beta_{i+1}} + \frac{1}{\gamma_{i+1}},
\]

so that

\[
(8) \quad \frac{1}{\alpha_i} + \frac{1}{\beta_i} + \frac{1}{\gamma_i} \geq 1 - \frac{1}{s} + \frac{1}{s} \left( \frac{1}{\alpha_{i+1}} + \frac{1}{\beta_{i+1}} + \frac{1}{\gamma_{i+1}} \right).
\]

But the quantity in parentheses in the second member of (8) exceeds \( 1 - 1/s \), the lower bound secured above for the first member. Hence

\[
\frac{1}{\alpha_i} + \frac{1}{\beta_i} + \frac{1}{\gamma_i} > 1 - \frac{1}{s} + \frac{1}{s} \left( 1 - \frac{1}{s} \right) = 1 - \frac{1}{s^2}.
\]

This gives a new lower bound for the quantity in parentheses in (8), which, when substituted, shows that the first member is not less than \( 1 - 1/s^2 \). As this process may be repeated indefinitely, we arrive at the

**FUNDAMENTAL RELATION**

\[
(9) \quad \frac{1}{\alpha_i} + \frac{1}{\beta_i} + \frac{1}{\gamma_i} \geq 1.
\]

When \( h = 4 \), the same procedure leads easily to the result that the sum of the reciprocals of the orders of the substitutions at the four critical points of \( \varphi_i^{-1} \) is at least 2. As each order is at least 2, each order must be precisely 2. Thus, all points of \( \varphi_i^{-1} \) which are not simple are of order 2. It follows, as at the beginning of this section, that \( \varphi_i^{-1} \) has precisely four simple points.

We suppose finally that \( h = 2 \). Two cases have to be considered:

(a) Either \( r \) or \( s \) exceeds 2. In this case, \( \varphi_i^{-1} \) and \( \psi_i^{-1} \) must have the same two critical points, else one of them would have more than two simple points, and these would be A-points. At both critical points, the branches of both functions are permuted in single cycles.

(b) \( r = s = 2 \). In this case, \( \varphi_i^{-1} \) and \( \psi_i^{-1} \) cannot have the same two critical points, else neither would have an A-point. If they had no critical point in common, each would have four A-points. Hence we may assume that \( \varphi_i^{-1} \) has simple branch points at each of two points \( a \) and \( b \), and that \( \psi_i^{-1} \) has one simple branch point at \( a \), and one at a third point \( c \).
In what follows, special attention has to be given to certain cases in which \( s \) is small. A device which would permit us to assume that \( r \) and \( s \) are arbitrarily large, one based, for instance, on replacing the permutable functions by iterates of themselves, would eliminate many painful paragraphs.

IV. THE MULTIPLICATION FORMULAS FOR \( \sigma^i \); THE POWERS OF \( \varepsilon \)

We consider the case in which \( h = 2 \), and in which \( r \) and \( s \) are not both 2.

In this case, \( \varphi^{-1}_i \) and \( \psi^{-1}_i \) each have the same two critical points, \( a_i \) and \( b_i \). Both branch points of either function are \( A \)-points of that function. Also, the values \( a_{i+1} \) and \( b_{i+1} \) which \( \varphi^{-1}_i \) assumes at its \( A \)-points are the same two values which \( \psi^{-1}_i \) assumes at its \( A \)-points, for these are the affixes of the critical points of \( \varphi^{-1}_{i+1} \) and \( \psi^{-1}_{i+1} \).*

Turning now to the sequence \((C)\), we shall prove that \( \sigma^{-1}_i \) has no critical points other than \( a_{i+1} \) and \( b_{i+1} \). Suppose that \( \sigma^{-1}_i \) has such critical points other than \( a_{i+1} \) and \( b_{i+1} \), and let \( m_i \) be the sum of its indices at those additional critical points. As \( m_i \leq 2r - 2 \), there is an \( i \) for which \( m_i \) is a maximum. Suppose that it is this \( i \) with which we are dealing. Consider the relation

\[
\sigma_i \varphi_i = \varphi_{i+1} \sigma_{i+1}.
\]

From a remark made at the beginning of § I, we see that the sum of the indices of \( \varphi^{-1}_i \) \( \sigma^{-1}_i \) at the critical points of \( \sigma^{-1}_i \) other than \( a_{i+1} \) and \( b_{i+1} \) is at least \( rm_i \). Turning to the second member of (10) we see that \( \sigma^{-1}_{i+1} \) has critical points whose affixes are not values of \( \varphi^{-1}_i \) at \( a_{i+1} \) and \( b_{i+1} \), that is, not \( a_{i+2} \) or \( b_{i+2} \), at which the sum of its indices is at least \( rm_i \). This contradicts the assumption that \( m_i \) is a maximum.

Thus the inverses of \( \varphi_i \sigma_i \) and \( \psi_i \sigma_i \) have the two critical points \( a_i, b_i \), at which all of their branches are permuted in single cycles. From this, and from the relation

\[
\sigma_i \psi_i = \psi_{i+1} \sigma_{i+1},
\]

it follows that the two critical points of \( \psi^{-1}_i \) are the values which \( \sigma^{-1}_i \) assumes at \( a_{i+1} \) and \( b_{i+1} \). Hence the inverse of

\[
F' = \varphi_i \sigma_i \psi_i \sigma_i
\]

* It is not necessary that \( \varphi^{-1}_i \) and \( \psi^{-1}_i \) should assume the same value at a particular \( A \)-point.
has only the two critical points $a_i$ and $b_i$. Also, from the condition of permutability, we see that the two values $a_i$ and $b_i$ are assumed by the inverses of both $\varphi_i \sigma_i$ and $\psi_i \sigma_i$ at $a_i$ and $b_i$. These same values are assumed by $F^{-1}$ at $a_i$ and $b_i$.

The relation

$$\varphi_i \sigma_i \psi_i \sigma_i = \sigma_{i-1} \varphi_{i-1} \sigma_{i-1} \psi_{i-1}$$

shows that $\sigma_{i-1}$ has no critical point other than $a_i$ and $b_i$, and that the inverse of $\varphi_{i-1} \sigma_{i-1} \psi_{i-1}$ has no critical points other than $a_{i-1}$ and $b_{i-1}$, the values (so we designate them) which $\sigma_{i-1}'$ assumes at its critical points. Hence the inverse of the function

$$\varphi_{i-1} \sigma_{i-1} \psi_{i-1} \sigma_{i-1} = \psi_{i-1} \sigma_{i-1} \varphi_{i-1} \sigma_{i-1}$$

has no critical points other than $a_{i-1}$ and $b_{i-1}$.

Thus the inverses of $\varphi_{i-1} \sigma_{i-1}$ and $\psi_{i-1} \sigma_{i-1}$ have $a_{i-1}$ and $b_{i-1}$ as their only critical points, and the values assumed by the inverses at their critical points are $a_{i-1}$ and $b_{i-1}$.

Continuing thus, we find that the inverses of the original permutable functions

$$\Phi = \varphi_0 \sigma_0, \quad \Psi = \psi_0 \sigma_0$$

have two critical points $a_0$ and $b_0$, and that the values of $\Phi^{-1}$ and $\Psi^{-1}$ at their critical points are $a_0$ and $b_0$.

The statement just made in regard to $\Phi(z)$ and $\Psi(z)$ holds also if no $\sigma_0$ exists; it is only necessary to discard those parts of the proof which involve a $\sigma$.

Let $\lambda(z)$ be any linear function such that $\lambda(a_0) = 0$, and $\lambda(b_0) = \infty$. Let

$$\Phi_1 = \lambda \Phi \lambda^{-1}, \quad \Psi_1 = \lambda \Psi \lambda^{-1}.$$ 

Then $\Phi_1^{-1}$ and $\Psi_1^{-1}$ have the two critical points 0 and $\infty$, and their values at these points are 0 and $\infty$.

It follows that

$$\Phi_1(z) = \eta \varepsilon^p, \quad \Psi_1(z) = \varepsilon \varepsilon^q,$$

where $p = \pm m$, and $q = \pm n$. If we multiply $\lambda(z)$ by a suitable constant, we will have $\eta = 1$. The condition of permutability then gives $\varepsilon \varepsilon^p = 1$.

Thus all pairs of permutable functions, of the type considered in this section, are found by transforming with a linear function the permutable pair $\varepsilon^p$ and $\varepsilon \varepsilon^q$ where $p$ and $q$ are positive or negative integers, and where $\varepsilon \varepsilon^{p-1} = 1$. 

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V. THE MULTIPLICATION FORMULAS FOR COS ε

This section and §§ VII, VIII, IX will handle the cases in which h = 3. We saw in § III that it is permissible to assume, when h = 3, that \( r \geq s \); where the contrary is not stated, this assumption will be understood to hold.

In all cases for which h = 3, we represent the critical points of \( \psi_i^{-1} \) by \( a_i, \beta_i, c_i \), and the orders of the substitutions which the branches of \( \psi_i^{-1} \) undergo at \( a_i, b_i, c_i \) by \( \alpha_i, \beta_i, \gamma_i \), respectively. The orders of the substitutions which the branches of \( \psi_i^{-1} \) undergo at \( a_i, b_i, c_i \) we denote by \( \alpha'_i, \beta'_i, \gamma'_i \), respectively.

We consider in this section the case in which, for some \( i \geq i \), two of the orders \( \alpha_i, \beta_i, \gamma_i \), say \( \alpha_i \) and \( \beta_i \), equal 2. The \( i \) used in this section is supposed to be of the type just described, and stays fixed throughout our work.

Consider first the case in which \( k = 3 \). We know that the critical points of \( \psi_i^{-1} \) are \( a_i, b_i, c_i \). We are going to show that \( \alpha'_i = \beta'_i = 2 \).

As \( r \geq 3 \), and as the branch points of \( \psi_i^{-1} \) at \( a_i \) and \( b_i \) are all simple, \( \psi_i^{-1} \) has at least four points (together) at \( a_i \) and \( b_i \). If \( \alpha'_i \) and \( \beta'_i \) both exceeded 2, all of these points would be \( A \)-points, whereas we know that \( \psi_i^{-1} \) has only three \( A \)-points. We may suppose thus that \( \alpha'_i = 2 \).

If \( r > 6 \), \( \psi_i^{-1} \) has at least four points at \( b_i \), so that \( \beta'_i \) is also 2. The cases in which \( r \leq 6 \), which create a type of nuisance of which there will be more later, we have to examine in detail.

Suppose that \( \beta'_i > 2 \). If \( r = 6 \), \( \psi_i^{-1} \) must have three branch points (simple), at \( b_i \); if it had fewer, it would have four or more \( A \)-points at \( b_i \). Hence, \( \alpha'_{i+1}, \beta'_{i+1}, \gamma'_{i+1} \), the orders of the substitutions at the critical points of \( \psi_{i+1}^{-1} \), are all equal. Thus, by (9), \( 3/\alpha_{i+1} \geq 1 \), and \( \alpha_{i+1} \) is either 3 or 2.

Suppose that \( \alpha'_{i+1} = 3 \). As the branch points of \( \psi_i^{-1} \) at \( b_i \) are all simple, the substitutions at the critical points of \( \psi_{i+1}^{-1} \) are all similar to the square of the substitution which the branches of \( \psi_i^{-1} \) undergo at \( b_i \), so that \( \beta'_i \) is 6, or 3. If \( \beta'_i = 6 \), we see, remembering that \( r \geq s \), that \( s = 6 \). This produces the absurdity that the sum of the indices of \( \psi_{i+1}^{-1} \) is 12, or more than \( 2s - 2 \). If \( \beta'_i = 3 \), we have \( s \geq 3 \). If \( s > 3 \), then, since \( \beta_i \) is prime to \( \beta'_i \), \( \psi_i^{-1} \) has at least two \( A \)-points at \( b_i \), so that \( \varphi_{i+1}^{-1} \) must have at least two critical points at which its branch points are simple. This contradicts the fact that \( \alpha'_{i+1}, \beta'_{i+1}, \gamma'_{i+1} \) all equal 3, for we saw above that one of them must be 2 if two of \( \alpha_{i+1}, \beta_{i+1}, \gamma_{i+1} \) are 2. Finally, if \( s = 3 \), \( \psi_i^{-1} \) has an \( A \)-point at \( a_i \), as well as at \( b_i \), so that the argument just used applies.

Suppose that \( \alpha'_{i+1} = 2 \). The index of \( \psi_{i+1}^{-1} \) cannot exceed \( s/2 \) at any critical point. Hence,

\[
\frac{3s}{2} \geq 2s - 2.
\]
or \( s \leq 4 \). We cannot have \( s = 3 \), as \( \psi_{i+1}^{-1} \) would have only three simple branch points in that case. If \( s = 4 \), there must be two simple branch points at each critical point of \( \psi_{i+1}^{-1} \). Hence \( \psi_{i+1}^{-1} \) would have an even number of \( A \)-points, whereas it must have three. This completes the proof that \( \beta_i' = 2 \) when \( r = 6 \).

When \( r = 5 \), \( \varphi_i^{-1} \) has at least one \( A \)-point at \( a_i \). If \( \beta_i' > 2 \), it would have at least three at \( b_i \). Thus \( \beta_i' = 2 \).

Suppose that \( r = 4 \) and that \( \beta_i' = 2 \). We must have \( s \leq 4 \), so that \( \beta_i' \leq 4 \).

If \( \varphi_i^{-1} \) has two simple branch points both at \( a_i \) and at \( b_i \), we may suppose that the substitutions at these points are (12) (34) and (13) (24) respectively. Hence the substitution at \( c_i \) is (14) (23). This, as seen above, leads to the absurdity that \( \varphi_i^{-1} \) has an even number of \( A \)-points.

Thus \( \varphi_i^{-1} \) must have two simple points, either at \( a_i \) or at \( b_i \). As \( \varphi_i^{-1} \) has at least two \( A \)-points at \( b_i \) when \( \beta_i' > 2 \), the simple points cannot be at \( a_i \), else \( \varphi_i^{-1} \) would have at least four \( A \)-points. Then \( \varphi_i^{-1} \) has a simple branch point and two simple points at \( b_i \). Suppose that \( s = 4 \). If \( \beta_i' \) were 4, \( \psi_i^{i+1} \) would have two critical points of index 3, and a third critical point, which is impossible, since the sum of its indices is 6. If \( \beta_i' \) were 3, we would have \( \alpha_i^{i+1} = \beta_i^{i+1} = \gamma_i^{i+1} = 3 \), and also \( \varphi_i^{-1} \) would have two critical points with only simple branch points, which come from the two \( A \)-points of \( \psi_i^{-1} \) at \( b_i \).

This was proved impossible above. If \( s = 3 \), and \( \beta_i' = 3 \), \( \psi_i^{i+1} \) would have three critical points of index 2, an impossibility.

Suppose that \( r = 3 \) and that \( \beta_i' > 2 \). We find the absurdity that \( \psi_i^{i+1} \) has two critical points of index 2 which come from \( b_i \), and a third critical point which comes from \( a_i \).

We have proved that \( \alpha_i' = \beta_i' = 2 \).

We shall now show that \( \varphi_i^{-1} \) and \( \psi_i^{-1} \) each have two simple points (two in all) at \( a_i \) and \( b_i \). Consider \( \varphi_i^{-1} \) for instance. If \( r \) is odd, the two simple points certainly exist. In the case where \( r \) is even, if there were no such simple points, the sum of the indices of \( \varphi_i^{-1} \) at \( a_i \) and \( b_i \) would be \( r \). Hence the index at \( c_i \) would be \( r - 2 \), and \( \varphi_i^{-1} \) would have precisely two points at \( c_i \). These would be the only possible \( A \)-points of \( \varphi_i^{-1} \), whereas there have to be three.

It follows from the above that, at \( c_i \), all of the branches of \( \varphi_i^{-1} \) are permuted in a single cycle. The same is true of \( \psi_i^{-1} \).

We examine now the case in which \( h = 3 \), \( a_i = \beta_i = 2 \), and \( k = 2 \).

As seen in § III, \( s \) must be 3 or 2. If \( s \) were 3, the two branch points of \( \psi_i^{-1} \) would be points of order 3. At least one of them would be situated either at \( a_i \) or at \( b_i \), and would be an \( A \)-point of \( \psi_i^{-1} \). Also \( \psi_i^{-1} \) would have 3 \( A \)-points at that critical point of \( \varphi_i^{-1} \) which is not a critical point of \( \psi_i^{-1} \).

Thus \( s = 2 \). If \( a_i \) and \( b_i \) were both critical points of \( \psi_i^{-1} \), \( \psi_i^{-1} \) would have only two \( A \)-points. Hence we may assume that the critical points of \( \psi_i^{-1} \) are \( b_i \) and \( c_i \).
We shall prove that \( r \) is odd, from which it will follow that \( \varphi_i^{-1} \) has one simple point at \( a_i \), one at \( b_i \), and that its branches are permuted in a single cycle at \( c_i \).

Suppose, contrarily, that \( r \) is even. We know that \( \varphi_i^{-1} \) cannot have more than two simple points (in all) at \( a_i \) and \( b_i \); if it did, its index at \( c_i \) would exceed \( r - 1 \). Then \( \varphi_i^{-1} \) can have no simple point at \( a_i \). If it had one, it would have two, and as \( \psi_i^{-1} \) has two \( A \)-points at \( a_i \), \( \varphi_i^{-1} \) would have two critical points with simple branch points and with four simple points, a condition as impossible for \( \varphi_{i+1}^{-1} \) as for \( \varphi_i^{-1} \).

It follows that \( \varphi_{i+1}^{-1} \) has two critical points with simple branch points and no simple points. Furthermore, it is permissible to replace \( i + 1 \) by \( i \), and to assume that \( \varphi_i^{-1} \) has no simple point at \( a_i \) or at \( b_i \).

This understood, it follows that \( \varphi_i^{-1} \) has just two points at \( c_i \), which must both be \( A \)-points. Consider the relation

\[
(11) \quad F^{-1} = \varphi_i \psi_{i+1} = \psi_i \varphi_{i+1}.
\]

From \( F = \varphi_i \psi_{i+1} \), we see that the two values which \( F^{-1} \) assumes at \( c_i \) are the two values which \( \psi_{i+1}^{-1} \) assumes at its critical points. As only one branch point of \( \psi_{i+1}^{-1} \) is an \( A \)-point, only one of the values of \( F^{-1} \) at \( c_i \) can be an affix of a critical point of \( \varphi_{i+2}^{-1} \), or, a fortiori, of \( \psi_{i+2}^{-1} \). Now \( \varphi_{i+1}^{-1} \) has two critical points with simple branch points and no simple points, at the points whose affixes are the values of \( \psi_i^{-1} \) at \( c_i \). Hence at the point whose affix is the value of \( \psi_i^{-1} \) at \( c_i \) (\( c_i \) is an \( A \)-point of \( \psi_i^{-1} \)), \( \varphi_{i+1}^{-1} \) has two points, which, as in the case of \( \varphi_i^{-1} \), must be \( A \)-points. This, since \( F = \psi_i \varphi_{i+1} \), entails the contradiction that the values of \( F^{-1} \) at \( c_i \) are both affixes of critical points of \( \psi_{i+2}^{-1} \). Thus \( r \) is odd.

When \( h = k = 3 \), the values of \( \varphi_i^{-1} \) and \( \varphi_i^{-1} \) at their simple points at \( a_i \) and \( b_i \) must be the same, namely, the affixes of the critical points with simple branch points of \( \varphi_{i-1}^{-1} \) and \( \psi_{i-1}^{-1} \). Similarly, \( \varphi_i^{-1} \) and \( \psi_i^{-1} \) must have the same single value at \( c_i \).

Suppose that, when \( k = 2 \), \( \varphi_i^{-1}(c_i) = c_{i+1} \) and \( \psi_i^{-1}(c_i) = c'_{i+1} \). We shall show that \( c_{i+1} = c'_{i+1} \).

We know that \( \psi_{i+1}^{-1} \) has a critical point at \( c_{i+1} \), and that \( \varphi_{i+1}^{-1} \) has a critical point at \( c'_{i+1} \) at which its sheets are permuted in a single cycle. Let \( c_{i+2} \) and \( c'_{i+2} \) be the values of \( \psi_{i+1}^{-1} \) and \( \varphi_{i+1}^{-1} \) at \( c_{i+1} \) and \( c'_{i+1} \) respectively. From (11), since \( F^{-1} \) has only one value at \( c_i \), we have \( c_{i+2} = c'_{i+2} \). But since \( \varphi_{i+1}^{-1} \) has an \( A \)-point where its branches are permuted in a single cycle, \( c_{i+2} \) is a critical point of \( \psi_{i+2}^{-1} \); and hence of \( \varphi_{i+2}^{-1} \). Thus \( \psi_{i+2}^{-1} \) must have an \( A \)-point at its critical point at \( c_{i+1} \). But the only branch point of \( \psi_{i+1}^{-1} \) which is an
A-point is the one at which the branches of \( \varphi_{i+1}^{-1} \) are permuted in a single cycle. Hence \( c_{i+1} = c'_i + c_{i+1}' \).

When \( k = 2 \), the value of \( \varphi_{i}^{-1} \) at its simple point at \( b_i \) is the affix of that critical point of \( \psi_{i+1}^{-1} \) at which the branches of \( \varphi_{i+1}^{-1} \) are permuted in pairs. We shall show that the value of \( \varphi_{i}^{-1} \) at its simple point at \( a_i \) is the affix of the second critical point of \( \varphi_{i+1}^{-1} \) where its branches are permuted in pairs.

As, by (11), \( F = \psi_i \varphi_{i+1} \), we see that \( F^{-1} \) has precisely two uniform branches at \( a_i \), whose values are the values of \( \varphi_{i+1}^{-1} \) at its simple points. One of these values is the affix of a critical point of \( \varphi_{i+1}^{-1} \) at which the branches of \( \varphi_{i+2}^{-1} \) are permuted in pairs. From \( F = \varphi_i \psi_{i+1} \) we see now that at the point whose affix is the value of \( \varphi_{i}^{-1} \) at its simple point at \( a_i \), both branches of \( \psi_{i+1}^{-1} \) are uniform, and the value of at least one of them is the affix of a critical point of \( \varphi_{i+2}^{-1} \). This can be so only if the value of \( \varphi_{i}^{-1} \) at its simple point at \( a_i \) is the affix of a critical point of \( \varphi_{i+1}^{-1} \) where the branches of \( \varphi_{i+1}^{-1} \) are permuted in pairs.

From what precedes, we see that when \( h = 3 \), and \( a_i = b_i = 2 \), both \( \varphi_{i}^{-1} \) and \( \psi_{i}^{-1} \) have two simple points at \( a_i \) and \( b_i \), and the values of \( \varphi_{i}^{-1} \) and \( \psi_{i}^{-1} \) at these simple points are the same. We shall call these values \( a_{i+1} \) and \( b_{i+1} \). Also, at \( c_i \), \( \varphi_{i}^{-1} \) and \( \psi_{i}^{-1} \) both have their branches permuted in a single cycle, and assume a common single value, which we shall call \( c_{i+1} \).

It is hardly necessary to mention the fact that for every \( j \) greater than the \( i \) used in our work above, \( \varphi_{j}^{-1} \) and \( \psi_{j}^{-1} \) will have the properties proved for \( \varphi_{i}^{-1} \) and \( \psi_{i}^{-1} \).

Precisely as in the preceding section, we can show that \( \sigma_i^{-1} \) has no critical points other than \( a_{i+1} \), \( b_{i+1} \), and \( c_{i+1} \).

We shall prove now that all branch points which \( \sigma_i^{-1} \) has at \( a_{i+1} \) and \( b_{i+1} \) are simple.

Suppose that for some \( j \geq i \), some of the branch points of \( \sigma_j^{-1} \) at \( a_{j+1} \) and \( b_{j+1} \) are not simple, and let \( m_j \) be the sum of the orders of the branch points which are not simple. Since \( m_j \leq 2t - 2 \), there is a \( j \) for which \( m_j \) is a maximum. We deal with such a \( j \).

Consider the relation

\[
\sigma_j \varphi_j = \varphi_{j+1} \sigma_{j+1}.
\]

We see that the inverse of \( \sigma_j \varphi_j \) has at \( a_{j+1} \) and \( b_{j+1} \) branch points which are not simple, and the sum of whose orders is at least \( rm_j \). As the critical points of \( \sigma_{j+1}^{-1} \) which are values of \( \varphi_{j+1}^{-1} \) at \( a_{j+1} \) and \( b_{j+1} \) are \( a_{j+2} \) and \( b_{j+2} \), the values of \( \varphi_{j+1}^{-1} \) at its simple points, and as the branch points of \( \varphi_{j+1}^{-1} \) at \( a_{j+1} \) and \( b_{j+1} \) are all simple, we see that \( \sigma_{j+1}^{-1} \) has branch points at \( a_{j+2} \) and \( b_{j+2} \) which are not simple, and the sum of whose orders is at least \( rm_j \). This contradicts the assumption that \( m_j \) is a maximum.
Thus, the inverse of \( \varphi_i \sigma_i \) has no critical point other than \( a_i, b_i, c_i \), and at \( a_i \) and \( b_i \) its branch points are all simple.

If every branch of \( \sigma_i^{-1} \) were permuted at \( a_{i+1} \) and at \( b_{i+1} \), \( \sigma_i^{-1} \) would have just two distinct values at \( c_{i+1} \) for its index at \( c_{i+1} \) would be \( t-2 \). Thus \( \varphi_i^{-1} \), since it has three critical points, would have at least one critical point which is a value assumed by \( \sigma_i^{-1} \) at some point other than \( c_{i+1} \). This means that the inverse of \( \varphi_i \) would either have more critical points than \( a_{i+1}, b_{i+1}, c_{i+1} \), or else it would have branch points of order greater than unity at \( a_{i+1} \) or \( b_{i+1} \). This is impossible by (12), according to what we know of the critical points of the second member of (12). Hence there are, at \( a_{i+1} \) and \( b_{i+1} \), precisely two places on the Riemann surface of \( \sigma_i^{-1} \) at which \( \sigma_i^{-1} \) is uniform. By (12), the value of \( \sigma_i^{-1} \) at these are \( a_i \) and \( b_i \). Also, at \( c_{i+1} \), the branches of \( \sigma_i^{-1} \) are permuted in a single cycle, and \( \varphi_i^{-1}(c_{i+1}) = c_i \).

Without difficulty, we see now that the inverse of

\[
F' = \varphi_i \sigma_i \psi_i \sigma_i
\]

has the two critical points \( a_i \) and \( b_i \) at which its branches are permuted in pairs, and the critical point \( c_i \) at which its branches are permuted in a single cycle. Also, at \( a_i \) and \( b_i \), there are two places on the surface of \( F'^{-1} \) at which \( F^{-1} \) is uniform, and the values of \( F^{-1} \) at these places are \( a_i \) and \( b_i \). Finally, \( F'^{-1}(c_i) = c_i \).

From the relation

\[
\varphi_i \sigma_i \psi_i \sigma_i = \sigma_{i-1} \varphi_{i-1} \sigma_{i-1} \psi_{i-1},
\]

we see that \( \sigma_{i-1}^{-1} \) has no critical points other than \( a_{i-1}, b_{i-1}, c_{i-1} \), that at \( a_i \) and \( b_i \) its branch points are all simple, and that at \( c_i \) its branches are permuted in a single cycle. Also, at \( a_i \) and \( b_i \), there are two places on the surface of \( \sigma_{i-1}^{-1} \) at which \( \sigma_{i-1}^{-1} \) is uniform, assuming certain values \( a_{i-1} \) and \( b_{i-1} \).

We let \( \sigma_{i-1}^{-1}(c_i) = c_{i-1} \). It is easy to see that the inverse of

\[
\varphi_{i-1} \sigma_{i-1} \psi_{i-1}
\]

has a critical point at \( c_{i-1} \) at which its branches are permuted in a single cycle. At \( a_{i-1} \) and \( b_{i-1} \) its branches are permuted in pairs, except that there are two places where the inverse is uniform and takes the values \( a_i \) and \( b_i \). Then the inverse of

\[
\varphi_{i-1} \sigma_{i-1} \psi_{i-1} \sigma_{i-1}
\]
has all its branches permuted in a single cycle at \( c_{i-1} \); and all its branches permuted in pairs at \( a_{i-1} \) and \( b_{i-1} \), except that there are two places where the inverse is uniform and takes the values \( a_{i-1} \) and \( b_{i-1} \). The value at \( c_{i-1} \) is \( c_{i-1} \).

Continuing thus, we prove that the inverse of

\[ \Phi \Psi = \Psi \Phi \]

has three critical points \( a_0, b_0, c_0 \), where it behaves in the manner already frequently described.

Also, \( \Phi^{-1} \) and \( \Psi^{-1} \) have the critical points \( a_0, b_0, c_0 \). It is obvious that \( \Phi^{-1}(c_0) = \Psi^{-1}(c_0) = c_0 \). Furthermore, as \( \Phi \) and \( \Psi \) are at least of degree 4, their inverses both actually have critical points at \( a_0 \) and \( b_0 \), so that the values which each inverse assumes at those places at \( a_0 \) and \( b_0 \) where it is uniform are \( a_0 \) and \( b_0 \).

When the sequence \( (\mu) \) does not exist, we may take \( i = 0 \), so that \( \phi_0 = \phi_1 = \Phi \), \( \psi_0 = \psi_1 = \Psi \). The values \( a_1 \), \( b_1 \), \( c_1 \), which \( \Phi^{-1} \) and \( \Psi^{-1} \) assume at their simple points and at \( c_0 \), are seen directly to be the same as \( a_0 \), \( b_0 \), \( c_0 \).

Let \( \lambda(x) \) be a linear function such that

\[ \lambda(a_0) = 1, \quad \lambda(b_0) = -1, \quad \lambda(c_0) = \infty. \]

Then the inverses of the two permutable functions

\[ \Phi_1 = \lambda \Phi \lambda^{-1}, \quad \Psi_1 = \lambda \Psi \lambda^{-1} \]

have simple branch points at 1 and \(-1\), and their branches are permuted in a single cycle at \( \infty \). The values of their uniform branches at 1 and \(-1\) are 1 and \(-1\), and \( \Phi_1^{-1}(\infty) = \Psi_1^{-1}(\infty) = \infty \).

Consider the function \( \cos \alpha \). Wherever it assumes either of the values 1 or \(-1\), it assumes it twice. It never assumes the value \( \infty \). Hence, if we operate on \( \cos \alpha \) with \( \Phi^{-1} \), the \( n \) values of \( \Phi^{-1}(\cos \alpha) \) are uniform \( \text{im kleinen} \), and therefore, also, uniform \( \text{im großen} \). As \( \Omega^{-1} \) assumes the value \( \infty \) only at \( \infty \), these \( n \) functions are entire.

Let \( f(z) \) be one of these entire functions. As \( \Phi^{-1}(\cos \alpha) \) assumes a value 1 or \(-1\) only when \( \cos \alpha \) is 1 or \(-1\), and then only through a uniform branch of \( \Phi^{-1} \), \( f(z) \) cannot assume a value 1 or \(-1\) unless it assumes it twice. Also, \( f(z) \) is never infinite.
Consider the function $\arccos z$. Its only finite critical points are 1 and $-1$, at which its branches are permuted in pairs. Hence, by the same reasoning used above, there are an infinite number of entire functions $\arccos f(z)$. Let $\beta(z)$ be one of these. We shall show that $\beta(z)$ is linear.

Wherever $\Phi_1^{-1}(z)$ is large, its modulus is of the order of $\sqrt{m|z|}$, where $m$ is the degree of $\Phi_1$. Now, since

$$|\cos z| \leq e^{|z|},$$

there is a $k$ such that

$$|f(z)| < k e^{|z|m}|,$$

for every $z$.

Now if

$$\beta(z) = u(x, y) + iv(x, y)$$

were not linear, there would be values of $z$ of large modulus for which

$$v(x, y) < -|z|,$$

and hence for which

$$|f(z)| = |\cos \beta(z)| \geq \frac{e^{|z|} - e^{-|z|}}{2}.$$

This contradicts the first inequality for $|f(z)|$.

Thus there is a relation

$$\cos z = \Psi_1(\cos (pz + q)),$$

or, what is the same, a relation

$$\cos (az + b) = \Psi_1(\cos z).$$

Similarly, we have

$$\cos (cz + d) = \Psi_1(\cos z).$$

As the first member of (13) has the primitive period $2\pi/a$, and as the second member has a period $2\pi$, we see that $a$ is an integer. In fact, we must have $a = \pm m$, where $m$ is the degree of $\Phi_1$. To determine $b$, we note that the first member of (13) must be, like the second, an even function of $z$. Hence, when

$$z_1 + z_2 = 2\pi,$$
we must have
\[ az_1 + az_2 + 2b = 2k \pi. \]

where \( k \) is some integer. It follows that \( 2b \) is a multiple of \( 2 \pi \), so that \( b \) is either 0 or \( \pi \) (neglecting multiples of \( 2 \pi \)). Similarly, \( c = \pm n \), and \( d \) is either 0 or \( \pi \).

Finally, since
\[
\cos(acz + ad + b) \quad \Phi_1 \Phi_1^*(\cos z),
\]
\[
\cos(acz + cb + d) \quad \Psi_1 \Phi_1^*(\cos z),
\]
we must have
\[(a-1)d \equiv (c-1)b \pmod{2\pi}.\]

As in the preceding section, all permutable pairs of functions of the type now considered can be found by transforming \( \Phi_1(z) \) and \( \Psi_1(z) \) with a linear function.

VI. THE MULTIPLICATION FORMULAS FOR \( \psi(z) \)

We are going to settle, in this section, the following two cases:

(a) \( h = 4 \),

(b) \( h = k = 2 \) and \( r = s = 2 \). (See next to last paragraph of § III.)

Let us examine Case (a). We must have \( k = 4, 3 \) or \( 2 \).

Suppose that \( k = 4 \). Then, by § III, \( \varphi_1^{-1} \) and \( \psi_1^{-1} \) must have the same four critical points, at which each has, in all, four simple points. Furthermore, the four values which \( \varphi_1^{-1} \) assumes at its simple points are the same that \( \psi_1^{-1} \) assumes at its simple points, for the four values are the affixes of the common critical points of \( \varphi_1^{-1} \) and \( \psi_1^{-1} \).

If \( k = 3 \), the degree \( s \) of \( \psi_1^{-1} \) must be \( 4 \), and \( \psi_1^{-1} \) must have three critical points at which its branches are permuted in pairs. The four simple points of \( \psi_1^{-1} \) are at a critical point \( w_0 \) of \( \varphi_1^{-1} \). As the four critical points of \( \varphi_1^{-1} \) will all have the same index that \( \varphi_1^{-1} \) has at \( w_0 \), that index must be \( (r-1)/2 \).

Hence \( r \) is odd, and \( \varphi_1^{-1} \) has one simple point at each of its four critical points.

Suppose that \( k = 2 \). We must have \( s = 2 \). Two of the critical points of \( \varphi_1^{-1} \) are critical points of \( \psi_1^{-1} \). The other two, call them \( w_1 \) and \( w_2 \), are not.

We shall show that the four values which \( \varphi_1^{-1} \) assumes at its simple points are the affixes of the four critical points of \( \varphi_1^{-1} \), that is, the four values of \( \psi_1^{-1} \) at \( w_1 \) and \( w_2 \).

Since \( \varphi_1^{-1} \) has two critical points of the type that \( \varphi_1^{-1} \) has at \( w_1 \), and two of the type that \( \varphi_1^{-1} \) has at \( w_1 \), it follows that \( \varphi_1^{-1} \) has (in all) two simple
points at \( w_1 \) and \( w_2 \). The other two simple points of \( \varphi_i^{-1} \) are \( A \)-points of \( \varphi_i^{-1} \), and the values which \( \varphi_i^{-1} \) takes at them are critical points of \( \varphi_{i+1}^{-1} \). We write

\[
F' = \varphi_i \psi_{i+1} = \psi_i \varphi_{i+1}.
\]

From the relation \( F' = \psi_i \varphi_{i+1} \), we see that, at \( w_1 \) and \( w_2 \), there are precisely four places on the Riemann surface of \( F^{-1} \) at which \( F^{-1} \) is uniform, and that, at these four places, the values of \( F^{-1} \) are the values which \( \varphi_{i+1}^{-1} \) assumes at its four simple points. Let \( u_1 \) and \( u_2 \) be the values of \( \varphi_i^{-1} \) at its simple points at \( w_1 \) and \( w_2 \). From what we have just seen, and from the relation \( F = \varphi_i \psi_{i+1} \), it follows that neither \( u_1 \) nor \( u_2 \) is a critical point of \( \psi_{i+1}^{-1} \), and that the values of \( \psi_{i+1}^{-1} \) at \( u_1 \) and \( u_2 \) are the four values of \( \varphi_{i+1}^{-1} \) at its simple points. As at least two of these values are critical points of \( \varphi_{i+1}^{-1} \), it follows that at least one of the points \( u_1 \) and \( u_2 \) is a critical point of \( \varphi_{i+1}^{-1} \). We have proved that at least three of the values of \( \varphi_i^{-1} \) at its simple points are critical points of \( \varphi_{i+1}^{-1} \). This implies, of course, that at least three of the values of \( \varphi_i^{-1} \) at its simple points are critical points of \( \varphi_{i+1}^{-1} \). Going back three sentences, we see that \( u_1 \) and \( u_2 \) are both critical points of \( \varphi_{i+1}^{-1} \), as was to be proved.

By similar reasoning, only more briefly, it can be shown that, when \( k = 3 \), \( \varphi_i^{-1} \) and \( \psi_i^{-1} \) assume the same four values at their simple points.

We shall now examine Case \( (b) \), and show that it may be amalgamated with Case \( (a) \).

Let \( a_i \) and \( b_i \) be the critical points of \( \varphi_i^{-1} \), and \( a_i \) and \( c_i \) those of \( \psi_i^{-1} \). The inverse of

\[
F' = \varphi_i \psi_{i+1} = \psi_i \varphi_{i+1}
\]

has the three critical points \( a_i, b_i, c_i \), at which its branches are permuted in pairs.

As \( \varphi_i^{-1} \) and \( \psi_i^{-1} \) have precisely one critical point in common, one, and only one, of the values of \( \varphi_i^{-1} \) at \( c_i \) equals a value of \( \psi_i^{-1} \) at \( b_i \). Let the values of \( \varphi_i^{-1} \) at \( c_i \) be \( a_{i+1} \) and \( c_{i+1} \), and the values of \( \psi_i^{-1} \) at \( b_i \) be \( a_{i+1} \) and \( b_{i+1} \). There is a point \( d_i \) at which \( \varphi_i^{-1} \) has the value \( b_{i+1} \). Evidently \( d_i \) is distinct from \( c_i \). We shall show that \( d_i \) is distinct from \( a_i \) and from \( b_i \).

As \( a_{i+1} \) and \( b_{i+1} \) are the critical points of \( \varphi_{i+1}^{-1} \), and \( a_{i+1} \) and \( c_{i+1} \) those of \( \psi_{i+1}^{-1} \), one and only one value of \( \varphi_{i+1}^{-1} \) at \( c_{i+1} \) equals a value of \( \psi_{i+1}^{-1} \) at \( b_{i+1} \). Let the values of \( \varphi_{i+1}^{-1} \) at \( c_{i+1} \) be \( a_{i+2} \) and \( c_{i+2} \), and those of \( \psi_{i+1}^{-1} \) at \( b_{i+1} \) be \( a_{i+2} \) and \( b_{i+2} \). From \( F = \varphi_i \psi_{i+1} \), we see that where \( \varphi_i^{-1} \) takes the value \( b_{i+1} \), \( F^{-1} \) has branches with values \( a_{i+2} \) and \( b_{i+2} \). But from \( F' = \psi_i \varphi_{i+1} \), we see that at the same point at which one branch of \( F^{-1} \) has the value \( a_{i+2} \),
another has the value \( c_{i+2} \). Hence where \( F^{-1} \) takes the value \( a_{i+2} \), it has at least three values, and therefore four. Thus \( F^{-1} \) cannot assume the value \( a_{i+2} \) at either \( a_i \) or \( b_i \), as it has only two distinct values at these points. This proves that \( d_i \) is distinct from \( a_i \) and \( c_i \).

Let \( b_{i+1} \) and \( d_{i+1} \) be the two values of \( \varphi_i^{-1} \) at \( d_i \). We shall prove that the values of \( \psi_i^{-1} \) at \( d_i \) are \( c_{i+1} \) and \( d_{i+1} \).

The relation \( F = \varphi_i \psi_{i+1} \) shows that two of the values of \( F^{-1} \) at \( d_i \) are \( a_{i+2} \) and \( b_{i+2} \). It follows from \( F = \psi_i \varphi_{i+1} \) that \( \psi_i^{-1} \) assumes the value \( c_{i+1} \) at \( d_i \).

Thus, since \( F = \psi_i \varphi_{i+1} \), three of the values of \( F^{-1} \) at \( d_i \) are \( a_{i+2}, b_{i+2}, c_{i+2} \). Hence, from \( F = \varphi_i \psi_{i+1} \), we see that \( \psi_i^{-1} \) assumes the value \( c_{i+2} \) at \( d_{i+1} \). But the proof above that \( \varphi_i^{-1} \) assumes the value \( b_{i+1} \) at the same point at which \( \psi_i^{-1} \) assumes the value \( c_{i+1} \) proves also that if \( \psi_i^{-1} \) assumes the value \( c_{i+2} \) at \( d_{i+1} \), \( \varphi_i^{-1} \) must assume the value \( b_{i+2} \) at \( d_{i+1} \). Hence, from \( F = \psi_i \varphi_{i+1} \), we find that \( \psi_i^{-1} \) assumes the value \( d_{i+1} \) at \( d_i \).

Returning to Case (a), we denote the critical points of \( \varphi_i^{-1} \) and \( \psi_i^{-1} \), for every \( i \geq i_1 \), by \( a_i, b_i, c_i, d_i \). This will permit us to treat Cases (a) and (b) together.

The functions \( \sigma \) are introduced as in the two preceding cases. It is seen immediately that \( \sigma_i^{-1} \) has no critical points other than \( a_{i+1}, b_{i+1}, c_{i+1}, d_{i+1} \) and that its branch points are all simple. This shows that the inverses of \( \varphi_i \sigma_i \) and \( \psi_i \sigma_i \) have no critical points other than \( a_i \), etc., that their branch points are all simple, and that there are exactly four places on the Riemann surfaces of the inverses at \( a_i \), etc., where the inverses are uniform.

When \( \epsilon = 4 \), the relation

\[
(14) \quad \sigma_i \varphi_i = \varphi_{i+1} \psi_{i+1}
\]

shows that the values which \( \sigma_i^{-1} \) assumes where it is uniform at \( a_{i+1} \), etc., are \( a_i \) etc. We shall show that the same holds when \( \epsilon = 2 \). First, the above relation shows that two of these values of \( \sigma_i^{-1} \) are \( a_i \) and \( b_i \). Similarly, the relation \( \sigma_i \psi_i = \psi_{i+1} \sigma_{i+1} \) shows that two of the values are \( a_i \) and \( c_i \). It remains to show that the fourth value is \( d_i \). If it were not, then since the values of \( \varphi_i^{-1} \) at \( d_i \) are \( b_{i+1} \) and \( d_{i+1} \), the inverse of \( \sigma_i \psi_i \) would take the values \( b_{i+1} \) and \( d_{i+1} \) at some point distinct from \( a_i \), etc. But the argument just given for \( \sigma_i^{-1} \) shows that \( \sigma_i^{-1} \) assumes the values \( a_{i+1}, b_{i+1}, c_{i+1}, d_{i+1} \) at \( a_{i+2} \) etc., which means that the inverse of \( \varphi_{i+1} \sigma_{i+1} \) assumes those three values at \( a_{i+1} \) etc. This, with (14), yields a contradiction.

It is visible now that the inverse of

\[
F = \varphi_i \sigma_i \psi_i \sigma_i
\]
has the four critical points \(a_i\), etc., that its branch points are all simple, that there are just four places on the surface of \(F^{-1}\) at \(a_i\), etc., where \(F^{-1}\) is uniform, and that the values of \(F^{-1}\) at these places are \(a_i\), etc.

We now work back to \(\Phi\) and \(\Psi\). From the relation

\[
F = \psi_i \sigma_i \psi_i \sigma_i = \sigma_{i-1} \psi_{i-1} \sigma_{i-1} \psi_{i-1},
\]

we see that \(\sigma_{i-1}^{-1}\) has no critical points other than \(a_i\), etc., that its branch points are all simple, and that there are, at \(a_i\), etc., just four places on the surface of \(\sigma_{i-1}^{-1}\) at which \(\sigma_{i-1}^{-1}\) is uniform. Let the values of \(\sigma_{i-1}^{-1}\) at these four places be \(a_{i-1}\), etc. We see by (15) that the inverse of \(\psi_{i-1} \sigma_{i-1} \psi_{i-1}\) has the critical points \(a_{i-1}\), etc. (all four, because it has at least eight branches), where its branches are permuted in pairs, except that there are four places at \(a_{i-1}\), etc., where the inverse is uniform and takes the values \(a_i\), etc. Then the inverse of

\[
\psi_{i-1} \sigma_{i-1} \psi_{i-1} \sigma_{i-1}
\]

has the four critical points \(a_{i-1}\), etc., with simple branch points, and has four places at \(a_{i-1}\), etc., where it is uniform and takes the values \(a_{i-1}\), etc.

Proceeding thus, we find that the inverse of

\[
\Phi \psi = \psi \Phi
\]

has four critical points \(a_0, b_0, c_0, d_0\), at which its branches are permuted in pairs, except that there are four places where the inverse is uniform, assuming the values \(a_0\), etc.

As to \(\Phi^{-1}\) and \(\psi^{-1}\), we see that they have no critical points other than \(a_0\), etc., and that their branch points are all simple. When \(\Phi^{-1}\) and \(\psi^{-1}\) each have four critical points, it is evident that their values where they are uniform at \(a_0\), etc., are \(a_0\), etc.

If \(m > 4\), \(\Phi^{-1}\) will have four critical points, so that the values of \(\psi\) where it is uniform at \(a_0\), etc., are \(a_0\), etc. Suppose that \(m > 4\), that \(n = 4\), and that \(\psi^{-1}\) has only three critical points. It is clear that at three of the places at \(a_0\), etc., at which \(\Phi^{-1}\) is uniform, it assumes values from among \(a_0\), etc.

If it did not assume one of these values at the fourth place, we would have the contradiction that the first member of (16) has four uniform branches at one of the points \(a_0\), etc., which do not assume the values \(a_0\), etc.

When \(m = 4\), we must also have \(n = 4\), so that \(\psi_i, \psi_i, \sigma_i\), are all of degree 2, and we may assume, above, that \(i = 0\). It was shown above that
in this case the values of the inverses of \( \varphi_i \sigma_i \) and \( \psi_i \sigma_i \), where they are uniform at \( a_i \), etc., are \( a_i \) etc.

Suppose that the sequence \((C)\) does not exist. We may take \( i = 0 \). As \( a_1 \), etc., play the same rôle in regard to \( \varphi_1 \) and \( \psi_1 \) as \( a_0 \), etc., do in regard to \( \varphi_0 \) and \( \psi_0 \), and as \( \varphi_0 = \varphi_1 = \varphi \), \( \psi_0 = \psi_1 = \psi \), we see immediately that the values which \( \varphi^{-1} \) and \( \psi^{-1} \) take at their uniform places at \( a_0 \), etc., are \( a_0 \), etc.

Let \( \lambda(z) \) be a linear function such that \( \lambda(a_0) = \infty \) and that

\[
\lambda(b_0) + \lambda(c_0) + \lambda(d_0) = 0.
\]

It does not matter, in this, which point is called \( a_0 \). We put

\[
e_1 = \lambda(b_0), \quad e_2 = \lambda(c_0), \quad e_3 = \lambda(d_0).
\]

We consider the two functions

\[
\varphi_1 = \lambda \varphi \lambda^{-1}, \quad \psi_1 = \lambda \psi \lambda^{-1},
\]

whose inverses have only simple branch points, which are found at \( e_1, e_2, e_3, \infty \).

Construct the elliptic function \( \varphi z \) such that \( \varphi(\omega_i) = e_i, \quad i = 1, 2, 3 \). This is possible because \( e_1 + e_2 + e_3 = 0 \). Furthermore, the orientation of the numbers \( e_i \) is of no importance.

As \( \varphi z \) assumes the values \( e_i \) and \( \infty \) twice wherever it assumes one of them, and as the branch points of \( \varphi_1^{-1} \) are all simple, the \( n \) values of \( \varphi_1^{-1}(\varphi z) \) are uniform \textit{im kleinen}, and hence also \textit{im großen}. Let \( f(z) \) represent one of the \( n \) meromorphic functions \( \varphi_1^{-1}(\varphi z) \). As \( \varphi_1^{-1} \) assumes a value \( e_i \) or \( \infty \) only where it is uniform, \( f(z) \) cannot assume a value \( e_i \) or \( \infty \) unless it assumes it twice.

Let \( \varphi^{-1} z \) be the inverse of \( \varphi z \). Then \( \varphi^{-1} z \) has the four critical points \( e_i \) and \( \infty \), and its branch points are all simple. Hence there are an infinity of meromorphic functions \( \varphi^{-1} \[ f(z) \] \). Let \( \beta(z) \) be one of these. Then

\[
f(z) = \varphi \[ \beta(z) \],
\]
or

\[
(17) \quad \varphi z = \varphi_1 \varphi \[ \beta(z) \].
\]

Now \( \varphi \[ \beta(z) \] \), being algebraically related to \( \varphi z \), is an elliptic function. Then \( \beta(z) \) must be entire, for if it had a pole with a finite affix \( \varphi \[ \beta(z) \] \) would have an essential singularity with a finite affix.
Differentiating (17), we find

\[ \phi'(z) = \Phi_i \phi[\beta(z)] \phi'[\beta(z)] \beta'(z). \]

Now \( \phi'[\beta(z)] \), being algebraically related to \( \phi[\beta(z)] \), is an elliptic function. It follows from (18) that \( \beta'(z) \) is an elliptic function. As \( \beta'(z) \) is entire, it must be a constant. Thus \( \beta(z) \) is linear.

There exists thus a relation

\[ \phi(z) = \Phi_1 [\phi(pz + q)], \]

or, what is the same, a relation

\[ \phi(az + b) = \Phi_1 (\phi z). \]

Similarly, there is a relation

\[ \phi(rz + d) = \Psi_1 (\phi z). \]

It remains only to characterize the constants. From (19) we see first, since \( 2 \omega_1 \) and \( 2 \omega_3 \) are periods of the first member, that

\[ 2a = 0, \quad 2a = 0 \pmod{2 \omega_1, 2 \omega_2}. \]

Also, since the first member of (19) has to be an even function of \( z \), like the second, we find

\[ 2b = 0 \pmod{2 \omega_1, 2 \omega_2}. \]

From (20), we find, similarly.

\[ 2c = 0, \quad 2c = 0, \quad 2b = 0 \pmod{2 \omega_1, 2 \omega_2}. \]

Finally, from the condition of permutability, we find

\[ (a - 1)d \equiv (c - 1)b \pmod{2 \omega_1, 2 \omega_2}. \]

Any set of constants which satisfy (21), (22), (23), (24) yield a pair of permutable functions. All pairs of permutable functions, of the type considered in this section, are found by transforming \( \Phi_1 \) and \( \Psi_1 \) with some linear function \( \lambda(z) \).
VII. THE MULTIPLICATION FORMULAS FOR $\varphi^2 z$
IN THE LEMNISCATIC CASE

We consider here the case in which $h = 3$, and in which, for some $i \geq i_1$, one of the numbers $\alpha_i, \beta_i, \gamma_i$ is 2, and the other two 4. Let, for instance, $\alpha_i = 2$, and $\beta_i = \gamma_i = 4$.

We take first the case of $k = 3$, in which $\varphi_{i-1}$ and $\psi_{i-1}$ have the common critical points $\alpha_i, b_i, c_i$. As before, we represent the orders of the substitutions which the branches of $\psi_{i-1}$ undergo at these points by $\alpha'_i, \beta'_i, \gamma'_i$, respectively.

We shall prove that $\alpha'_i = 2$, and that, when $s > 4$, $\beta'_i = \gamma'_i = 4$. When $s = 4$, one of $\beta'_i$ and $\gamma'_i$ is 2, and the other is 4. We cannot have $s = 3$.

First let $r$ be odd. Since $\alpha_i, \beta_i, \gamma_i$ are divisors of 4, $\varphi_{i-1}$ must have at least one simple point at each of $\alpha_i, b_i, c_i$. If $\alpha'_i$ were not 2, or if $\beta'_i$ and $\gamma'_i$ were not divisors of 4, $\varphi_{i-1}$ would have $\omega_i$ points in addition to the three simple points.

Suppose that one of $\beta'_i$ and $\gamma'_i$ is 2 rather than 4. Let it be $\gamma'_i$, for instance. Unless $s \leq 6$, $\psi_{i-1}$ will surely have more than three $A$-points at $c_i$. If $s = 6$, $\psi_{i-1}$ must have three simple branch points $c_i$. This leads to the result that $\varphi_{i+1}$ has three critical points at which its branches are permuted in pairs, a situation proved impossible in § V. If $s = 5$, $\psi_{i-1}$ would have at least three $A$-points at $c_i$, and at least one at $a_i$. If $s = 3$, $\psi_{i-1}$ would have at least two $A$-points at $c_i$, and at least one at each of $a_i$ and $b_i$.

Thus, when $\gamma'_i = 2$, we have $s = 4$. We must have $\beta'_i = 4$, for the case of $\alpha'_i = \beta'_i = \gamma'_i = 2$ is known to be impossible.

When $s = 4$, it is necessary, since $\beta'_i = 4$, that $\psi_{i-1}$ have two simple points, either at $a_i$, or at $c_i$. They must be at $c_i$, otherwise $\psi_{i-1}$ would have four $A$-points.

We take now the case in which $r$ is even. It is evident that $r > 4$, so that $r$ is at least 6.

First we prove that $\alpha'_i = 2$. If $\alpha'_i > 2$, $r$ must be 6, and $\varphi_{i-1}$ must have three simple branch points at $a_i$. As $\varphi_{i-1}$ has branch points of order 3 at $b_i$ and at $c_i$, $\varphi_{i-1}$ must have two simple points at $b_i$ or at $c_i$, or the sum of its indices would be 11 instead of 10. Thus, $\varphi_{i-1}$ would have at least five $A$-points. Hence $\alpha'_i = 2$.

As $r \geq 6$, $\varphi_{i-1}$ has at least four points at $b_i$ and $c_i$. Then either $\beta'_i$ or $\gamma'_i$ must be a divisor of 4. Let it be $\beta'_i$, for instance.

Let $\beta'_i = 2$. The very proof used above for $r$ odd shows that $s = 4$. We know that $\gamma'_i$ cannot equal 2 in this case, and it will be seen below that $\gamma'_i \neq 3$. Hence, if it were possible for $s$ to be 4, we would have $\gamma'_i = 4$. This information will be used below in proving that either $r$ or $s$ must be odd; we shall know thus that $\beta'_i$ cannot be 2.
Suppose that $s > 4$, so that $\beta_i = 4$. By (9), we must have $\gamma'_i = 3$ or $4$. We show that $\gamma'_i$ cannot be $3$ for any value of $s$. If $\gamma'_i$ were $3$, at least two of the orders $\alpha_{i+1}', \beta_{i+1}', \gamma'_{i+1}$ of the substitutions at the critical points of $\psi_i^{-1}$ must equal $3$. The orders $\alpha_{i+1}', \beta_{i+1}', \gamma_{i+1}$, at the critical points of $\varphi_i^{-1}$, are certainly divisors of $4$. They cannot all be $4$. If two of them are $2$, we see from § 5 that two of $\alpha_{i-1}', \beta_{i-1}', \gamma_{i-1}$, etc. must be $2$. If one of $\alpha_{i-1}', \beta_{i-1}', \gamma_{i-1}$, etc. is $2$ and the other two $4$, the argument above shows that two of $\alpha_{i-1}', \beta_{i-1}', \gamma_{i-1}$, etc. are divisors of $4$. Hence $\gamma'_i = 4$.

We shall now examine the Riemann surfaces of $\varphi_i^{-1}$ and $\psi_i^{-1}$. It will suffice to deal with $\varphi_i^{-1}$.

Suppose first that $r$ is odd. Then the indices of $\varphi_i^{-1}$ at $a_i$, $b_i$, $c_i$ are not greater, respectively, than

$$\frac{r - 1}{2}, \quad \frac{3(r - 1)}{4}, \quad \frac{3(r - 1)}{4}.$$

As the sum of the three indices is $2r - 2$, the upper bounds just given must be the actual values of the indices. Thus $\varphi_i^{-1}$ has one simple point and $(r - 1)/2$ simple branch points at $a_i$, and one simple point and $(r - 1)/4$ branch points of order $3$ at $b_i$ and at $c_i$.

Let $r$ be even. We shall prove that $\varphi_i^{-1}$ has two simple points, either at $b_i$ or at $c_i$. We know that $\alpha_{i+1}', \beta_{i+1}', \gamma'_{i+1}$ are divisors of $4$. They cannot all be $2$, as we have seen several times. Thus one of them at least must be $4$. This is possible only if $\varphi_i^{-1}$ has either at $b_i$ or at $c_i$ a point whose order is prime to $4$. Such a point has to be a simple point if $\beta_i$ and $\gamma_i$ equal $4$. Also, since $r$ is even, the simple points of $\varphi_i^{-1}$ must come in pairs.

Suppose that the two simple points are at $c_i$. If $r \equiv 2$, mod $4$, there must be a simple branch point at $b_i$. If $r \equiv 0$, mod $4$, there must be a simple branch point at $c_i$, in addition to the two simple points. The two simple points and the simple branch point are the $A$-points of $\varphi_i^{-1}$.

We show now that either $r$ or $s$ is odd. Suppose that both are even. By what precedes, we may suppose that $\psi_i^{-1}$ has two simple points at $c_i$. Thus $\varphi_i^{-1}$ would have two critical points with similar substitutions of order $4$, whereas the preceding paragraph shows that the substitutions cannot be similar when $r$ is even.

We consider now the case of $k = 2$. We must have $s = 3$, or $s = 2$. If $s$ were $3$, $\psi_i^{-1}$ would have an $A$-point at each of its critical points, which would be points of order $3$, and three $A$-points at that critical point of $\varphi_i^{-1}$ which is not a critical point of $\psi_i^{-1}$. Hence, $s = 2$. If the critical points of $\psi_i^{-1}$ were $b_i$ and $c_i$, they would be $A$-points of $\psi_i^{-1}$ and $\psi_i^{-1}$ would also
have two $A$-points at $a_i$. Thus we may assume that the critical points of $\psi_{i-1}$ are $a_i$ and $c_i$.

We shall show that, when $k = 2$, $r$ is odd. Suppose that $r$ is even. If $\psi_{i+1}$ is to have critical points, $\psi_{i-1}$ must have two simple points, either at $a_i$ or at $c_i$. Suppose that they are at $a_i$. Consider the relation

$$ F = \psi_i \psi_{i+1} = \psi_i \psi_{i-1}. $$

It follows from $F = \psi_i \psi_{i+1}$ that among the values of $F^{-1}$ at $a_i$ are the two values which $\psi_{i-1}$ assumes as its critical points. One of these values is the affix of a critical point of $\psi_{i+2}$ with a substitution of order 2, and hence the affix of a critical point of $\psi_{i-2}$. Now, from $F = \psi_i \psi_{i+1}$, since the affixes of the critical points of $\psi_{i-1}$ are values of $\psi_{i+1}$ at its simple points, it follows that the value of $\psi_{i-1}$ at $a_i$ is the affix of a critical point of $\psi_{i+1}$, a falsity.

If the two simple points are at $c_i$, $\psi_{i-1}$ will have two simple points at its critical point with substitution of order 2, something just seen to be impossible.

Thus, when $k = 2$, $r$ is odd, so that $\psi_{i-1}$ has one simple point at each of $a_i, b_i, c_i$. Furthermore, it is easy to show, by the method already frequently used, that the value $a_{i-1}$ which $\psi_{i-1}$ assumes at its simple point at $a_i$ is the value of $\psi_{i-1}$ at $c_i$, and that the values $b_{i-1}$ and $c_{i-1}$ which $\psi_{i-1}$ assumes at its simple points at $b_i$ and $c_i$ are the two values of $\psi_{i-1}$ at $b_i$.

Suppose that $r$ is odd. If $s$ is odd, $\varphi_{i-1}$ and $\psi_{i-1}$ have the same value, $a_{i-1}$, at their simple points at $a_i$, and the same pair of values $b_{i-1}$ and $c_{i-1}$ at their simple points at $b_i$ and $c_i$. If $s$ is even, $\psi_{i-1}$ takes the values $b_{i+1}$ and $c_{i+1}$ at its simple points, and the value $a_{i+1}$ at its simple branch point which is an $A$-point.

In what remains to be done, it is unnecessary to use the condition that $r \leq s$. Accordingly, we work under the legitimate and convenient assumption that $r$ is odd.

The details from this point on are entirely analogous to the corresponding details in the three cases already treated. It is the easiest matter to prove that $\varphi_i^{-1}$ has no critical points other than $a_{i+1}, b_{i+1}, c_{i+1}$, and that the orders of the substitutions which its branches undergo at those points are divisors of 2, 4 and 4 respectively. Also we work back readily to $\Phi^{-1}$ and $\Psi^{-1}$. These have no critical points other than three certain points $a_0, b_0, c_0$, at which their branches undergo substitutions whose orders are divisors of 2, 4 and 4 respectively. If the degree of either permutable function is odd, the branches of its inverse are permuted in pairs at $a_0$, except one which is uniform and has the value $a_0$. At $b_0$ and $c_0$ its branches are permuted in fours,
except that, both at $b_0$ and at $c_0$, there is a place on the surface of the inverse where the inverse is uniform, and the values of the inverse at these places are (as a pair) $b_0$ and $c_0$. If the degree is even, the branches are permuted in pairs at $a_0$. Either at $b_0$ or at $c_0$, there are two uniform branches whose values are $b_0$ and $c_0$, and a simple branch point, where the value assumed is $c_0$.

To identify $\Phi$ and $\Psi$, we choose any number $\omega$, different from zero, and construct $\varphi(x|\omega,i\omega)$. It is well known that in this lemniscatic case, $e_2 = 0$, and $e_3 = -e_1$. We consider now $\varphi^3x$. Where it assumes the value $\infty$, namely, at the points congruent to the origin, it assumes it four times. Similarly, the value 0 is assumed only at the points congruent to $\omega x$, and then four times, while the value $e_1^2$ is assumed twice at all points congruent to $\omega x$ and $\omega y$. There are no values other than $\infty$, 0, and $e_1^2$ which are assumed more than once by $\varphi^3x$ at any point.

We now take a linear function $\lambda(x)$ such that

$$
\lambda(a_0) = e_1^2, \quad \lambda(b_0) = 0, \quad \lambda(c_0) = \infty,
$$

and deal with

$$
\Phi_1 = \lambda \Phi \lambda^{-1}, \quad \Psi_1 = \lambda \Psi \lambda^{-1}.
$$

Precisely as in the preceding section, we find that

$$
\varphi^3(ax + b) = \Phi_1(\varphi^3x), \quad \varphi^3(cx + d) = \Psi_1(\varphi^3x),
$$

where

$$
2a\omega_i \equiv 0, \quad 2c\omega_i \equiv 0 \pmod{2\omega_1, 2\omega_3} \quad (i = 1, 3).
$$

To determine $b$, we note that, in the lemniscatic case, $\varphi^3i\varepsilon = \varphi^3\varepsilon$. Hence, if

$$
\varepsilon_1 \equiv i\varepsilon_2 \pmod{2\omega_1, 2\omega_3},
$$

we must have,

$$
a\varepsilon_1 + b \equiv i a\varepsilon_2 + i b \pmod{2\omega_1, 2\omega_3}.
$$
Multiplying the first congruence through by \( a \), and subtracting the result from the second, we have

\[
b(1 - i) \equiv 0 \pmod{2 \omega_1, 2 \omega_3}.
\]

A similar condition holds for \( d \). Furthermore, as in the preceding cases

\[
(a - 1) d \equiv (c - 1) b \pmod{2 \omega_1, 2 \omega_3}.
\]

**VIII. The Multiplication Formulas for \( \varphi'z \) in the Equianharmonic Case**

We take, for \( h = 3 \), the case in which, for some \( i \geq i_1, \alpha_i = \beta_i = \gamma_i = 3 \). We cannot have \( r \equiv 2 \pmod{3} \), for in that case, the sum of the indices of \( \varphi_i^{-1} \) could be at most \( 2r - 4 \).

If \( r \equiv 1 \pmod{3} \), there is one simple point at each critical point of \( \varphi_i^{-1} \), and the other points are all of order 3.

If \( r \equiv 0 \pmod{3} \), there are three simple points at one of the critical points, and none at either of the others.

Let \( k = 3 \). Then \( \alpha_i' = \beta_i' = \gamma_i' = 3 \), else \( \varphi_i^{-1} \) would have more \( A \)-points than its three simple points. The surface of \( \psi_i^{-1} \) is of one of the types described above.

If \( k = 2 \), we must have \( s = 3 \), for if \( s \) were 2, \( \psi_i^{-1} \) would have four \( A \)-points. When \( k = 2 \), we must have \( r \equiv 1 \pmod{3} \), and it can be shown, as in the preceding sections, that \( \varphi_i^{-1} \) and \( \psi_i^{-1} \) assume the same three values at their simple points.

It is most easy now to introduce \( \sigma \) and to work back to \( \Phi \) and \( \Psi \). The inverses of these two functions have no critical points other than certain three points \( a_0, b_0, c_0 \). On the Riemann surface of the inverse of either function, there are precisely three places at \( a_0, b_0, c_0 \), at which the inverse is uniform, and the values of the inverse at these places are \( a_0, b_0, c_0 \).

To identify \( \Phi \) and \( \Psi \), we construct \( \varphi(z | \omega, e^{\pm \frac{2\pi i}{3}} \omega) \) where \( \omega \) is any number different from zero. As \( \varphi(z | \omega, \omega_1, \omega_2) \) is a homogeneous function of degree \( -2 \) in \( z, \omega_1 \) and \( \omega_2 \), and as \( \varphi(z | \omega, e^{\pm \frac{2\pi i}{3}} \omega) \) is identical with \( \varphi(z | e^{2\pi i} \omega, -\omega) \), we have, in the present case,

\[
\varphi e^{\frac{2\pi i}{3} z} = e^{\frac{2\pi i}{3}} \varphi z.
\]
Differentiating, we find

$$\frac{2\pi i}{\nu} \frac{\zeta}{\nu} = \nu' \zeta, \quad \frac{2\pi i}{\nu} \frac{\zeta}{\nu} = \nu'' \zeta,$$

$$\frac{2\pi i}{\nu} \frac{\zeta}{\nu} = \nu^\prime\prime\prime \zeta.$$

Hence, if

$$(25) \quad \frac{2\pi i}{\nu} \frac{\zeta}{\nu} = \zeta \quad (\text{modd } 2 \omega, 2 e^{\frac{\pi i}{\lambda} \omega}).$$

we will have, except for $\zeta \equiv 0$, when $\nu' \zeta = \infty$,

$$(26) \quad \nu'' \zeta = \nu^\prime\prime\prime \zeta = 0.$$

We consider the following two solutions of (25):

$$z_1 = \frac{2 \omega}{1 - e^{\frac{\pi i}{\lambda}}} \quad z_2 = \frac{2 \omega + 2 e^{\frac{\pi i}{\lambda} \omega}}{1 - e^{\frac{\pi i}{\lambda}}}.$$

We cannot have $z_1 \equiv z_2$, modd $2 \omega, 2 e^{\frac{\pi i}{\lambda} \omega}$, for

$$z_2 - z_1 = 2 \omega \frac{e^{\frac{\pi i}{\lambda}}}{1 - e^{\frac{\pi i}{\lambda}}}.$$

and the modulus of the second member of the last congruence is less than that of $2 \omega$, the period of smallest modulus. We may suppose that $z_1$ and $z_2$ lie within the same parallelogram.

From (26), it follows that the values $\nu' (z_1)$ and $\nu' (z_2)$ are each assumed at least three times by $\nu' (z)$ at $z_1$ and $z_2$. But since $\nu'' (z)$ vanishes just four times in a parallelogram, these values are assumed precisely three times each. Furthermore, no other value, except $\infty$, is ever assumed more than once at a point. Also, as $\nu' (z)$ assumes every value just three times in a parallelogram, $\nu' (z_1)$ and $\nu' (z_2)$ are not equal.
We now take a linear $\lambda(z)$ such that

$$
\lambda(a_0) = \infty, \quad \lambda(b_0) = \phi'(z_1), \quad \lambda(c_0) = \phi'(z_2),
$$

and let

$$
\Phi_i = \lambda \phi \lambda^{-1}, \quad \Psi_i = \lambda \psi \lambda^{-1}.
$$

We find

$$
\phi'(az+b) = \Phi_i(\psi'z), \quad \psi'(cz+d) = \Psi_i(\psi'z),
$$

where

$$
2a \omega \equiv 0, \quad 2c \omega \equiv 0 \quad \text{(modd } 2\omega, 2e^{\frac{2\pi i}{8}}\omega).
$$

$$
\left(1 - e^{\frac{2\pi i}{8}}\right)b \equiv 0, \quad \left(1 - e^{\frac{2\pi i}{8}}\right)d \equiv 0 \quad \text{(modd } 2\omega, 2e^{\frac{2\pi i}{8}}\omega),
$$

and

$$
(a - 1)d \equiv (c - 1)b \quad \text{(modd } 2\omega, 2e^{\frac{2\pi i}{8}}\omega).
$$

**IX. THE MULTIPLICATION FORMULAS FOR $\psi^2z$ IN THE EQUIANHARMONIC CASE**

Referring to (9), we find that the only case still to be treated is that, under $h = 3$, in which, for some $i \geq i_1$, one and only one of $\alpha_i$, $\beta_i$, $\gamma_i$ equals 2, and a second equals 3. We let $\alpha_i = 2$, $\beta_i = 3$, $\gamma_i = 2$. We may assume that $\alpha_{j} = 2$, $\beta_{j} = 3$, $\gamma_{j} \neq 2$ for every $j > i$, for the work of the preceding sections shows that a different set of values of $\alpha_{j}$, etc., would imply other values than those assumed for $\alpha_{i}$, etc.

We prove first that $k = 3$. If $k$ were 2, we would have $s = 3$ or $s = 2$. If $s = 3$, $\psi_{-1}$ would have three simple points at that critical point of $\phi_{-1}$ which is not a critical point of $\phi_{-1}$. This leads to the contradiction that $\alpha_{i+1} = \beta_{i+1} = \gamma_{i+1}$. Suppose that $s = 2$. Then $a_i$ must be a critical point of $\psi_{-1}$, for if the critical points of $\psi_{-1}$ were $b_i$ and $c_i$, $\psi_{-1}$ would have an $A$-point at each of these two points, in addition to two $A$-points at $a_i$. The only way in which $\alpha_{i+1}$ can equal 2 is for $c_i$ to be the second critical point of $\psi_{-1}$, and for $\gamma_i$ to equal 4. Then $\alpha_{i+1} = 2$, $\beta_{i+1} = \gamma_{i+1} = 3$. which is impossible, for the argument just presented shows that one of $\beta_{i+1}$ and $\gamma_{i+1}$ must be 4, as $\gamma_i$ is. Thus $k = 3$.

We must have, of course, $3 \leq \gamma_i \leq 6$. We shall show that $\gamma_i = 3$ when $r = 4$, and that otherwise $\gamma_i = 6$. 
Suppose first that \( r_i = 3 \). We cannot have \( r \equiv 2, \mod 3 \), for \( \varphi_i^{-1} \) would have four simple points at \( b_i \) and \( c_i \), which would be \( A \)-points. Suppose that \( r \equiv 0, \mod 3 \). If there are no simple points at \( b_i \) or at \( c_i \), the sum of the indices of \( \varphi_i^{-1} \) at \( b_i \) and \( c_i \) is \( 4r/3 \). Hence the index at \( a_i \) is \( 2r/3 - 2 \). This means, since \( \varphi_i^{-1} \) has only simple branch points at \( a_i \), that

\[
\frac{4r}{3} - 4 \leq r,
\]

or \( r \leq 12 \). If \( r = 12 \), \( \varphi_i^{-1} \) has six simple branch points at \( a_i \). Hence, if \( \varphi_i^{-1} \) has an \( A \)-point at \( a_i \), it has six, and if it has one at \( b_i \) (or at \( c_i \)), it has four. If \( r = 9 \), there is a simple point and four simple branch points at \( a_i \). The simple point must be the only \( A \)-point at \( a_i \), else there would be five. But if there were an \( A \)-point at \( b_i \) (or at \( c_i \)) there would be three, which is impossible because of the \( A \)-point at \( a_i \). When \( r = 6 \), there are two simple points at \( a_i \), and the impossibility follows as in the preceding cases. In the case in which \( r \equiv 0, \mod 3 \), and in which there are simple points at \( b_i \), or \( c_i \), we see that there must be just three simple points, either at \( b_i \) or at \( c_i \). This would require that the index of \( \varphi_i^{-1} \) at \( a_i \) be \( 2r/3 \), which is impossible, since \( \varphi_i^{-1} \) has only simple branch points at \( a_i \).

Thus, when \( r = 3 \), we have \( r \equiv 1, \mod 3 \), so that there is a simple point at \( b_i \) and at \( c_i \). The sum of the indices at \( b_i \) and \( c_i \) is \( 4(r - 1)/3 \), so that the index at \( a_i \) is \( 2(r - 1)/3 \). Then

\[
\frac{4(r - 1)}{3} \leq r,
\]

so that \( r = 4 \). There are two simple branch points at \( a_i \).

In the case of \( r = 4 \), we must have \( s = 4 \), or \( s = 3 \). The branches of \( \psi_i^{-1} \) must undergo at \( a_i \) a substitution of order 2, else \( \varphi_i^{-1} \) would have two \( A \)-points at \( a_i \), in addition to the two simple points at \( b_i \) and \( c_i \). Then we cannot have \( s = 4 \), for in that case, either none or two of \( a_i+1 \) etc. would equal 2, according as \( \psi_i^{-1} \) had none or two simple points at \( a_i \). Thus \( s = 3 \), and \( \psi_i^{-1} \) has a simple point and a simple branch point at \( a_i \). At either \( b_i \) or \( c_i \), \( \psi_i^{-1} \) must have a branch point of order 2, else \( \varphi_i^{-1} \) would have four \( A \)-points. Let it be at \( b_i \). Then \( \psi_i^{-1} \) has a simple point and a simple branch point at \( c_i \).

We show now that the values 4 and 5 are impossible for \( r_i \), so that \( r_i = 6 \) when \( r > 4 \).

We may suppose that \( \alpha_{i+1} = 2, \beta_{i+1} = 3 \). We shall show first that if \( r_i \) were 4 or 5, \( r_{i+1} \) would equal \( r_i \). This will follow as soon as we show.
that \( \gamma_{i+1} \neq 3 \), for, from the way in which \( \alpha_{i+1} \) etc. depend upon \( \alpha_i \) etc., it is clear that \( \gamma_{i+1} \) cannot exceed 4 if \( \gamma_i = 4 \), and that \( \gamma_{i+1} \) cannot be 4, or more than 5, if \( \gamma_i = 5 \).

That \( \gamma_{i+1} \neq 3 \) when \( r > 4 \) was proved above. Let \( r = 4 \). We have to consider the possibility of \( \gamma_i = 4 \). In that case, \( \varphi_i^{-1} \) must have two simple points at \( a_i \) and one at \( b_i \). That \( \varphi_i^{-1} \) may have no other \( A \)-points, it is necessary that \( \beta_i = 3 \). Hence, as \( s \leq 4 \), \( \psi_i^{-1} \) cannot have more than one \( A \)-point at \( b_i \). Thus there cannot be two of the numbers \( \alpha_{i+1} \) etc. which equal 3, and as \( \beta_{i+1} = 3 \), \( \gamma_{i+1} \neq 3 \). This finishes the proof that \( \gamma_{i+1} = \gamma_i \).

When \( \gamma_i = 5 \), we see directly that for \( \alpha_{i+1} \) etc. to equal 2, 3, 5, respectively, \( \psi_i^{-1} \) must have one and only one \( A \)-point at each of \( a_i \), etc. Also when \( \gamma_i = 4 \), \( \psi_i^{-1} \) must have one and only one point at \( c_i \) whose order is prime to 4, so that \( s \) is odd, and \( \psi_i^{-1} \) must also have an \( A \)-point at \( a_i \).

Thus, if \( \gamma_i \) is either 4 or 5, \( \psi_i^{-1} \) has a single \( A \)-point at each of \( a_i \), etc. We shall show that these \( A \)-points are simple points, and that \( \alpha_i \) etc. are respectively equal to 2, 3 and \( \gamma_i \).

First, we cannot have \( \alpha_i = \beta_i = \gamma_i = 3 \), for the work of the preceding section shows that \( \psi_i^{-1} \) would have three simple points at \( a_i \), etc., and \( \psi_i^{-1} \) would have other \( A \)-points, either at \( a_i \), or at \( c_i \). Hence, by (9), one of \( \alpha_i \), etc., is 2. Now, neither \( \beta_i \) nor \( \gamma_i \) can equal 2, else \( \psi_i^{-1} \) would have more than one \( A \)-point at \( b_i \) or \( c_i \) respectively. Thus, \( \alpha_i = 2 \), so that \( \psi_i^{-1} \) has a simple point at \( a_i \). Now \( \psi_i^{-1} \) must have more than one point at \( c_i \), else, since it has only simple branch points at \( a_i \), it would have only simple branch points at \( b_i \), and \( \beta_i \) would equal 2. As \( \psi_i^{-1} \) has just one \( A \)-point at \( c_i \), the orders of all of its points but one at \( c_i \) must be multiples of \( \gamma_i \). Since \( \beta_i > 2 \), we must have \( \gamma'_i < 2 \gamma_i \), for \( \gamma_i > 3 \), and the sum of the reciprocals of \( \alpha_i \) etc. is at least unity. Thus \( \gamma'_i = \gamma_i \), and there is a simple point at \( c_i \). By similar reasoning, we find that \( \beta'_i = 3 \), and that there is a simple point at \( b_i \).

Now, the sum of the indices of \( \psi_i^{-1} \) is

\[
\frac{s-1}{2} + \frac{2(s-1)}{3} + \frac{(\gamma_i-1)(s-1)}{\gamma_i},
\]

a quantity inferior to \( 2s - 2 \), if \( \gamma_i \) is 4 or 5.

We have proved that \( \gamma_i = 6 \) when it is not 3.

Before discussing the surfaces of \( \varphi_i^{-1} \) and \( \psi_i^{-1} \), for \( \gamma_i = 6 \), we shall show that \( \alpha'_i = 2 \) and \( \beta'_i = 3 \). If \( \alpha'_i \) is not two, the only way in which \( \varphi_i^{-1} \) can have fewer than four \( A \)-points at \( a_i \) is for \( \varphi_i^{-1} \) to have just six branches, which are permuted in three pairs at \( a_i \). This implies, however, since \( \gamma_i = 6 \), that \( \varphi_i^{-1} \) has simple points at \( b_i \). Hence \( \alpha'_i = 2 \). Again, if \( \beta'_i \neq 3 \), every point of \( \varphi_i^{-1} \)
at $b_i$ is an 4-point. Hence, $r < 10$, and $r \neq 8$. Also, if $r$ is odd, $\varphi_i^{-1}$ has a simple point at $a_i$, so that $r$ cannot be 9 or 7. Finally, if $r$ be 6, $\varphi_i^{-1}$ must have no simple points at $b_i$, and this requires that $\varphi_i^{-1}$ have four simple points at $a_i$. Hence $\beta_i' = 3$.

In what follows, we shall not use the condition that $r \geq 8$. Thus all results obtained for $\varphi_i^{-1}$ will hold for $\psi_i^{-1}$, when $r' = 6$.

We shall show first that we cannot have $r \equiv 2$, mod 3. We may assume that $a_{i+1}, \beta_{i+1}, y_{i+1}$ are 2, 3 and 6 respectively. We know then that $a_{i+1} = 2$, $\beta_{i+1} = 3$. Now, as $\varphi_i^{-1}$ has two simple points at $b_i$ if $r \equiv 2$, mod 3, we must have $y_{i+1}' = 3$. But if $r \equiv 2$, mod 3, not all of the points of $\varphi_i^{-1}$ at $c_{i+1}$ can have orders which are multiples of 3. This fact, together with the fact that $\varphi_i^{-1}$ has two simple points at $b_{i+1}$, shows that $a_{i+2} = \beta_{i+2} = y_{i+2}' = 3$, whereas we must have $a_{i+2}' = 2$.

Thus, $r \equiv 0, 1, 3$ or 4, mod 6. We examine the surface of $\varphi_i^{-1}$ for each of these cases.

First, suppose that $r \equiv 0$, mod 6. Every point of $\varphi_i^{-1}$ at $b_i$ is of order 3. Hence the index of $\varphi_i^{-1}$ at $b_i$ is $2r/3$. At $a_i$, $\varphi_i^{-1}$ will have either none or two simple points. Hence its index at $a_i$ is either $r/2$ or $r/2 - 1$. Thus the index of $\varphi_i^{-1}$ at $c_i$ is either $5r/6 - 2$ or $5r/6 - 1$; we shall see that it has the former value, so that there are no simple points at $a_i$.

Suppose that $\varphi_i^{-1}$ has, at $c_i$, $w$ points of order 6, $x$ points of order 3, $y$ points of order 2 and $z$ simple points. Then

$$(27) \quad 6w + 3x + 2y + z = r.$$ 

If the index at $c$ were $5r/6 - 1$, we would have

$$(28) \quad 5w + 2x + y = \frac{5r}{6} - 1.$$ 

Multiplying through by $6/5$ in (28), and subtracting the result from (27), we find

$$\frac{3x}{5} + \frac{4y}{5} + z = \frac{6}{5}.$$ 

The only solution of this equation in positive integers is $x = 2$, $y = 0$, $z = 0$. This means that $\varphi_i^{-1}$ must have two points of order 3 at $c_i$, and that all its
other points at \( c_i \), if such exist, are of order 6. If \( \gamma'_i \) were not 3, \( \varphi_{i-1} \) would have at least two \( A \)-points at \( c_i \), in addition to the two simple points at \( a_i \). But if \( \gamma'_i = 3 \), \( \varphi_{i-1} \) will have no \( A \)-points other than the two at \( a_i \). Thus there are no simple points at \( a_i \), and we have

\[
5w + 2x + y = \frac{5r}{6} - 2,
\]

so that

\[
\frac{3x}{5} + \frac{4y}{5} + z = \frac{12}{5}.
\]

The solution \( x = 4 \), \( y = z = 0 \) leads to an absurdity by the argument just given. The only other solution is \( x = y = z = 1 \).

Hence, when \( r \equiv 0 \), mod 6, all branches of \( \varphi_{i-1} \) must be permuted in pairs at \( a_i \), and in triples at \( b_i \). At \( c_i \), there is a simple point, a point of order 2, and one of order 3. If there are other points at \( c_i \), they are of order 6.

We prove now that \( \gamma'_i = 6 \). By (9), \( 2 \leq \gamma'_i \leq 6 \). If \( \gamma'_i \) were 2, 3 or 4, \( \varphi_{i-1} \) would have a critical point with at least three simple points which would arise from the critical point of \( \varphi_{i-1} \) at \( c_i \). This is impossible, for the critical points of \( \varphi_{i-1} \) are evidently similar to those of \( \varphi_{-1} \). If \( \gamma'_i \) were 5, \( \alpha_{i+1} \) etc. would have the impossible values 5, 5, 5. Thus, \( \gamma'_i = 6 \).

In the case of \( r \equiv 1 \), mod 6, \( \varphi_{i-1} \) has one simple point at \( a_i \), and one at \( b_i \).

We see as above, only with less trouble, that \( \varphi_{i-1} \) has one simple point at \( c_i \), and that its other points at \( c_i \) are of order 6; also that \( \gamma'_i = 2, 3, \) or 6, according as \( s = 3, s = 4 \) or \( s > 4 \).

Similarly, when \( r \equiv 3 \), mod 6, \( \varphi_{i-1} \) has a single simple point at \( a_i \), and none at \( b_i \). At \( c_i \), it has a simple point, a simple branch point, and all its other points are of order 6. As before, \( \gamma'_i \) is a divisor of 6. It is impossible for \( s \) to be 3. When \( s = 4, \gamma'_i = 3 \), and when \( s > 4, \gamma'_i = 6 \).

We suppose finally that \( r \equiv 4 \), mod 6. At \( b_i \), \( \varphi_{i-1} \) must have just one simple point, and its index is \( 2(r-1)/3 \). The index at \( a_i \) would be \( r/2 - 1 \) if there were two simple points at \( a_i \), otherwise \( r/2 \). Hence the index of \( \varphi_{i-1} \) at \( c_i \) is either

\[
\frac{5r}{6} - \frac{1}{3} \quad \text{or} \quad \frac{5r}{6} - \frac{4}{3},
\]

we shall show that the index has the latter value, so that \( \varphi_{i-1} \) has no simple points at \( a_i \).
Suppose that the index had the first value. We would have

\[ 5w + 2x + y = \frac{5r}{6} - \frac{1}{3}, \]

and using (27), we find

\[ \frac{3x}{5} + \frac{4y}{5} + z = \frac{2}{5}, \]

which has no solutions. Hence the second value is the true one, and

\[ \frac{3x}{5} + \frac{4y}{5} + z = \frac{8}{5}. \]

This equation has the solutions \( x = 0, y = 2, z = 0 \), and \( x = 1, y = 0, z = 1 \). We show that the solution \( y = 2 \) is impossible. First, \( \gamma_i \) is a divisor of 6, else \( \varphi_i^{-1} \) would have at least three \( A \)-points at \( c_i \) in addition to the simple point at \( b_i \). But as all the points of \( \varphi_i^{-1} \) at \( c_i \) would be of even order if \( y = 2 \), the only way for \( \alpha_{i+1} \) to equal 2 would be for \( \gamma_i \) to be divisible by 4. Hence \( y \neq 2 \). Thus, \( \varphi_i^{-1} \) has one simple point and one point of order 3 at \( c_i \), and its other points at \( c \) are of order 6.

If \( s = 3, \gamma_i = 2 \). Also, \( s = 4 \) is impossible. Finally, \( \gamma_i = 6 \) when \( s > 3 \).

We have already called attention to the fact that when \( \gamma_i = 6 \) the above discussion of the surface of \( \varphi_i^{-1} \) applies also to the surface of \( \psi_i^{-1} \).

The surfaces of \( \varphi_i^{-1} \) and \( \psi_i^{-1} \) being recognized, we can use the methods of the foregoing sections to work back to the surfaces of \( \Phi^{-1} \) and \( \Psi^{-1} \). There are, indeed, several cases to be examined, and diophantine equations like those used above must be employed in discussing the surface of \( \sigma_i^{-1} \), but no new ideas have to be introduced.

We find that the degrees of \( \Phi^{-1} \) and \( \Psi^{-1} \) are congruent to 0, 1, 3 or 4, mod 6, and that the inverse of each has three critical points, called below \( a_0, b_0, c_0 \). We shall describe \( \psi \); similar remarks will apply to \( \Phi \).

If \( n \equiv 0, \text{mod } 6 \), the branches of \( \Psi^{-1} \) are all permuted in pairs at \( a_0 \), and all in triples at \( b_0 \). At \( c_0 \), \( \Psi^{-1} \) has one uniform branch with value \( c_0 \), one simple branch point at which \( \Psi^{-1} = b_0 \), and one branch point of order 2 at which \( \Psi^{-1} = a_0 \). If \( n > 6 \), the remaining branches are permuted in sixes at \( c_0 \).

If \( n \equiv 1, \text{mod } 6 \), \( \Psi^{-1} \) has one uniform branch at \( a_0 \) whose value is \( a_0 \), and the other branches of \( \Psi^{-1} \) are permuted in pairs at \( a_0 \); at \( b_0 \), there is a uniform
branch whose value is \( b_0 \), and the other branches are permuted in triples; at \( c_0 \), there is one uniform branch whose value is \( c_0 \), while the other branches are permuted in sixes.

If \( n \equiv 3 \pmod{6} \), there is one uniform branch at \( a_0 \) whose value is \( a_0 \); at \( c_0 \) there is one uniform branch whose value is \( c_0 \), and one simple branch point for which \( \psi^{-1} = b_0 \). If \( n > 3 \), the other branches are permuted in sixes at \( c_0 \).

If \( n \equiv 4 \pmod{6} \), \( \psi^{-1} \) has one uniform branch at \( b_0 \) whose value is \( b_0 \). At \( c_0 \), \( \psi^{-1} \) has one uniform branch whose value is \( c_0 \), and one branch point of order 2 at which \( \psi^{-1} = a_0 \). Also if \( n > 4 \), the other branches are permuted in sixes at \( c_0 \).

To identify \( \Phi \) and \( \Psi \), we use the function \( \varphi(z \mid \omega, e^{\pi i/3} \omega) \) of the preceding section. Putting \( \varphi \omega = e_1 \), we have, from the homogeneity formulas,

\[
\begin{align*}
\varphi^2 &= e_2 = \varphi \omega^2 = e^{\frac{2\pi i}{3}} e_1, \\
\varphi^3 &= e_3 = \varphi \omega^3 = e^{\frac{4\pi i}{3}} e_1.
\end{align*}
\]

The values \( e_i \) cannot be zero, else \( \varphi \omega \) would have six zeros in a parallelogram. Hence, as \( \varphi' \omega \) vanishes only when \( \omega \) is a half-period, \( \varphi \omega \) vanishes only once wherever it vanishes.

We consider now the function \( \varphi^3 \omega \). Wherever it assumes the values \( e_1, 0, \) and \( \infty \), it assumes them 2, 3 and 6 times respectively. There are no other multiple values.

We take a linear \( \lambda(z) \) such that

\[
\lambda(a_0) = e_1, \quad \lambda(b_0) = 0, \quad \lambda(c_0) = \infty,
\]

and introduce \( \Phi_1 \) and \( \Psi_1 \). We find

\[
\varphi^3(a \omega + b) = \Phi_1(\varphi^3 \omega), \quad \varphi^3(c \omega + d) = \Psi_1(\varphi^3 \omega),
\]

where

\[
\begin{align*}
2a\omega &\equiv 0, \quad 2c\omega \equiv 0 \pmod{2\omega, 2e^{\frac{\pi i}{3}} \omega}; \\
b \left(1-e^{\frac{\pi i}{3}}\right) &\equiv 0, \quad d \left(1-e^{\frac{\pi i}{3}}\right) \equiv 0 \pmod{2\omega, 2e^{\frac{\pi i}{3}} \omega}; \\
(a-1)d &\equiv (c-1)b \pmod{2\omega, 2e^{\frac{\pi i}{3}} \omega}.
\end{align*}
\]

The second congruence is found from the equation \( \varphi^3 \omega^{1/3} = \varphi^3 \omega \).
X. PERMUTABLE FUNCTIONS WITH A COMMON ITERATE

The results of the preceding section are based upon the assumption that every function of the sequence \((C)\) is of degree less than \(m\) and less than \(n\); that is, that for no \(i\) is one of the functions \(\sigma_i \varphi_i, \sigma_i \psi_i\) a rational function of the other. The removal of this assumption will lead to a new class of permutable pairs of functions.

It will be convenient, in what follows, to represent \(\varphi_i \sigma_i\) by \(\Phi_i\) and \(\psi_i \sigma_i\) by \(\Psi_i\). The preceding sections deal with the sequences

\[
\Phi_0, \Phi_1, \Phi_2, \ldots, \Phi_i, \ldots,
\]

\[
\Psi_0, \Psi_1, \Psi_2, \ldots, \Psi_i, \ldots.
\]

Let us suppose now that for \(i \geq i_0\) (this \(i_0\) is not to be confused with the \(i_0\) of § III), one of the functions \(\Phi_i, \Psi_i\) is a rational function of the other. It fixes the ideas to suppose that \(i_0\) is the smallest number of this type. If we assume that \(m \geq n\), we will be sure that it is \(\Phi_i\) which is a rational function of \(\Psi_i\). Let \(\Phi_{i_0} = \beta_0 \Psi_{i_0}\). The permutability of \(\Phi_{i_0}\) and \(\Psi_{i_0}\) gives

\[
\beta_0 \Psi_{i_0} \Phi_{i_0} = \Psi_{i_0} \beta_0 \Phi_{i_0},
\]

or

\[
\beta_0 \Psi_{i_0} = \Psi_{i_0} \beta_0,
\]

so that \(\Psi_{i_0}\) is permutable with \(\beta_0\).

We distinguish the following two cases:

- \((a)\) \(\beta_0\) is of degree greater than unity;
- \((b)\) \(\beta_0\) is linear.

Suppose that we have met Case \((a)\). It will be convenient to replace the pair of symbols \(\Psi_{i_0}\) and \(\beta_0\) by the pair \(\Phi_{10}\) and \(\Psi_{10}\), and we suppose the new symbols to replace the old in such a way that the degree of \(\Phi_{10}\) is not less than that of \(\Psi_{10}\). We have thus

\[
\Phi_{i_0} = \Phi_{10} \Psi_{10} = \Psi_{10} \Phi_{10},
\]

and \(\Psi_{i_0}\) is either \(\Phi_{10}\) or \(\Psi_{10}\).
We deal now with $\Phi_{10}$ and $\Psi_{10}$ exactly as we dealt originally with $\Phi_0$ and $\Psi_0$, and obtain the sequences

$$
\Phi_{10}, \Phi_{11}, \Phi_{12}, \ldots, \Phi_{1i}, \ldots
$$

$$
\Psi_{10}, \Psi_{11}, \Psi_{12}, \ldots, \Psi_{1i}, \ldots
$$

The following two cases are here possible:

(a) There is no $i$ such that $\Phi_{1i}$ is a rational function of $\Psi_{1i}$;
(b) For $i \geq i_1$, $\Phi_{1i}$ is a rational function of $\Psi_{1i}$.

If Case (b) is at hand, we deal with $\Phi_{1i}$ and $\Psi_{1i}$ exactly as we treated $\Phi_i$ and $\Psi_i$ above.

The process we are employing leads finally to two sequences of permutable functions, rational and non-linear,

$$
\Phi_{p0}, \Phi_{p1}, \Phi_{p2}, \ldots, \Phi_{pi}, \ldots
$$

$$
\Psi_{p0}, \Psi_{p1}, \Psi_{p2}, \ldots, \Psi_{pi}, \ldots
$$

(30)

the common degree of the functions in the first sequence being at least equal to that of the functions in the second, and the two sequences having one of the two following properties:

(I) There is no $i$ such that $\Phi_{pi}$ is a rational function of $\Psi_{pi}$;
(II) For $i \geq i_p$, $\Phi_{pi}$ is a linear function of $\Psi_{pi}$.

Case (I), which does not yield any new types of functions, is quickly disposed of.

We know that $\Phi_{p0}$ and $\Psi_{p0}$ come from one of the several types of multiplication theorems discussed in the preceding sections. To take an example which is typical, suppose that the periodic function involved is $\cos \varepsilon$. Then the inverses of the three functions

$$
\Phi_{p0}, \quad \Psi_{p0}, \quad \Phi_{p0} \Psi_{p0}
$$

have no critical points other than certain three points $a_0$, $b_0$, $c_0$, at the first two of which their branches are permuted in pairs, except that there are two places on the surface of each inverse at $a_0$ and $b_0$ where each inverse is uniform and assumes the values $a_0$ and $b_0$; at $c_0$, the branches of each inverse are permuted in a single cycle, and the single value of each inverse is $c_0$. Now, since

$$
\Phi_{p-1,i_{p-1}} = \Phi_{p0} \Psi_{p0},
$$
and since \( U_{p-1, lp+1} \) is either \( \Phi_{lp0} \) or \( U_{p0} \), we can work back to \( \Phi_{p-1,0} \), \( U_{p-1,0} \), and show by the familiar process that these last two functions come from the multiplication formulas for \( \cos \beta \). Continuing in this fashion, we find that \( \Phi_0 \) and \( U_0 \) are also given by the multiplication formulas for \( \cos \beta \).

An examination of all other possibilities shows similarly that every pair of permutable functions which comes under Case (I) is given by one of the multiplication formulas of the preceding sections.

We take now Case (II), in which \( \Phi_{plp} \) is a linear function of \( U_{plp} \). For brevity, we represent these two functions by \( \Phi_\infty \) and \( U_\infty \), respectively. We have

\[
\Phi_\infty = \beta U_\infty,
\]

where \( \beta(\beta) \) is linear.

As above, \( U_\infty \) and \( \beta \) are permutable. We have thus to determine the circumstances under which a rational function of degree greater than unity is permutable with a linear function. This question has been solved by Julia,* but it will not hurt to give a brief treatment of it here.

The linear function \( \beta(\beta) \) has either two fixed points or one. In the first case, if \( \lambda(\beta) \) is a linear function which carries the fixed points to 0 and \( \infty \) respectively, \( \lambda \beta \lambda^{-1} \) will be of the form \( e^z \). In the second case, if \( \lambda(\beta) \) carries the single fixed point to \( \infty \), \( \lambda \beta \lambda^{-1} \) will be of the form \( z + h \). Thus, if \( \Phi_0 \) and \( U_0 \) and all of the functions of the several sequences are transformed with \( \lambda^{-1} \), we may suppose that \( \beta(\beta) \) is either \( sx \) or \( z + h \).

If \( \beta(\beta) \) were of the form \( z + h \). with \( h \neq 0 \), we would have

\[
U_\infty(z + h) = U_\infty(z) + h,
\]

so that, indicating differentiation with an accent, we find

\[
U_\infty'(z + h) = U_\infty'(z).
\]

Hence \( U_\infty'(\beta) \) would be periodic, and, being rational, would be a constant. This would require that \( U_\infty(\beta) \) be linear.

Thus \( \beta(\beta) \) must have two fixed points, and

\[
(31) \quad U_\infty(\varepsilon z) = \varepsilon U_\infty(z).
\]

* Loc. cit., p. 177.
Differentiating, we find

$$W'(\varepsilon \varepsilon) = W'_\infty(\varepsilon).$$

If $\varepsilon$ were not a root of unity, $W'_\infty(\varepsilon)$ would assume, for each of the distinct points $\varepsilon^r z_1 (z_1 \neq 0, \infty; r = 1, 2, \ldots)$, the same value as at $z_1$. Hence, $W'_\infty(\varepsilon)$, being rational, would be a constant, and $W'_\infty(\varepsilon)$ would be linear.

Thus, $\varepsilon$ is a root of unity, let us say, a primitive $r$th root of unity. We read directly from (31) that $W'_\infty(\varepsilon)/\varepsilon$ is a rational function of $\varepsilon^r$, that is,

$$W'_\infty(\varepsilon) = \varepsilon R(\varepsilon^r),$$

where $R(\varepsilon)$ is a rational function.

Referring now to the definitions, given in the introduction, of the operations of the first and second types, we see immediately that if the pair of non-linear permutable functions $\Phi(\varepsilon)$ and $\Psi(\varepsilon)$ do not come from the multiplication theorems of the periodic functions, there exists a linear $\Lambda(\varepsilon)$ such that $\lambda \Phi \lambda^{-1}$ and $\lambda \Psi \lambda^{-1}$ can be obtained by repeated operations of the first and second types, starting from a pair of functions

$$\varepsilon R(\varepsilon^r), \quad \varepsilon \varepsilon R(\varepsilon^r),$$

where $R(\varepsilon)$ is rational, and where $\varepsilon$ is a primitive $r$th root of unity.

As we do not determine all cases in which operations of the first type are possible, the process just described can certainly not be accepted as furnishing a neat characterization of the pairs of functions which come under Case (II). Nevertheless, we shall progress much further in the study of this case. In particular, we shall settle completely the case in which $\Phi(\varepsilon)$ and $\Psi(\varepsilon)$ are polynomials.

We begin by proving that if the pair of permutable functions $\Phi(\varepsilon)$ and $\Psi(\varepsilon)$ come under Case (II), in particular, if $\Phi(\varepsilon)$ and $\Psi(\varepsilon)$ do not come from the multiplication theorems of the periodic functions, some iterate of $\Phi(\varepsilon)$ is identical with some iterate of $\Psi(\varepsilon)$.

We have shown that

$$\Phi_{\pi \varepsilon} = \varepsilon \Psi_{\pi \varepsilon}. $$

We denote the $n$th iterate of any function $F(\varepsilon)$ by $F^{(n)}(\varepsilon)$. As $\Psi_{\pi \varepsilon}$ is permutable with $\varepsilon \varepsilon$, we have

$$\Phi_{\pi \varepsilon}^{(n)} = \varepsilon^n \Psi_{\pi \varepsilon}^{(n)}.$$
and, in particular, since \( \epsilon^r = 1 \),

\[
(32) \quad \Phi^{(r)}_{\Phi_{\Phi}} = \Psi^{(r)}_{\Phi_{\Phi}}.
\]

We have regard now to the manner in which the two functions of (32) are obtained from those which precede them in (30). We write

\[
(33) \quad \Phi_{\Phi_{\Phi}} = \sigma \varphi, \quad \Psi_{\Phi_{\Phi}} = \sigma \psi,
\]

our failure to attach subscripts to \( \sigma \), \( \varphi \) and \( \psi \) causes no confusion. Thus (32) may be written

\[
\sigma \varphi \ldots \sigma \varphi = \sigma \psi \ldots \sigma \psi
\]

so that, operating on both sides of the last equation with \( \varphi \), and replacing \( \epsilon \) by \( \sigma(\epsilon) \), we have

\[
\varphi \sigma \varphi \ldots \sigma \varphi \sigma = \varphi \sigma \psi \ldots \sigma \psi \sigma.
\]

As \( \varphi \sigma \) and \( \psi \sigma \) are permutable, we find

\[
\varphi \sigma \varphi \ldots \sigma \varphi \varphi = \psi \ldots \sigma \psi \sigma \varphi \sigma.
\]

Removing \( \varphi \sigma \) from the beginning of each member of this equation, we have

\[
\varphi \sigma \ldots \varphi \sigma = \psi \sigma \ldots \psi \sigma,
\]

that is,

\[
\Phi^{(r)}_{\Phi_{\Phi_{\Phi_{\Phi}}}} = \Psi^{(r)}_{\Phi_{\Phi_{\Phi_{\Phi}}}}.
\]

Continuing thus, we see that the \( r \)th iterates of \( \Phi_{\Phi_{\Phi_{\Phi}}} \) and \( \Psi_{\Phi_{\Phi_{\Phi}}} \) are equal. Now, since

\[
\Phi_{\Phi_{\Phi_{\Phi}}} = \Phi_{\Phi_{\Phi_{\Phi_{\Phi}}}}.
\]
we see directly that

\[ \Phi^{(r)}_{p-1, i_{p-1}} = \Psi^{(2r)}_{p-1, i_{p-1}}. \]

There is nothing to prevent us from working back through the sequence which precedes (30) as we did through (30), and thence onward through the earlier sequences. We find thus that some iterate of \( \Phi(z) \) is identical with some iterate of \( \Psi(z) \).

We turn for a while to the case in which \( \Phi(z) \) and \( \Psi(z) \) are polynomials. In a paper published a few years ago,\(^*\) we determined all pairs of polynomials which have an iterate in common, and the results there produced could be used to settle quickly the problem now before us. However, we shall gain a better insight into the nature of the fractional permutable functions which have an iterate in common by studying the polynomial case from the group-theoretic point of view of the present paper, and of our paper Prime and composite polynomials.

From a result stated at the bottom of page 54 of the paper just mentioned, it follows that, if \( \Phi(z) \) and \( \Psi(z) \) are polynomials, we may suppose that the functions in the sequences (29) to (30) inclusive are all polynomials.

We prove now that when \( \Phi(z) \) and \( \Psi(z) \) are polynomials, then, in (30), \( \Phi_{p0} \) is a linear function of \( \Psi_{p0} \). Suppose that this is not so, and that the first pair of functions in (30) which are linear functions of each other are \( \Phi_{pi_p} \) and \( \Psi_{pi_p} \), where \( i_p > 0 \). We have already seen that if \( \Phi(z) \) and \( \Psi(z) \) are subjected to a suitable linear transformation (in this case integral), we will have

\[ \Phi_{pi_p} = \varepsilon \Psi_{pi_p}, \]

or, by (33),

\[ \Phi_{pi_p} = \psi \phi = \varepsilon \sigma \psi. \]

Now the algorithm which produces the sequence (30) supposes that no rational \( \beta(z) \) of degree greater than 1 exists, such that \( \phi = \xi \beta, \psi = \xi \beta. \) We have shown, however, (loc. cit., p. 56) that if \( \Phi_{pi_p} \) has two decompositions of the types shown in (34), in which \( \phi \) and \( \psi \) are of the same degree, \( \phi \) must be a linear function of \( \psi \). This completes the proof that \( \Phi_{p0} \) is a linear function of \( \Psi_{p0}. \)

\(^*\) These Transactions, vol. 21 (1920), p. 313.
We may suppose, thus, that

\[ \phi_{\rho_0} = \varepsilon z R(z'), \quad \psi_{\rho_0} = z R(z'), \]

where \( \varepsilon \) is an \( r \)th root of unity, and where \( R(z) \) is a polynomial.\(^*\)

Hence

\[ (35) \quad \phi_{p-1, i_{p-1}} = \varepsilon \psi_{\rho_0}^{(2)}, \quad \psi_{p-1, i_{p-1}} = \varepsilon_1 \psi_{\rho_0}, \]

where \( \varepsilon_1 \), if not equal to \( \varepsilon \), is unity.

We shall show now that \( \phi_{p-1, 0} \) is a rational function of \( \psi_{p-1, 0} \), so that these two functions may be considered to be given by (35). If this were not the case, we would have

\[ (36) \quad \phi_{p-1, i_{p-1}} = \sigma \varphi, \quad \psi_{p-1, i_{p-1}} = \sigma \psi, \]

where no non-linear \( \beta(z) \) exists such that \( \varphi = \sigma \beta, \psi = \varepsilon \beta \). From (35), and the second equation of (36), we find

\[ (37) \quad \phi_{p-1, i_{p-1}} = \varepsilon \varepsilon_1^{-1} \psi_{\rho_0} \psi_{p-1, i_{p-1}} = \varepsilon \varepsilon_1^{-1} \psi_{\rho_0} \sigma \psi. \]

Thus, by (37) and the first equation of (36), \( \varphi \) and \( \psi \) would determine systems of imprimitivity of the group of the inverse of \( \phi_{p-1, i_{p-1}} \) with not more than one letter in common. For this it would be necessary (loc. cit., p. 57), that the degrees of \( \varphi \) and \( \psi \) be prime to each other. This produces the contradiction that the degree of \( \phi_{p-1, i_{p-1}} \) is not divisible by that of \( \psi_{p-1, i_{p-1}} \). Hence, we have

\[ \phi_{p-2, i_{p-1}} = \varepsilon_2 \psi_{\rho_0}^{(3)}, \quad \psi_{p-2, i_{p-1}} = \varepsilon_2 \psi_{\rho_0}^{(j)}, \]

where \( \varepsilon_2 \) and \( \varepsilon_3 \) are \( r \)th roots of unity, and where \( j \) is 1 or 2.

Continuing thus, we find that if \( \phi(z) \) and \( \psi(z) \) are permutable polynomials (non-linear), which do not come from the multiplication formulas of \( \varepsilon^r \) or \( \cos z \), there exist a linear \( \lambda(z) \), and a polynomial

\[ G(z) = z R(z'), \]

\(^*\) The case of \( r = 1 \) does not require a separate statement, for we may suppose, using a suitable linear transformation, that all polynomials met are divisible by \( z \).
such that
\[ \Phi = \lambda^{-1}(\epsilon_1 G^{(r)} \lambda), \quad \Psi = \lambda^{-1}(\epsilon_2 G^{(r)} \lambda) \]

where \(\epsilon_1\) and \(\epsilon_2\) are \(r\)th roots of unity.

Furthermore, this necessary condition for permutability is immediately seen to be sufficient. In fact, if \(R(z)\) is any rational function, integral or fractional, the above formulas will give a pair of permutable rational functions.

When we seek explicit formulas for the permutable pairs of fractional functions which do not come from the multiplication theorems of the periodic functions, things do not go through smoothly. For instance, it is not necessary that one of the functions \(\Phi\) and \(\Psi\) should be a rational function of the other, as is always the case for polynomials.

We shall give an example of such a case. Let
\[ \varphi(z) = \frac{z^3 + 2}{\varepsilon z + 1}, \quad \psi(z) = \frac{z^3 + 2}{z + 1}, \quad \sigma(z) = \frac{z^3 - 4}{z - 1}, \]
where \(\varepsilon\) is a primitive third root of unity. We shall see below that \(\Phi = \varphi \sigma\) and \(\Psi = \psi \sigma\) are permutable. We observe at present that \(\Phi\) is not a linear function of \(\Psi\); if it were, \(\varphi\) would be a linear function of \(\psi\), which is not so, because \(\psi\) is infinite for \(z = -1\) and \(z = \infty\), whereas \(\varphi\) is infinite for \(z = \infty\), but not for \(z = -1\). We have
\[ \sigma \varphi = \varepsilon z \frac{z^3 - 8}{z^3 - 1}, \quad \sigma \psi = z \frac{z^3 - 8}{z^3 - 1}. \]

As \(\varphi(z) = \psi(\varepsilon z)\), and as \(\sigma \psi(\varepsilon z) = \varepsilon \sigma \varphi(z)\), we have
\[ \varphi \sigma \psi \sigma = \psi(\varepsilon \sigma \psi \sigma) = \psi \sigma \psi(\varepsilon \sigma) = \psi \sigma \varphi \sigma. \]
This means that \(\Phi\) and \(\Psi\), which, we repeat, are not linear functions of each other, are permutable.

It can be shown that \(\Phi\) and \(\Psi\) do not come from the multiplication theorems of the periodic functions. We shall escape the calculations connected with this question by modifying the example. Let \(\beta(z)\) be any rational function such that \(\beta(\varepsilon z) = \varepsilon \beta(z)\), where \(\varepsilon\) is a primitive third root of unity. We consider the functions
\[ \Phi = \varphi \beta \sigma, \quad \Psi = \psi \beta \sigma, \]
where $\alpha$, $\varphi$, and $\psi$ are the three functions of the second degree used above.  
As above, $\Phi$ is not a linear function of $\Psi$.  Also,

$$q \beta \alpha \psi \beta \alpha = \psi (\epsilon \beta \alpha \psi \beta \alpha) = \psi \beta \alpha \psi (\epsilon \beta \alpha) = \psi \beta \alpha \psi \beta \alpha,$$

so that $\Phi$ and $\Psi$ are permutable.

It is easily seen, since $\beta(z)$ is of a very general type, that, by suitably choosing $\beta(z)$, we can make the critical points of $\Phi^{-1}$ and of $\Psi^{-1}$ numerous and complicated at pleasure.  This means that $\Phi$ and $\Psi$ cannot come from the multiplication theorems of the periodic functions, for, if they did, their inverses would have at most four critical points.

In the above example, $\Phi$ and $\Psi$ have the same third iterate.

Concerning the pairs of permutable fractional functions which come neither from the multiplication theorems of the periodic functions, nor from the iteration of a function, the only information we have, in addition to the fact that the functions of the pair have an iterate in common, is that contained in the statement on page 443.  We think that the example given above makes it conceivable that no great order may reign in this class.

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